Probability

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Warmup

Problem (Uncertain source)

An $n \times n \times n$ cube is painted black and then cut into $1 \times 1 \times 1$ cubes, one of which is then selected and rolled. What is the probability that the rolled cube comes up black?

(If it makes you feel better, you may assume $n = 3$.)

Problem (From personal experience)

Six fair six-sided dice are rolled. What is the probability that (at least) three of the outcomes are the same?

Solution

I see an $n \times n \times n$ cube and I want it painted black

All n^3 small cubes have a top face, but n^2 of them have the top face painted black, which is $\frac{1}{n}$ of all the cubes. The same goes for the other five directions. Therefore $\frac{1}{n}$ of all faces of all small cubes are painted black.

Choosing a random small cube and rolling it is equivalent to choosing a random face of a random small cube, and we know $\frac{1}{n}$ of these are black, so the probability is $\frac{1}{n}$.

Solution

Six fair six-sided dice are rolled...

This is really hard unless you reverse the problem: what is the probability that there are no triples?

Then there are 0, 1, 2, or 3 doubles. If there are k doubles, there are $\binom{6}{k}$ ways of choosing which numbers they are, $\binom{6-k}{k}$ ways of choosing which numbers are left out, and $\frac{6!}{2^k}$ ways of arranging these outcomes. So this probability is

$$\frac{\binom{6}{0}\binom{6}{0}6! + \binom{6}{1}\binom{5}{1}\frac{6!}{2} + \binom{6}{2}\binom{4}{2}\frac{6!}{4} + \binom{6}{3}\binom{3}{3}\frac{6!}{8}}{6^6} = \frac{6!}{6^6}\left(1 + 15 + \frac{45}{2} + \frac{5}{2}\right)$$

which simplifies to $\frac{205}{324}$, so our answer is $\frac{119}{324}$: just over $\frac{1}{3}$.

PIE[1]

Problem (1996 AIME, Problem 6.)

Five teams play each other in a round-robin tournament; each game is random, with either team having a 50% probability of winning (there are no draws). What is the probability that every team will win at least once, but no team will be undefeated?

Problem (2001 AIME II, Problem 9.)

The unit squares in a 3×3 grid are colored red and blue at random, and each color is equally likely. What is the probability that a 2×2 square will be red?

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[1] Principle of Inclusion-Exclusion. Not to be confused with π.

Solution

1996 AIME, Problem 6

The Principle of Inclusion-Exclusion in one of its simplest cases states: let L denote "some team loses all games" and W denote "some team wins all games"; then

$$\Pr[\neg L \wedge \neg W] = 1 - \Pr[L] - \Pr[W] + \Pr[L \wedge W].$$

(Ask if the notation is unclear.)

We have $\Pr[L] = \Pr[W] = 5 \cdot \frac{1}{2^4}$ and $\Pr[L \wedge W] = 5 \cdot 4 \cdot \frac{1}{2^7}$, so our answer is $1 - \frac{5}{16} - \frac{5}{16} + \frac{5}{32} = \frac{17}{32}$.

Solution

2001 AIME II, Problem 9

There are four possible 2×2 squares.

- For a single square, the probability that it's all red is $\frac{1}{2^4}$.
- For two squares, the probability both are red is $\frac{1}{2^6}$ in the four cases when they're adjacent, and $\frac{1}{2^7}$ in the two cases when they're diagonally opposite.
- For three squares, the probability all are red is $\frac{1}{2^8}$.
- The probability all four squares are red is $\frac{1}{2^9}$.

So the probability we want is

$$4 \times \frac{1}{2^4} - \left(4 \times \frac{1}{2^6} + 2 \times \frac{1}{2^7}\right) + 4 \times \frac{1}{2^8} - \frac{1}{2^9} = \frac{95}{512}.$$

Recursions

Problem (1990 AIME, Problem 9.)

A fair coin is tossed 10 *times. What is the probability that no two consecutive tosses are heads?*

Problem (1994 AIME, Problem 9.)

A deck of 12 *cards:* $A\heartsuit$, $2\heartsuit$, $\ldots$, $6\heartsuit$, *and* $A\spadesuit$, $2\spadesuit$, $\ldots$, $6\spadesuit$ *is shuffled. You play a solitaire game: drawing cards from the deck, and discarding two that match in value. However, if you ever hold three non-matching cards, you lose.*

What is the probability of winning?

