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Foundations of Probability: Solutions

Western PA ARML Practice — *April 10, 2016*

1 Theoretical exercises

1. *Prove that* $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$. *What happens if the events* A *and* B *are disjoint?*

   Converting the probability into a sum of $\Pr(\omega)$ as ω ranges over various sets, we see that each $\omega \in A \cup B$ is counted exactly once on the right-hand side.

   If A and B are disjoint, then $\Pr[A \cap B] = 0$, so we have $\Pr[A \cup B] = \Pr[A] + \Pr[B]$.

2. *Prove the subadditivity of probability.*

   We prove that

   $$\Pr[A_1 \cup A_2 \cup \cdots \cup A_n] \le \Pr[A_1] + \Pr[A_2] + \cdots + \Pr[A_n]$$

   by induction on n. The base case $n = 1$ is trivial, but we'll need to do $n = 2$ separately anyway, so let's make that another base case. In fact, $n = 2$ follows from the previous problem, since

   $$\Pr[A_1 \cup A_2] = \Pr[A_1] + \Pr[A_2] - \Pr[A_1 \cap A_2] \le \Pr[A_1] + \Pr[A_2].$$

   For the induction step, let $B = A_1 \cup A_2 \cup \cdots \cup A_{n-1}$. Then we have (by the $n = 2$ case) that $\Pr[B \cup A_n] \le \Pr[B] + \Pr[A_n]$. Now use the induction hypothesis to bound $\Pr[B]$ by $\Pr[A_1] + \cdots + \Pr[A_{n-1}]$, and we have the statement we want.

3. *A certain school has 500 clubs, each with at least 10 members (some people are in multiple clubs). Prove that it's possible to split the school into two groups (say, the Blue group and the Green group), not necessarily of equal size, such that every club has at least one member from each group.*

   Choose a Blue/Green split by assigning each student to a random group independently. Let A_i be the event that all the members of the i^{th} club are in one group. Then $\Pr[A_i] \le 2 \cdot 2^{-10} = \frac{1}{512}$, since there are at least 10 members.

   By the previous problem, the probability that any club has this property is

   $$\Pr[A_1 \cup \cdots \cup A_n] \le \Pr[A_1] + \cdots + \Pr[A_n] \le 500 \cdot \frac{1}{512} < 1.$$

   Therefore with positive probability, the random assignment works: so there must be some valid assignment.

4. *Suppose that A_1, A_2, and A_3 are pairwise independent events:* $\Pr[A_i \cap A_j] = \Pr[A_i] \cdot \Pr[A_j]$ *for all* $1 \leq i < j \leq 3$. *Prove or disprove:*

$$\Pr[A_1 \cap A_2 \cap A_3] = \Pr[A_1] \cdot \Pr[A_2] \cdot \Pr[A_3].$$

This is false. Let our probability space be given by three independent coinflips x, y, and z. Let A_1 be the event that $x \neq y$, A_2 be the event that $x \neq z$, and A_3 be the event that $y \neq z$.

We can check that $\Pr[A_i] = \frac{1}{2}$ and that $\Pr[A_i \cap A_j] = \frac{1}{4}$ for any i and j (for example, $A_i \cap A_j$ can happen if $(x, y, z) = (\mathrm{H}, \mathrm{T}, \mathrm{T})$ or if $(x, y, z) = (\mathrm{T}, \mathrm{H}, \mathrm{H})$), so the events are pairwise independent.

However, $\Pr[A_1 \cap A_2 \cap A_3] = 0$: when three coins are tossed, two of them have to be the same. This is different from $\Pr[A_1] \cdot \Pr[A_2] \cdot \Pr[A_3] = \frac{1}{8}$.

5. *We say that events A and B are* positively dependent *if* $\Pr[A \mid B] > \Pr[A]$. *Does this imply that* $\Pr[B \mid A] > \Pr[B]$? *Prove this, or give a counter-example.*

We can rewrite $\Pr[A \mid B] > \Pr[A]$ as $\frac{\Pr[A \cap B]}{\Pr[B]} > \Pr[A]$, or $\Pr[A \cap B] > \Pr[A] \cdot \Pr[B]$. (We're assuming $\Pr[B] > 0$, but if $\Pr[B] = 0$, we can't condition on it anyway.)

This inequality is symmetric in A and B, so it's the same as $\Pr[B \mid A] > \Pr[B]$.

