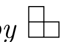
Writing Proofs Misha Lavrov

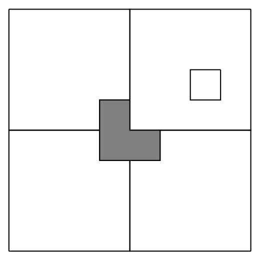
Induction: Solutions

Western PA ARML Practice *March 6, 2016*

1. *Prove that a $2^n \times 2^n$ chessboard with any one square removed can always be covered by*  *shaped tiles.*

   Solution 1: We induct on n. For $n = 0$, a $2^n \times 2^n$ chessboard with one square removed is empty, and therefore already covered. (If the $n = 0$ case is awkwardly trivial, the $n = 1$ case is also easy to verify.)

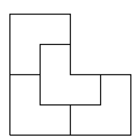
   Suppose we can tile a $2^{n-1} \times 2^{n-1}$ chessboard with any square removed. Take a $2^n \times 2^n$ chessboard with a square missing and divide it into four quarters. As shown below, place one tile in the center so that each quarter is missing one square:

   By the induction hypothesis, each of the quarters can now be tiled, which gives us a way of tiling the $2^n \times 2^n$ chessboard. This means that we know how to tile the chessboard for all n.

   Solution 2: The setup and base case of the proof are similar, but the induction step is different. Given a $2^n \times 2^n$ chessboard with any square removed, divide it up into a $2^{n-1} \times 2^{n-1}$ grid of 2×2 squares. The 2×2 square with a 1×1 square missing can be filled by a single tile, and then we are left with a $2^{n-1} \times 2^{n-1}$ grid of 2×2 squares, with one 2×2 square missing.

   Using four tiles, we can make a copy of the original tile at twice the scale:

   If we want to use the scaled-up tiles to cover the $2^{n-1} \times 2^{n-1}$ grid of 2×2 squares missing, that task is equivalent to covering a $2^{n-1} \times 2^{n-1}$ chessboard with one square missing by tiles of the ordinary size. By the induction hypothesis, this can be done; therefore we can tile the $2^n \times 2^n$ chessboard with a square missing as well. By induction, this is possible for all n.

2. *Chicken McNuggets come in boxes of 6, 9, and 20 nuggets. Prove that for any integer $n > 43$, it is possible to buy exactly n nuggets with a combination of these boxes.*

   We prove six base cases:

   - $44 = 20 + 9 + 9 + 6$, so $n = 44$ is possible.

- $45 = 9 + 9 + 9 + 9 + 9$, so $n = 45$ is possible.
- $46 = 20 + 20 + 6$, so $n = 46$ is possible.
- $47 = 20 + 9 + 9 + 9$, so $n = 47$ is possible.
- $48 = 6 + 6 + 6 + 6 + 6 + 6 + 6 + 6$, so $n = 48$ is possible.
- $49 = 20 + 20 + 9$, so $n = 49$ is possible.

For all larger n, if buying exactly $n - 6$ nuggets is possible, then buying exactly n nuggets is possible: just buy $n - 6$, and then another box of 6 nuggets. Having proven the result for six consecutive values of n, the result for all larger n follows.

If we also allow 4-nugget boxes, then what is the largest integer n such that it's impossible to buy exactly n nuggets?

We can check that $n = 11$ is impossible. If we buy three boxes, then we get at least $4+4+4 = 12$ nuggets, so we just need to check all possibilities for at most two boxes: none of 4, 6, 9, $4+6$, $4+9$, $6+9$ equal 11, and of course buying a 20-nugget box would put us over the target immediately.

We prove that all values of $n \geq 12$ are possible in the same way as before:

- Buying exactly 12 is possible since $12 = 4 + 4 + 4$.
- Buying exactly 13 is possible since $13 = 9 + 4$.
- Buying exactly 14 is possible since $14 = 6 + 4 + 4$.
- Buying exactly 15 is possible since $15 = 9 + 6$.
- From there, we can induct on n: to buy $n > 15$ nuggets, buy $n - 4$ nuggets and then a box of 4 more.

3. *Find the mistake in the following proofs:*

   (a) The first proof is missing a base case. It's true that if $n - 1 = n$, then $n = n + 1$, but $n - 1 = n$ is never true for any n.

   (b) The second proof does not prove enough base cases for the induction step to work. To show that ℓ_1 and ℓ_2 are the same line, we say that they both pass through A_2 and A_n. However, when $n = 3$, these are the same point, which says nothing about ℓ_1 and ℓ_2; the induction step only holds for $n \geq 4$.

4. *Prove that for all $n \geq 1$, the sum of the first n odd numbers is a perfect square.*

   We show that $1 + 3 + \cdots + (2n - 1) = n^2$ by induction on n.

   For $n = 1$, both sides of the equation say $1 = 1$.

   Now assume the statement holds for $n \geq 1$; we show it holds for $n+1$. If $1+3+\cdots+(2n-1) = n^2$, then $1 + 3 + \cdots + (2n + 1) = n^2 + (2n + 1)$: we're just adding $2n + 1$ to both sides. But $n^2 + (2n + 1)$ factors as $(n + 1)^2$, which was what we wanted.

   By induction, the equation holds for all n.

5. *Prove that for all $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$.*

   We show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ by induction on n.

   For $n = 1$, we have $\frac{1}{1^2} \leq 2 - \frac{1}{1}$, which is true, since $1 \leq 1$.

