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Counting Strategies: Solutions

Western PA ARML Practice	*December 6, 2015*

1. *A* para-palindrome *is a word like FOOLPROOF or BATHTUB: it's off from being a palindome by virtue of a single letter being wrong. Of course, we can also extend this to numbers, declaring 31415 to be a para-palindrome. How many 6-digit numbers are para-palindromes?*

   The answer is 24 300.

   There are three possible para-palindrome patterns: $ABCDBA$, $ABCCDA$, and $ABCCBD$, where A, B, C, and D are dgiits. In each case, there are 9 choices for A (since $A \neq 0$), 10 choices for B and C, and 9 choices for D (since it must be different from its mirror opposite). This means 8 100 para-palindromes in each case, and 24 300 total.

2. *A chess piece starts out in the bottom left corner of an 8×8 chessboard and can take steps either one square right or one square up.*

   (a) *How many paths can the chess piece take to get to the top right square?*

   The answer is $\binom{14}{7} = 3\,432$.

   This is a well-known problem: the chess piece takes 14 steps, 7 of which are left steps and 7 of which are up steps. Each order of those steps gives a different path, and there are $\binom{14}{7}$ ways to pick which steps are left steps.

   (b) *How many of those paths avoid the 2×2 center of the chessboard?*

   The answer is $\binom{14}{7} - 2\binom{7}{3}^2 = 982$.

   Let A be the starting square, Z the ending squares, and M and N be the two squares in the center which are halfway between A and Z. (On a chessboard, these are e4 and d5.)

   Any path through the center must pass through M or N. There are $\binom{7}{3}$ paths from A to M, and $\binom{7}{3}$ paths from M to Z, so $\binom{7}{3}^2$ paths from A to Z pass through M. Similarly, $\binom{7}{3}^2$ paths from A to Z pass through M, and excluding those gives $\binom{14}{7} - 2\binom{7}{3}^2$ paths that avoid the center.

3. *(AMC 10 2014) Four fair six-sided dice are rolled. What is the probability that at least three of the four dice show the same value?*

   The answer is $\frac{7}{72}$.

   There are $\binom{4}{1} \cdot 6 \cdot 5 = 120$ outcomes with exactly three dice with the same value: choose which die is different, choose the value of the three dice, and choose a different value of the die which is left over. There are also 6 outcomes in which all four dice are the same. So the probability is $\frac{126}{1296} = \frac{7}{72}$.

4. *(AIME 2003) An integer between 1000 and 9999, inclusive, is called* balanced *if the sum of its two leftmost digits equals the sum of its two rightmost digits. How many balanced integers are there?*

   The answer is $\sum_{k=1}^{9} k(k+1) + \sum_{k=1}^{9} k^2 = 615$.

   When the sum of each pair of digits is $k \leq 9$, there are $k+1$ choices for the rightmost digits: $(0,k),(1,k-1),\ldots,(k,0)$. There are only k choices for the leftmost digits, since $(0,k)$ is not possible. So there are $k(k+1)$ balanced integers in this case, and together these cases give the first summation above.

   If the sum of each pair of digits is $s \geq 10$, let $k = 19 - s$. Then there are k^2 choices for each pair: $(9, 10-k), \ldots, (10-k, 9)$. So there are k^2 balanced integers in this case, and together these cases give the second summation above.

5. *(AIME 2002) Count the number of sets $\{A, B\}$, where A and B are* nonempty *subsets of $\{1,2,3,\ldots,10\}$ with no elements in common.*

   The answer is $\frac{3^{10}+1}{2} - 2^{10} = 28\,501$.

   For each $i \in \{1,2,3,\ldots,10\}$, there are 3 possibilities: $i \in A$, $i \in B$, or neither. This gives 3^{10} cases. However, in 2^{10} of the cases, $A = \emptyset$, which is ruled out; in 2^{10} more of the cases, $B = \emptyset$. Finally, there is the $\{\emptyset, \emptyset\}$ case. This yields $3^{10} - 2 \cdot 2^{10} + 1$ by inclusion-exclusion, but we must divide by 2 since the set $\{A, B\}$ is unordered.

6. *How many ways are there to rearrange the letters of "FROUFROU" such that no two identical letters are adjacent? ("FROUFROU" itself counts.)*

   The answer is 864.

   With the intent of using inclusion-exclusion, we figure out the following things:

   - There are $\frac{8!}{2^4}$ ways to rearrange the letters of "FROUFROU": they can be rearranged in 8! ways, but this counts each arrangement 2^4 times, because 4 pairs of letters are identical.

   - If we require that the two O's are adjacent, there are $\frac{7!}{2^3}$ rearrangements: we think of these as rearrangements of F, F, R, R, U, U, and the two-letter block OO, with 3 pairs of identical letters.