2 Conditional probability

1. *You take a quarter out of your pocket and flip it 10 times; 7 of the flips land heads and 3 land tails. However, you realize that you left home with five quarters in your pocket. Four were fair coins, and the remaining coin was a trick coin that lands heads with a $\frac{2}{3}$ chance. What is the probability that the coin you were flipping was a trick coin?*

Initially, the odds of the coin being a trick coin were $1 : 4$. Each time the coin lands heads, we multiply this by $\frac{2}{3} : \frac{1}{2} = 4 : 3$, and each time the coin lands tails, we multiply this by $\frac{1}{3} : \frac{1}{2} = 2 : 3$. After 7 flips that land heads and 3 flips that land tails, the odds are $4^7 \cdot 2^3 : 4 \cdot 3^{10}$, which simplifies to $2^{15} : 3^{10}$ or $32768 : 59049$, so the probability is

$$\frac{32768}{32768 + 59049} = \frac{32768}{91817}.$$

2. *I draw two cards from a standard 52-card deck. You ask me: "Is at least one of your cards an ace?" and I say "Yes." What is the probability that both of my cards are aces?*

Let (x, y) be the pair of cards. There are $4 \cdot 51$ outcomes in which x is an ace, $4 \cdot 51$ outcomes in which y is an ace, but the $4 \cdot 3$ outcomes in which both were aces are counted twice.

Altogether the probability that at least one card is an ace (call this event A) is $\frac{4 \cdot 51 + 4 \cdot 51 - 4 \cdot 3}{52 \cdot 51} = \frac{33}{221}$. The probability that both cards are aces (call this event $2A$) is $\frac{4 \cdot 3}{52 \cdot 51} = \frac{1}{221}$.

Having learned that A occurred, the new probability that $2A$ occurred is $\Pr[2A \mid A] = \frac{\Pr[2A \cap A]}{\Pr[A]} = \frac{\Pr[2A]}{\Pr[A]} = \frac{1/221}{33/221} = \frac{1}{33}$.

3. *The Miller–Rabit test for primality is the most widely used way to test if a number is prime. It has a one-sided guarantee: if it says "NO" on input n, then n is composite. However, if it says "YES" on input n, then n may be prime or composite. We have the guarantee that*

$$\Pr[\textit{Test says "YES" on } n \mid n \textit{ is composite}] \le \frac{1}{2}.$$

Each time the test is performed, it has an independent chance of making this error. The Prime Number Theorem says that a randomly chosen number between 1 and N has approximately a $\frac{1}{\ln N}$ chance of being prime. Suppose I choose a random number between 1 and a googol (10^{100}). I keep running the Miller–Rabin test on it and it keeps saying "YES". How many times do I need to run it before being 99% certain that my number is prime?

Initially, the probability that the number is prime is $\frac{1}{\ln 10^{100}} = \frac{1}{100 \ln 10}$, so the odds of prime to composite are approximately $1 : 100 \ln 10$ or $1 : 230$. (Due to the difference between odds and probability, this is more like $1 : 229$, but such minor details won't matter.)

Each time the test says "YES", the odds are multiplied by $2 : 1$, since a "YES" is twice as likely for a prime number.

A probability of 99% corresponds to odds of $99 : 1$, or $22671 : 229$. After 15 tests, our odds become $16384 : 229$, which is not quite enough (it leads to a probability of 98.6%); after 16 tests, our odds become $32768 : 229$, for a probability of 99.3%.

4. *A murder is committed in a town of 1000 people. You pick a random suspect off the street, and test their fingerprints. Your fingerprint test is 99% accurate: there is a 99% chance of a match for two fingerprints from the same person, but only a 1% chance of a match for two fingerprints from different people.*

   (a) *If the random suspect's fingerprints match the murder weapon, what is the probability that they are guilty?*

   The odds that a random suspect in this town is guilty are $1 : 999$. However, a match on the fingerprints is 99 times more likely for a guilty person, updating the odds to $99 : 999$, for a probability of $\frac{99}{99+999} = \frac{11}{122}$.

   (b) *Eyewitness testimony also puts the suspect at the crime scene. It's known that eyewitnesses will always identify the true criminal correctly, but will also identify an unrelated person as having been at the crime scene 10% of the time. What is the new probability that they are guilty?*

   Positive eyewitness testimony is 10 times more likely for a guilty person, so we multiply the odds again by $10 : 1$, and they become $990 : 999$ (still more likely to be innocent than not, by a hair). The probability is $\frac{990}{990+999} = \frac{110}{221}$.