   Now assume the inequality holds for some $n \geq 1$; we'll show it holds for $n+1$. Starting from

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n},$$

   add $\frac{1}{(n+1)^2}$ to both sides. On the left-hand side, we get the sum we want to bound. On the right-hand side, we get

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} = 2 - \frac{(n+1)^2 - n}{n(n+1)^2} = 2 - \frac{1}{n+1} \cdot \frac{n^2+n+1}{n^2+n} < 2 - \frac{1}{n+1}.$$

   Therefore the inequality continues to hold for $n+1$ if it holds for n. By induction, the inequality holds for all n, and so in particular, the sum is bounded above by 2.

6. *In a chess tournament, each pair of players played exactly one game, and incredibly, none of them ended in draws.*

   Prove that there is a participant in the tournament (call him Bobby) such that every other player either lost their game with Bobby, or lost a game with someone else that lost to Bobby.

   We induct on n, the number of participants in the tournament. For $n = 1$, the statement is trivial, and so we let this be our base case.

   Suppose that the statement holds for n players; let's try to show that it holds for $n+1$ players. Think of the resulting tournament as one played by n players, with an extra player then showing up and playing all of them.

   By the induction hypothesis, the first n players can be divided into 3 groups: $\{\text{Bobby}\} \cup S \cup T$, where Bobby is (one possible choice of) the participant we call Bobby, S is the set of players that lose to Bobby, and T is the set of players that beat Bobby (and must therefore lose to someone in S).

   If the extra player loses to Bobby, then they can also be added to S, and the claim continues to hold. If the extra player wins against Bobby, but loses to someone in S, then they can be added to T, and the claim continues to hold. The remaining case is when the extra player wins against Bobby *and* wins against every player in S.

   In that case, partition the $n+1$ players into three new groups: $\{\text{new player}\}$, $S' := S \cup \{\text{Bobby}\}$, and T. Then everyone in S' lost their game with the new player; everyone in T lost a game to someone in S' (in particular, to someone in S.) So the claim continues to hold, with the new player becoming Bobby in place of the old Bobby.

   By induction, the claim holds for all n.

7. *(VTRMC 2012) Define a sequence (a_n) for n a positive integer inductively by $a_1 = 1$ and*

$$a_n = \frac{n}{\prod_{d|n, d<n} a_d}, \quad \textit{where the product ranges over all proper divisors } d \textit{ of } n.$$

Thus $a_2 = 2$, $a_3 = 3$, $a_4 = 2$, etc. Find a_{999000}.

We will prove that the following general formula holds:

$$a_n = \begin{cases} n & \text{if } n \text{ is prime,} \\ p & \text{if } n = p^k \text{ for some prime } p, \\ 1 & \text{if } n \text{ has at least two distinct prime factors.} \end{cases}$$

Since $999000 = 2^3 \cdot 3^3 \cdot 5^3 \cdot 37$, this means that $a_{999000} = 1$.

We prove the formula by an unusual induction on n. Our base case will be every prime n. Then we will show that if the formula holds for all proper divisors of n, it must also hold for n.

Suppose that n is prime. Then its only proper divisor is 1, so $a_n = \frac{n}{a_1} = \frac{n}{1} = n$. The base case is shown.

Now, suppose that the formula holds for all proper divisors of n. There are two cases to consider. First, if n is a prime power—if $n = p^k$ for some prime p, then its proper divisors are $1, p, p^2, \ldots, p^{k-1}$. So we have

$$a_n = \frac{n}{a_1 \cdot a_p \cdot a_{p^2} \cdots a_{p^{k-1}}} = \frac{n}{1 \cdot p \cdots p} = \frac{n}{p^{k-1}} = p.$$

If n is not a prime power, then it has a prime factorization $p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_m}$. For all its divisors d, $a_d = 1$ unless d is a power of a prime, by the inductive hypothesis, so we can ignore a_d in the formula. If d *is* a power of a prime, then it has the form $d = p_i^k$ for $k \le e_i$, and $a_d = p_i$. There are exactly e_i proper divisors d such that $a_d = p_i$: they are $p_i, p_i^2, \ldots, p_i^{e_i}$. So the product of a_d over all divisors d picks up a factor of exactly $p_i^{e_i}$ for every prime p_i in the factorization of n. This means that it exactly equals n, and the formula tells us that $a_n = \frac{n}{n} = 1$.

By induction, the formula holds for all n, and in particular for $n = 999000$.

8. *A binary sequence such as 0011010 is written on a blackboard. In a step, you are allowed to change the first number (from 0 to 1 or vice versa) or the number after the first occurrence of 1. (Starting with 0011010, you could get to 1011010 or to 0010010.) Prove that you can change any sequence to any other sequence of the same length.*

We prove the statement "Any sequence of length n can be changed to any other sequence of length n" by induction on n. The base case, $n = 1$, is easy to prove: we can change 0 to 1 or 1 to 0 by an application of the first rule.

Now assume that the statement holds for some n. A corollary is that in a sequence of length $n+1$, we can change the first n bits to anything we like: the same rules that go between two n-bit sequences let us go between the first n bits of two longer sequences.

Take two arbitrary sequences $x_0 x_1 \cdots x_n$ and $y_0 y_1 \cdots y_n$. We'll show how to get from one to the other.

- If $x_n = y_n$, then we can use the n-bit method (which we know exists by the induction hypothesis) to change $x_0 x_1 \cdots x_{n-1}$ to $y_0 y_1 \cdots y_{n-1}$.

- If $x_n \neq y_n$, then we can use the n-bit method to change the first n bits of $x_0 x_1 \cdots x_n$ to be $000 \cdots 01$, so that we get the sequence $000 \cdots 01 x_n$. The second rule (which lets us change the bit after the first 1), can be used to change this to $00 \cdots 01 y_n$. Now we can use the n-bit method again to change the first n bits to $y_0 y_1 \cdots y_{n-1}$, and we are left with the target sequence.

By induction, we can change any sequence to any other sequence of the same length, regardless of the length.