   - If we require that the two O's and the two U's are adjacent, there are $\frac{6!}{2^2}$ rearrangements: we think of these as rearrangements of F, F, R, R, and the two-letter blocks OO and UU, with 2 pairs of identical letters.

   - If we require that the two R's, the two O's, and the two U's are adjacent, there are $\frac{5!}{2}$ rearrangements: we think of these as rearrangements of F, F, RR, OO, and UU, with 1 pair of identical letters.

   - Finally, if we require that all pairs of identical letters are adjacent, there are 4! ways to rearrange those two-letter blocks.

Putting these together, we get

$$\frac{8!}{2^4} - \binom{4}{1}\frac{7!}{2^3} + \binom{4}{2}\frac{6!}{2^2} - \binom{4}{3}\frac{5!}{2} + \binom{4}{4}4! = 864.$$

7. *A total of 7 beads of two colors are used to make a bracelet. How many different bracelets can be made? (Bracelets can be rotated or flipped over.)*

   The answer is 18.

   We divide the 128 ways to arrange the 7 beads in a fixed circle into three cases:

   - There's the 2 arrangements of 7 beads of a single color, which are unchanged when flipped or rotated.
   - For each of 7 axes of symmetry, there are 2^4 arrangements of beads which are symmetric about that axis; $2^4-2 = 14$ arrangements, excluding the first case, and $7 \cdot 14$ arrangements total.

     These can be divided into groups of 7 that produce the same bracelet (just rotated), so only 14 unique bracelets arise from arrangements in this case.

     (It's also worth noting that if an arrangement of beads is symmetric about any two axes, then all beads must be the same.)
   - This leaves $128 - 2 - 98 = 28$ arrangements of beads which are not symmetric about any axis. These can be divided into groups of 14 that produce the same bracelet (just rotated or flipped), so only 2 unique bracelets arise from arrangements in this case.

   Altogether, there are $2 + 14 + 2 = 18$ bracelets.

8. *(AIME 1996) Five teams play each other in a round-robin tournament; each game is random, with either team having a 50% probability of winning (there are no draws). What is the probability that every team will win at least once, but no team will be undefeated?*

   The answer is $\frac{17}{32}$.

   Let W be the event that a team is undefeated, and L be the event that a team loses all its games. We want find

   $$\Pr[\neg W \text{ and } \neg L] = 1 - \Pr[W] - \Pr[L] + \Pr[W \text{ and } L].$$

   For a fixed team, the probability that it wins all its games is $\frac{1}{16}$; since two teams can't win all their games at the same time (they play each other), the probability that any team does this is just $\Pr[W] = \frac{5}{16}$. Similarly, $\Pr[L] = \frac{5}{16}$ by exactly the same argument.

   If we fix a lucky team and an unlucky team, the probability that the lucky team wins all its games and the unlucky team loses all its games is $\frac{1}{2^7}$: there are 7 games, which all have to go right. So $\Pr[W \text{ and } L] = \frac{5 \cdot 4}{2^7} = \frac{5}{32}$.

   So the probability we want is $1 - \frac{5}{16} - \frac{5}{16} + \frac{5}{32} = \frac{17}{32}$.

9. *(AIME 2001) The squares in a 3×3 grid are colored red and blue at random, and each color is equally likely. What is the probability that a 2×2 square will be entirely red?*

   The answer is $\frac{95}{512}$.

   There are four 2×2 squares that could be entirely red. The probability that this happens for one square is $\frac{1}{2^4}$; the probability for two adjacent 2×2 squares is $\frac{1}{2^6}$ (since they overlap in 2 places) and for two diagonally opposite 2×2 squares it is $\frac{1}{2^7}$ (since they overlap in 1 place. The probability that three 2×2 squares are all red is $\frac{1}{2^8}$, and the probability that all four 2×2 squars are red is $\frac{1}{2^9}$. By inclusion-exclusion, we get

$$4 \cdot \frac{1}{2^4} - \left(4 \cdot \frac{1}{2^6} + 2 \cdot \frac{1}{2^7}\right) + 4 \cdot \frac{1}{2^8} - \frac{1}{2^9} = \frac{1}{4} - \frac{1}{16} - \frac{1}{64} + \frac{1}{64} - \frac{1}{512} = \frac{95}{512}.$$

10. *(Putnam 1985) Find the number of ordered triples (A_1, A_2, A_3) of sets with the property that*

    (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, *and*

    (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

    The answer is 6^{10}.

    For each $i \in \{1, 2, 3, \ldots, 10\}$, at least 1 and at most 2 of the statements $i \in A_1$, $i \in A_2$, $i \in A_3$ must be true, which gives 6 possibilities.