

## The Invariance Principle: Solutions

*Western PA ARML Practice**February 28, 2016*

1. An  $8 \times 8$  chessboard is colored in the usual way, but that's boring, so you decide to fix this. You can take any row, column, or  $2 \times 2$  square, and reverse the colors inside it, switching black to white and white to black.

*Prove that it's impossible to end up with 63 white squares and 1 black square.*

Any of these operations changes the total number of white squares by an even number. In the case of a row or column, it goes from having  $k$  white squares to having  $8 - k$  white squares, for a change of  $8 - 2k = 2(4 - k)$ . In the case of a  $2 \times 2$  square, we go from  $k$  white squares to  $4 - k$  white squares, for a change of  $4 - 2k = 2(2 - k)$ .

Since we start with an even number of white squares, the number of white squares always remains even, so it is impossible to end with 63 white squares.

2. The numbers  $1, 2, \dots, 100$  are written on a blackboard. You may choose any two numbers  $a$  and  $b$  and erase them, replacing them with the single number  $a + b - 1$ . After 99 steps, only a single number will be left. What is it? After each operation, the total of all the numbers decreases by 1. We begin with a total of  $1 + 2 + \dots + 100 = 5050$ , so after 99 steps, the total will be  $5050 - 99 = 4951$ . Since there's only one number left, that number must be equal to 4951.
3. Suppose you instead replace  $a$  and  $b$  by the product  $ab + a + b$ . What number will be left at the end?

In this case, the product  $(a_1 + 1)(a_2 + 1) \cdots (a_n + 1)$  remains unchanged, since the two factors  $(a + 1)$  and  $(b + 1)$  are replaced by a single factor of  $(ab + a + b + 1) = (a + 1)(b + 1)$ . Initially, this product is  $2 \cdot 3 \cdots 100 = 101!$ . At the end, the product is just one more than the single remaining number, so we will be left with  $101! - 1$ .

4. At a party, some pairs of people shake hands. We call a person odd who has shaken hands with an odd number of other guests. Prove that there is an even number of odd people at the party.

The total number of ordered pairs of people who have shaken hands is even: each handshake between  $A$  and  $B$  produces two ordered pairs  $(A, B)$  and  $(B, A)$ . We can also find this total by adding up the number of times each guest has shaken hands. If the total is even, an even number of the summands are odd.

5. A room is initially empty. Every minute, either two people enter or one person leaves. After exactly  $3^{3^3}$  minutes, could the room contain exactly  $3^{3^3} + 1$  people?

Every minute, the number of people changes by 2 modulo 3. The room starts out empty, so after a number of minutes that's a multiple of 3, the number of people in the room will be divisible by 3, so it cannot be  $3^{3^3} + 1$ .

6. A herd of 100 cows is divided into four pens: 10 cows in the north pen, 20 cows in the east pen, 30 cows in the south pen, and 40 cows in the west pen.

The pens are connected through a gateway we can use to let three cows out of one pen and distribute them between the others. For instance, if we let three cows out of the south pen, we end up with 11 cows in the north pen, 21 cows in the east pen, 27 cows in the south pen, and 41 cows in the west pen.

Prove that we can never use this gateway to split the herd into four equal groups, with 25 cows in each of the four pens.

If we reduce the number of cows in each pen modulo 4, the initial state is  $(2, 0, 2, 0)$  and the final state we want is  $(3, 3, 3, 3)$ . However, opening the gateway always adds  $(-1, -1, -1, -1)$  modulo 4 to the state, so we can never make all four numbers equal modulo 4.

7. (St. Petersburg) A teacher wrote down three positive real numbers on the blackboard and told Dima to decrease one of them by 3%, decrease another by 4%, and increase the last by 5%. Dima wrote down the results in his notebook. It turned out that he wrote down the same three numbers that are on the blackboard, just in a different order. Prove that Dima must have made a mistake.

If the initial numbers are  $x$ ,  $y$ , and  $z$ , the correct solutions are  $0.97x$ ,  $0.96y$ , and  $1.05z$ , so their product changes from  $xyz$  to  $0.97776xyz$ . Therefore the answers can never be the same as the original numbers in a different order.

8. (Engel) There is a positive integer in each square of a rectangular table. In each move, you may double each number in a row or subtract 1 from each number of a column. Prove that you can reach a table of zeroes by a sequence of these permitted moves.

We reduce the columns to zero one at a time, by iteratively applying the following process:

- If there are any entries equal to 1, double their rows.
- Subtract 1 from each number in the column until there are entries equal to 1 again.

Until *all* the entries in the column are equal to 1, this process decreases the total, so eventually we will reduce to the all-ones case. At that point, subtracting one from each entry reduces each element of the column to 0.

9. (a) The integers  $1, 2, \dots, n$  are written down in that order. At each step, you may swap any two integers: for example, if  $n = 6$ , you can begin by changing  $1, 2, 3, 4, 5, 6$  to  $1, 2, 5, 4, 3, 6$  by swapping 3 and 5.