Solution

1990 AIME, Problem 9

Let a_n be the *number* of sequences of n coinflips with no two consecutive heads. Either the last is a tail (and the previous $n-1$ are also such a sequence), or the last two flips are tail and head (and the previous $n-2$ are also such a sequence). So $a_n = a_{n-1} + a_{n-2}$. We identify these as Fibonacci numbers; furthermore, $a_1 = 2$ and $a_2 = 3$, so $a_{10} = 144$ and the probability we want is

$$\frac{144}{2^{10}} = \frac{9}{64}.$$

Solution

1994 AIME, Problem 9

To win, the top 3 cards must contain a match, and then the sequence of cards when this match is removed must still be winning. Let P_k be the probability of winning when the deck has k matching pairs. The probability that the top 3 cards contain a match is $\frac{3}{2k-1}$, so

$$P_k = \frac{3}{2k-1} P_{k-1} = \frac{3}{2k-1} \times \frac{3}{2k-3} \times \cdots \times \frac{3}{3},$$

and then we stop because $P_1 = 1$. In particular,

$$P_6 = \frac{3}{11} \times \frac{3}{9} \times \frac{3}{7} \times \frac{3}{5} \times \frac{3}{3} = \frac{9}{385}.$$

Weird probability spaces

Problem (1988 AIME, Problem 5.)

A positive divisor of 10^{99} *is chosen uniformly*[2] *at random. What is the probability that it is an integer multiple of* 10^{88}*?*

Problem (2004 AIME I, Problem 10.)

A circle of radius 1 *is randomly placed in the* 15×36 *rectangle* $ABCD$*, in such a way that the circle lies completely within the rectangle. What is the probability that the circle does not touch the diagonal* AC*?*

[2] probability slang for "all outcomes are equally likely".

Solution

1998 AIME, Problem 5

You should think of the random positive divisor as first choosing a power of 2 from $2^0, 2^1, \ldots, 2^{99}$, and then a power of 5 from $5^0, 5^1, \ldots, 5^{99}$, and multiplying them together.

The result is a multiple of 10^{88} if the power of 2 chosen was at least 2^{88}, and the power of 5 chosen was at least 5^{88}. The probability of each is $1 - \frac{88}{100} = \frac{3}{25}$, so the probability we want is $\frac{9}{625}$.

Solution

2004 AIME I, Problem 10.

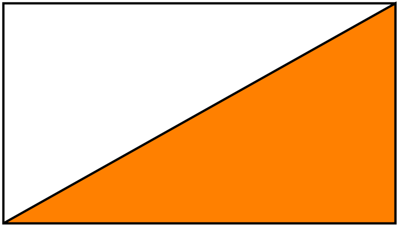
Many solutions are possible; here is the simplest I have found.

Solution

2004 AIME I, Problem 10.

Many solutions are possible; here is the simplest I have found.

1. The sides of the triangle are 36 and 15.

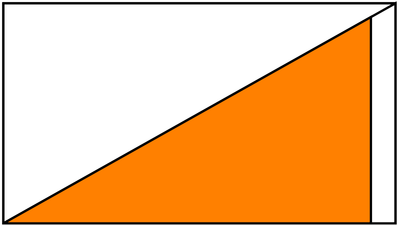


Solution

2004 AIME I, Problem 10.

Many solutions are possible; here is the simplest I have found.

1. The sides of the triangle are 36 and 15.
2. Maintaining proportion, the sides are 35 and $175/12$.

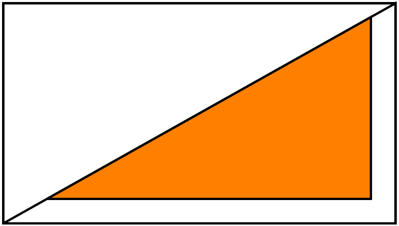


Solution

2004 AIME I, Problem 10.

Many solutions are possible; here is the simplest I have found.

1. The sides of the triangle are 36 and 15.
2. Maintaining proportion, the sides are 35 and 175/12.
3. Now the sides are 163/5 and 163/12; the diagonal height is 163/13.

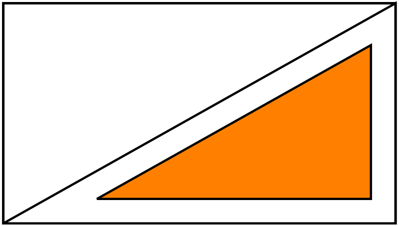


Solution

2004 AIME I, Problem 10.

Many solutions are possible; here is the simplest I have found.

1. The sides of the triangle are 36 and 15.
2. Maintaining proportion, the sides are 35 and $175/12$.
3. Now the sides are $163/5$ and $163/12$; the diagonal height is $163/13$.



The diagonal height goes down by 1, to $150/13$, so the sides scale to 30 and $25/2$. The area of the triangle and its mirror image is $30 \times 25/2 = 375$. TThe circle's center can be anywhere in a rectangle of area $34 \times 13 = 442$. So the probability is $\frac{375}{442}$.

Unrelated hard problem

Compute $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} \binom{k+10}{k}$.