Number Theory

ARML Homework #1

September 22, 2013

Homework disclaimer: I believe that math is interesting, and I try to think of problems that I think are interesting. If you think they're interesting too, then you will want to do them. I'll leave it at that. If you solve some problems and want feedback on your answers, email them to me at `mlavrov@andrew.cmu.edu`. (For problems that aren't proofs, my feedback will be much more useful if you show work.)

1. (This has little to do with what we covered last week, but it's a well-known problem.)

   The number $n!$ (which we call "n factorial" as opposed to saying "ENN" loudly) is defined to be the product $n! = 1 \cdot 2 \cdot 3 \cdots (n-2) \cdot (n-1) \cdot n$. For example, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

   Find the number of zeroes at the end of 100!. (You don't want to compute 100!, believe me.)

2. We can also define the operation $x \bmod y$ for real numbers x and y, by saying that $x \bmod y$ is the number you get by subtracting y from x until the result is less than y (but still positive). For example, $\pi \bmod 1 = \pi - 3 \approx 0.14$.

   By using the rule $\gcd(x, y) = \gcd(y, x \bmod y)$ for $x \geq y$, start computing $\gcd(\pi, 1)$. Keep track of the combination of π and 1 that gives you each of the intermediate answers. What happens?

   (I'm not cruel, so you should probably use a calculator for this problem.)

3. (a) Prove that for any integer x, either $x^2 \equiv 0 \pmod 4$ or $x^2 \equiv 1 \pmod 4$.

   (b) If $x^2 \equiv b \pmod{12}$, find all the possible values of b between 0 and 11. (These are called the *quadratic residues* mod 12.)

4. A *Pythagorean triple* is a triple of integers (a, b, c) such that $a^2 + b^2 = c^2$. Prove that for any Pythagorean triple (a, b, c), the product abc is divisible by 30.

   (Hint: consider the equation $a^2 + b^2 \equiv c^2$ mod 2, mod 3, mod 5.)

   If you're feeling ambitious, prove that abc is divisible by 60 instead.

5. Prove that the following rule for divisibility by 7 works:

   Take the last digit of a number, double it, and subtract it from the rest of the number. Repeat until the result is clearly divisible by 7 or not. For example, 2415 is divisible by 7, because

$$2415 \quad \Rightarrow \quad 241 - 2 \cdot 5 = 231 \quad \Rightarrow \quad 23 - 2 \cdot 1 = 21 \quad \Rightarrow \quad 2 - 2 \cdot 1 = 0.$$