Geometry

Circles

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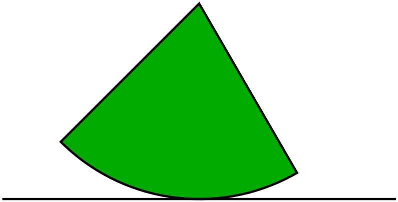
ARML Practice 12/08/2013

Warm-up problems

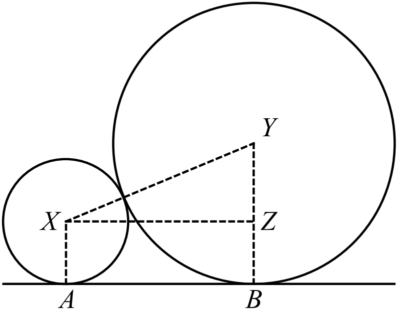
1. A circular arc with radius 1 inch is rocking back and forth on a flat table. Describe the path traced out by the tip.



2. A circle of radius 4 is externally tangent to a circle of radius 9. A line is tangent to both circles and touches them at points A and B. What is the length of AB?

Warm-up solutions

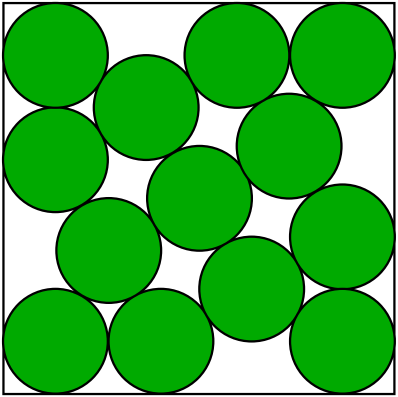
1. A horizontal line 1 inch above the table.
2. Let X and Y be the centers of the circle. Lift AB to XZ as in the diagram below.



Then $XY = 4 + 9 = 13$, $YZ = 9 - 4 = 5$, and $XZ = AB$. Because $XZ^2 + YZ^2 = XY^2$, we get $XZ = AB = 12$.

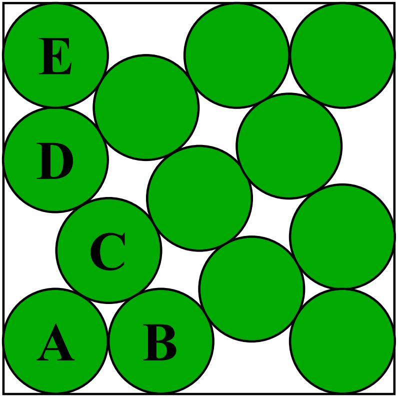
Many tangent circles

Shown below is the densest possible packing of 13 circles into a square. If the radius of a circle is 1, find the side length of the square.



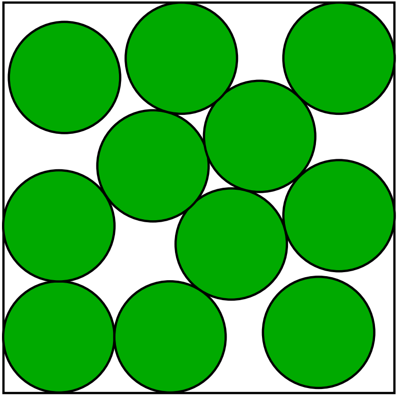
Many tangent circles: Solution

ABC is equilateral with side length 2, so C is $\sqrt{3}$ units above A. ACD is isosceles, so D is $\sqrt{3}$ units above C, and finally E is 2 units above D. So $AE = 2 + 2\sqrt{3}$, and the square has side $4 + 2\sqrt{3}$.



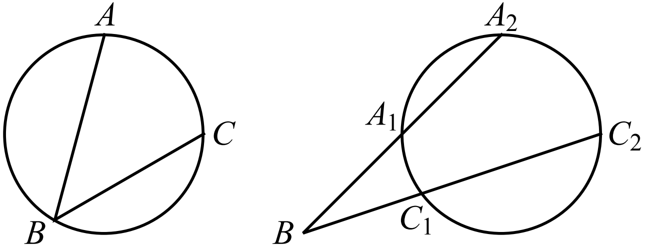
More tangent circles

For a real challenge, try eleven circles. Yes, two of those are loose. The solution is approximately but not quite 7; you can use this to check your answer.



Facts about angles

Definition. We say that the measure of an arc of the circle is the measure of the angle formed by the radii to its endpoints.



