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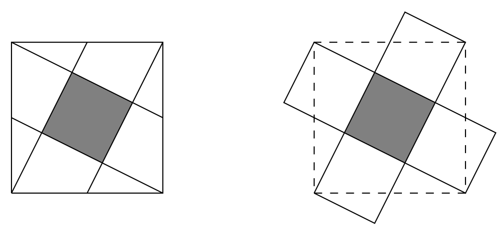
	The Area Method (Solutions)	
Western PA ARML Practice		*October 18, 2015*

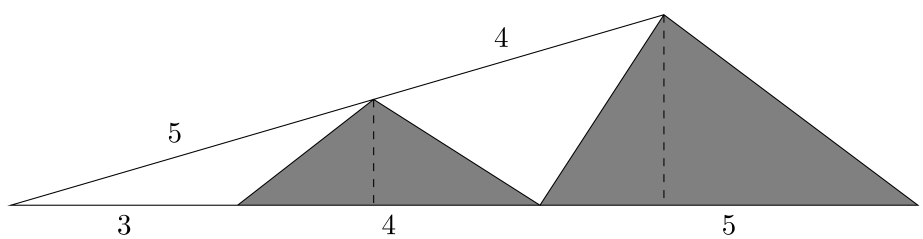
1 Warm-up problems

1. *In square $ABCD$, line segments are drawn from A to the midpoint of BC, from B to the midpoint of CD, from C to the midpoint of DA, and from D to the midpoint of AB. The four segments form a smaller square within square $ABCD$. If $AB = 1$, what is the area of the smaller square?*



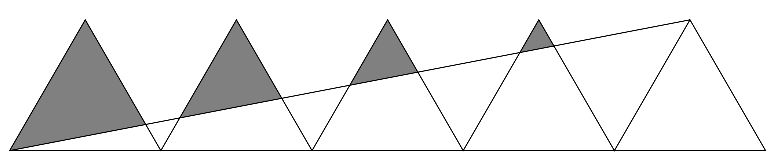
We can rearrange the pieces of square $ABCD$, as shown above, to form five squares congruent to the small one. Therefore the area of the small square is $\frac{1}{5}$.

2. *In the diagram below, what is the ratio of the areas of the two shaded triangles?*



Draw heights of the two triangles (as shown above). Similar triangles allow us to conclude that the large triangle has height $\frac{9}{5}$ of the height of the small triangle. Its base is $\frac{5}{4}$ of the base of the small triangle. Therefore its area is $\frac{9}{4}$ of the small triangle.

3. *In the diagram below, what is the ratio of the shaded area to the area of one of the five congruent triangles?*

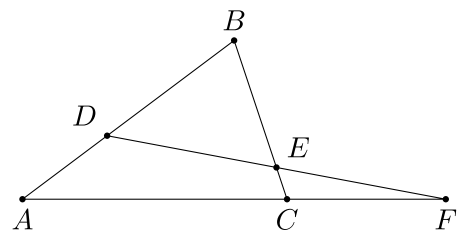


By similar triangles or otherwise, we observe that the biggest shaded region extends $\frac{4}{5}$ of the way along the right side of its triangle, so it is $\frac{4}{5}$ of the area of one of the triangles. The next shaded region extends $\frac{3}{4}$ of the way along the left, and $\frac{3}{5}$ of the way along the right, so it is

$\frac{9}{20}$ of the area of one of the triangles. The next shaded region extends $\frac{1}{2}$ of the way along the left, and $\frac{2}{5}$ of the way along the right, so it is $\frac{1}{5}$ of the area of one of the triangles. The last shaded region extends $\frac{1}{4}$ of the way along the left, and $\frac{1}{5}$ of the way along the right, so it is $\frac{1}{20}$ of the area of one of the triangles. Adding these up, we get $\frac{4}{5} + \frac{9}{20} + \frac{1}{5} + \frac{1}{20} = \frac{3}{2}$.

2 The area method

1. *(ARML 1996) In* $\triangle ABC$*,* $AB = AC = 115$*,* $AD = 38$*, and* $CF = 77$*. Compute* $\frac{S_{CEF}}{S_{DBE}}$.



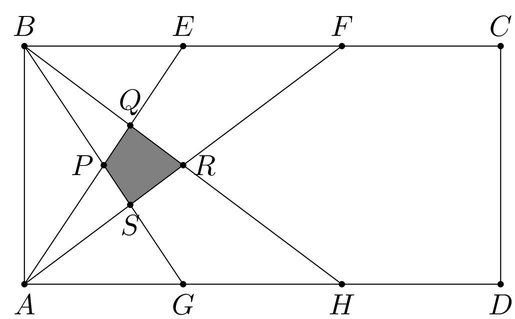
We have $\frac{DE}{FE} = \frac{S_{DBC}}{S_{FBC}} = \frac{\frac{77}{115}S_{ABC}}{-\frac{77}{115}S_{ABC}} = -1$, so $DE = EF$.

Therefore $S_{CEF} = \frac{1}{2}S_{CDF} = \frac{1}{2} \cdot \frac{77}{192}S_{ADF} = \frac{1}{2} \cdot \frac{77}{192} \cdot \frac{38}{115}S_{ABF}$.

On the other hand, $S_{DBE} = \frac{1}{2}S_{DBF} = \frac{1}{2} \cdot \frac{77}{115}S_{ABF}$.

