Polynomials

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My favorite system of equations

Problem (Solve in your head, ideally in under 30 seconds)

The area of a rectangle is 18 *and the length of a diagonal is* 8. *Find the perimeter.*

Problem (AIME I 2003/4.)

Given that $\log_{10}\sin x + \log_{10}\cos x = -1$ *and that* $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, *find* n.

My favorite system of equations

Solutions

1. If $ab = 18$ and $a^2 + b^2 = 8^2 = 64$, then

$$(a+b)^2 = a^2 + b^2 + 2ab = 64 + 2 \cdot 18 = 100,$$

so $a + b = 10$ and the perimeter is 20.

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1. If $ab = 18$ and $a^2 + b^2 = 8^2 = 64$, then

$$(a+b)^2 = a^2 + b^2 + 2ab = 64 + 2 \cdot 18 = 100,$$

   so $a + b = 10$ and the perimeter is 20.

2. This is a similar system of equations if we let $a = \sin x$ and $b = \cos x$. Then

$$\begin{cases} a^2 + b^2 = 1 \\ ab = 10^{-1} = 0.1 \end{cases}$$

   so $a + b = \sqrt{1.2}$ and

$$\log_{10}(a+b) = \log_{10}\sqrt{1.2} = \frac{1}{2}\left(\log_{10} 12 - 1\right).$$

Sum of roots

Problem (ARML 2010/I-4.)

For real numbers α, B, *and* C, *the zeros of* $T(x) = x^3 + x^2 + Bx + C$ *are* $\sin^2\alpha$, $\cos^2\alpha$, *and* $-\csc^2\alpha$. *Compute* $T(5)$.

Problem (AIME I 2001/3.)

Find the sum of the roots, real and non-real, of the equation

$$x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0,$$

given that there are no multiple roots.

Sum of roots

Solutions

1. If $T(x) = (x - \sin^2\alpha)(x - \cos^2\alpha)(x + \csc^2\alpha)$, then the coefficient of x^2 is $-\sin^2\alpha - \cos^2\alpha + \csc^2\alpha = -1 + \csc^2\alpha$. But the coefficient of x^2 is 1. So $\csc^2\alpha = 2$, and $T(x) = (x - \frac{1}{2})^2(x + 2)$ so $T(5) = 567/4$.

Sum of roots

Solutions

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2. The equation expands as

$$\binom{2001}{1}\frac{1}{2}x^{2000} - \binom{2001}{2}\frac{1}{4}x^{1999} + \cdots = 0.$$

As we shall see on the next slide, the sum of the roots is the negative ratio of the first two coefficients here, which is 500.

Vieta's formulae

Let $x_1, \ldots, x_n$ be the roots, with multiplicity, of

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Then the following holds:

$$\begin{cases} \sum_i x_i = x_1 + x_2 + \cdots + x_n = -\frac{a_{n-1}}{a_n}. \\ \sum_{i<j} x_i x_j = x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n = \frac{a_{n-2}}{a_n}. \\ \sum_{i<j<k} x_i x_j x_k = -\frac{a_{n-3}}{a_n}. \\ \sum_{i<j<k<\ell} x_i x_j x_k x_\ell = \frac{a_{n-4}}{a_n}. \\ \ldots \\ x_1 x_2 \cdots x_n = (-1)^n \frac{a_0}{a_n}. \end{cases}$$

More practice

Problem (AIME 1996/5.)

Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ *are* a, b, *and* c, *and that the roots of* $x^3 + rx^2 + sx + t = 0$ *are* $a + b$, $b + c$, *and* $c + a$. *Find* t.

Problem (AIME I 2005/8.)

The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ *has three real roots. Find their sum.*

(Plus some nonsense about turning it into an integer 000-999 that you don't have to worry about. Aren't you glad? –Misha)

More practice

Solutions

1. Since $a + b + c = -3$, the roots of $x^3 + rx^2 + sx + t = 0$ are $-3 - a$, $-3 - b$, and $-3 - c$. The coefficient t is the negative of the product of these: $-(-3 - a)(-3 - b)(-3 - c)$ which is the negative of $x^3 + 3x^2 + 4x - 11$ evaluated at -3.

   We can compute this to be $-(-27 + 27 - 12 - 11) = 23$.

More practice

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   We can compute this to be $-(-27+27-12-11)=23$.

2. Let $y=2^{111x}$; then the equation becomes $\frac{1}{4}y^3+4y=2y^2+1$. The product of the roots of y is 4; however, this product is also $2^{111x_1}\cdot 2^{111x_2}\cdot 2^{111x_3}=2^{111(x_1+x_2+x_3)}$. Solving, we see that $x_1+x_2+x_3=\frac{2}{111}$.

Newton-Girard formulae

Problem (Degree 2 Newton-Girard formula)

Let $x_1, x_2, \ldots, x_n$ *be the roots of the polynomial*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Find $s_2 := \sum_i x_i^2 = x_1^2 + x_2^2 + \cdots x_n^2.$

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Solution

We have $(\sum_i x_i)^2 = \sum_i x_i^2 + 2\sum_{i<j} x_i x_j$. *Substituting in the things we know, we get*

$$\left(-\frac{a_{n-1}}{a_n}\right)^2 = s_2 + 2\frac{a_{n-2}}{a_n}.$$

Practice with Newton and Girard

Problem (ARML 2006/I-6.)

Determine the sum of the y-coordinates of the four points of intersection of $y = x^4 - 5x^2 - x + 4$ *and* $y = x^2 - 3x$.

Problem (AIME I 2004/7.)

Let C *be the coefficient of* x^2 *in the product*

$$(1 - x)(1 + 2x)(1 - 3x)(\cdots)(1 + 14x)(1 - 15x).$$

Find $|C|$.

(Note the alternating signs. –Misha)

Practice with Newton and Girard

Solutions

1. Let $(x_1, y_1), \ldots, (x_4, y_4)$ be the four points. Then $x_1, \ldots, x_4$ are the solutions to $x^4 - 5x^2 - x + 4 = x^2 - 3x$. So we can compute $\sum_i x_i = 0$ and $\sum_{i<j} x_i x_j = -6$.

   However, we want $\sum_i y_i = \sum_i (x_i^2 - 3x_i)$. The $\sum_i x_i$ cancels, it's 0; from the N-G formula, $\sum_i x_i^2 = 0^2 - 2(-6) = 12$.

Practice with Newton and Girard

Solutions

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   However, we want $\sum_i y_i = \sum_i (x_i^2 - 3x_i)$. The $\sum_i x_i$ cancels, it's 0; from the N-G formula, $\sum_i x_i^2 = 0^2 - 2(-6) = 12$.

2. Let $y_1, \dots, y_{15} = -1, 2, -3, \dots, -15$. Then we want to find $\sum_{i<j} y_i y_j$. We can get this from $\sum_i y_i = -8$, and $\sum_i y_i^2$ which is $\frac{1}{6}(15)(15+1)(2 \cdot 15 + 1) = 1240$ by the well-known formula. Solving $64 = 1240 - 2S$, we get $S = 588$.