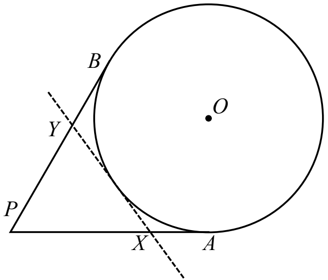
Theorem 1. If A, B, and C are points on a circle, $\angle ABC = \frac{1}{2}\widehat{AC}$.

Theorem 2. If lines from point B intersect a circle at A_1, A_2 and C_1, C_2, $\angle A_1BC_1 = \angle A_2BC_2 = \frac{1}{2}\left(\widehat{A_2C_2} - \widehat{A_1C_1}\right)$.

Angles inside circles: problems

1. A hexagon $ABCDEF$ (not necessarily regular) is inscribed in a circle. Prove that $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F$.

2. In the diagram below, PA and PB are tangent to the circle with center O. A third tangent line is then drawn, interesecting PA and PB at X and Y. Prove that the measure of $\angle XOY$ does not change if this tangent line is moved.

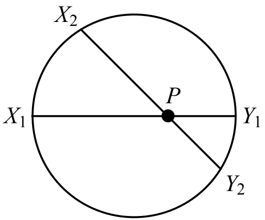


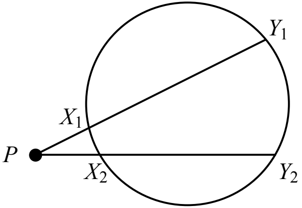
Solutions

1. By Theorem 1, $\angle A = \frac{1}{2}\left(\overset{\frown}{BC} + \overset{\frown}{CD} + \overset{\frown}{DE} + \overset{\frown}{EF}\right)$ and similarly for other angles. If we add up such equations for $\angle A + \angle C + \angle E$, we get $\overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{CD} + \overset{\frown}{DE} + \overset{\frown}{EF} + \overset{\frown}{AF} = 360^\circ$, and the same for $\angle B + \angle D + \angle F$.
2. Let Z be the point at which XY is tangent to the circle. Then $\triangle XAO$ and $\triangle XZO$ are congruent, because $AO = ZO$, $XO = XO$, and $\angle XAO = \angle XZO = 90^\circ$. $\angle ZOX = \angle XOA$, and both are equal to $\frac{1}{2}\angle ZOA$. Similarly, $\angle ZOY = \angle YOB$, and both are equal to $\frac{1}{2}\angle ZOB$. Adding these together, we get $\angle XOY = \frac{1}{2}\angle AOB$, which does not depend on the position of the tangent line.

Power of a point

Theorem 3. Two lines through a point P intersect a circle at points X_1, Y_1 and X_2, Y_2 respectively. Then $PX_1 \cdot PY_1 = PX_2 \cdot PY_2$.





One way to think about this is that the value $PX \cdot PY$ you get by choosing a line through P does not depend on the choice of line, only on P itself. This value is called the "power of P".

Power of a point: problems

1. Assume P is inside the circle for concreteness. Prove that $\triangle PX_1X_2$ and $\triangle PY_2Y_1$ are similar. Deduce Theorem 3.
2. Two circles (not necessarily of the same radius) intersect at points A and B. Prove that P is a point on AB if and only if the power of P is the same with respect to both circles.
3. Three circles (not necessarily of the same radius) intersect at a total of six points. For each pair of circles, a line is drawn through the two points where they intersect. Prove that the three lines drawn meet at a common point, or are parallel.

Power of a point: solutions

1. $\angle X_1PX_2$ and $\angle Y_1PY_2$ are vertical, and therefore equal. $\angle PX_1X2$ and $\angle PY_2Y_1$ both intercept arc $\overset{\frown}{X_2Y_1}$, so they are both equal by Theorem 1. So the triangles are similar.

   Therefore $\frac{PX_1}{PX_2} = \frac{PY_2}{PY_1}$, and we get Theorem 3 by cross-multiplying.

2. If P is on AB, the line PA intersects either circle at A and B, so the power of P is $PA \cdot PB$.

   But if P is not on AB, the line PA does not pass through B: it intersects one circle at X and another at Y, so the power of P is $PA \cdot PX$ for one circle and $PA \cdot PY$ for the other.

3. If any two lines intersect, the point of intersection will have the same power for all three circles, so it lies on all three lines by the previous problem.