Taking the ratio and simplifying, we get $\frac{38}{192} = \frac{19}{96}$.

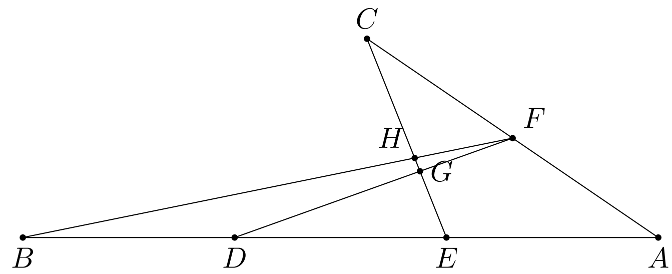
2. *(ARML 2000) In rectangle* $ABCD$*,* G *and* H *are trisection points of* AD*, and* E *and* F *are trisection points of* BC*. If* $AB = 360$ *and* $BC = 450$*, compute the area of* $PQRS$.



We find the area $\triangle PQR$ first, then double due to symmetry.

R is the midpoint of BH, and $\frac{BQ}{HQ} = \frac{S_{BAE}}{S_{HAE}} = -\frac{\frac{1}{6}S_{ABCD}}{\frac{1}{3}S_{ABCD}} = \frac{1}{2}$, so Q is $\frac{1}{3}$ of the way from B to H. Therefore $QR = \frac{1}{6}BH$, and $S_{PQR} = \frac{1}{6}S_{PBH} = \frac{1}{12}S_{GBH} = \frac{1}{72}S_{ABCD}$. Doubling, we get $S_{PQRS} = \frac{1}{36}S_{ABCD} = \frac{1}{36} \cdot 360 \cdot 450 = 4500$.

3. *(ARML 2012) Given noncollinear points A, B, C, segment AB is trisected by points D and E, and F is the midpoint of segment AC. DF and BF intersect CE at G and H, respectively. If $S_{EDG} = 18$, compute S_{FGH}.*



We begin by locating G and H along CE.

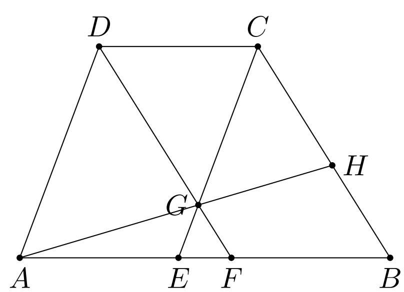
By the intersection lemma, $\frac{CG}{EG} = \frac{S_{CDF}}{S_{EDF}}$. We have $S_{CDF} = \frac{1}{2}S_{CDA}$, while $S_{EDF} = \frac{1}{2}S_{ADF} = \frac{1}{4}S_{ADC}$, so the ratio is $2:1$ and G is $\frac{2}{3}$ of the way from C to E.

By the intersection lemma again, $\frac{CH}{EH} = \frac{S_{CBF}}{S_{EBF}}$. WE have $S_{CBF} = \frac{1}{2}S_{CBA}$, while $S_{EBF} = \frac{2}{3}S_{ABF} = \frac{1}{3}S_{ABC}$, so the ratio is $3:2$ and H is $\frac{3}{5}$ of the way from C to E.

In particular, $GH = (\frac{2}{3} - \frac{3}{5})EC = \frac{1}{15}EC$.

Therefore $S_{EDG} = \frac{1}{3}S_{EDC} = \frac{1}{9}S_{ABC}$, while $S_{FGH} = \frac{1}{15}S_{FEC} = \frac{1}{30}S_{AEC} = \frac{1}{90}S_{ABC}$, which is $\frac{1}{10}S_{EDG} = \frac{18}{10} = \frac{9}{5}$.

4. *(ARML 2015) In trapezoid $ABCD$ with bases AB and CD, $AB = 14$ and $CD = 6$. Points E and F lie on AB such that $AD \parallel CE$ and $BC \parallel DF$. Segments DF and CE intersect at G, and AG intersects BC at H. Compute $\frac{S_{CGH}}{S_{ABCD}}$.*



From parallelograms, we get $AE = DC = FB = 6$, leaving $EF = 2$. Therefore $\frac{S_{CDF}}{S_{EDF}} = \frac{6}{2} = 3$, so G is $\frac{3}{4}$ of the way from C to E; similarly, it is $\frac{3}{4}$ of the way from D to F.

We have $\frac{CH}{BH} = \frac{S_{CAG}}{S_{BAG}}$ by the intersection lemma. By the vertex sliding lemma, $S_{CAG} = \frac{3}{4}S_{CAE} = \frac{3}{4}\cdot\frac{6}{14}S_{CAB} = \frac{9}{28}S_{ABC}$. By another application of vertex sliding, $S_{BAG} = \frac{1}{4}S_{BAC}$, so $\frac{CH}{BH} = -\frac{9}{7}$.

Since $CH = \frac{9}{16}CB$ and $CG = \frac{3}{4}CE$, we have $S_{CGH} = \frac{27}{64}S_{CEB}$. If h is the height of $ABCD$, then $|S_{CEB}| = \frac{1}{2}\cdot 8\cdot h = 4h$, while $|S_{ABCD}| = \frac{6+14}{2}\cdot h = 10h$, so $S_{CGH} = \frac{27}{64}\cdot\frac{4}{10}S_{ABCD}$, whcih simplifies to $\frac{27}{160}$.