Prove that you can never return to the original order after an odd number of swaps.

Solution #1: We keep track of the number of “inversions”: pairs  $\{i, j\}$  that occur in the wrong order in the list (if  $i < j$  and  $i$  is after  $j$  in the list, or vice versa). Initially, there are no inversions.

Swapping two elements of the list adds an odd number to the number of inversions. If we swap  $a$  and  $b$ , then:

- If  $x$  is before both of them in the list, or after both of them, then the number of inversions with  $x$  is unaffected.
- If  $x$  is between them in the list, then both pairs  $\{x, a\}$  and  $\{x, b\}$  change their state of being an inversion or not.
- Finally,  $\{a, b\}$  changes its state.

After an odd number of swaps, the number of inversions is also odd, so we cannot be in the initial state.

Solution #2: We begin by proving a weaker claim: we can't return to the original order after an odd number of swaps of adjacent elements, without swapping any non-adjacent pairs.

Here, if  $a < b$ , then  $a$  and  $b$  start out in that order, and only change places if we actually swap  $a$  and  $b$  while they're adjacent. If we make some number of swaps and end up at the initial state, then we must have swapped  $a$  and  $b$  an even number of times. This is true for any  $a$  and  $b$ , so we must have made an even number of swaps total.

Next, we show that this weaker claim implies the original. To do this, note that we can swap two elements that are  $k$  steps apart through a sequence of  $2k - 1$  swaps of adjacent elements: make  $k - 1$  swaps to put them next to each other, swap them, and then reverse the  $k - 1$  swaps.

So if we had a sequence of an odd number of swaps that returned to the original state, we could replace each swap by an odd number of adjacent swaps. An odd number of odd numbers added together is odd, so we'd have an odd number of adjacent swaps returning to the original state, which is impossible.

- (b) *The 15-puzzle is a sliding puzzle with fifteen square tiles, numbered 1 through 15, arranged in a  $4 \times 4$  square. In the late 19th century, Sam Loyd offered a \$1000 prize for anyone that could swap the 14 and 15 tiles by sliding the tiles around. Prove that this is impossible, and so the prize would never have to be paid out.*

Imagine the empty space as a tile labeled "16". Then each step of solving the puzzle is a swap of the numbers  $\{1, 2, \dots, 16\}$ . If we begin by swapping 14 and 15, and then solve the puzzle, the total number of swaps is even by part (a), so the number of moves made in the puzzle is odd.

However, after an odd number of moves, the empty space can't end up back in the lower right corner: if you imagine coloring the  $4 \times 4$  square in a checkerboard coloring, each move changes the empty space from black to white, so after an odd number of moves the color will be different from the original. Therefore the solution we want can't exist.

10. (Putnam 2008) *Start with a sequence  $a_1, a_2, \dots, a_n$  of positive integers. If possible, choose two indices  $j < k$  such that  $a_j$  does not divide  $a_k$ , and replace  $a_j$  and  $a_k$  by  $\gcd(a_j, a_k)$  and  $\text{lcm}(a_j, a_k)$ , respectively. Prove that if this process is repeated, it must eventually stop, and the final sequence does not depend on the choices made.*

Let  $p$  be a prime, and let  $p^{e_1}, p^{e_2}, \dots, p^{e_n}$  be the highest powers of  $p$  that divide  $a_1, a_2, \dots, a_n$ ,

respectively. Then replacing  $a_j$  and  $a_k$  by their GCD and LCM means replacing  $p^{e_j}$  and  $p^{e_k}$  by  $p^{\min\{e_j, e_k\}}$  and  $p^{\max\{e_j, e_k\}}$ , putting the two powers of  $p$  in increasing order.

Each such step either leaves the  $p$ -power sequence unchanged or makes it more ordered (reducing the number of pairs that are in decreasing order); the latter can only happen finitely many times, since we start with at most  $\binom{n}{2}$  out-of-order pairs. Moreover, each step involves reordering the powers of  $p$  for at least one prime  $p$ : if they were already in order for every  $p$ , then  $a_j$  would divide  $a_k$ . Therefore only finitely many steps are possible. The final sequence is obtained by sorting the  $p$ -power sequences for every  $p$  and then multiplying them together.

11. *Seven squares of an  $8 \times 8$  grid are shaded. At each step, we shade in each unshaded square that has at least two shaded neighboring squares (horizontally or vertically). Prove that this process cannot end in the entire grid being shaded.*

Our invariant here is the total perimeter of the shaded squares (not counting borders between two shaded squares). Initially, this is at most  $7 \cdot 4 = 28$ . When a square is shaded by the given rules, the total perimeter cannot increase: the newly shaded square is adjacent to at least 2 already shaded squares, so we lose at least 2 borders and gain at most 2.

We cannot end with the entire grid being shaded, because then the total perimeter would have increased to 32, which cannot happen.