This is not a Power Round

The Clubs of Oddberg

ARML Practice 5/26/2013

1 Background

The town of Oddberg (population: 1000) is known for its many clubs. The citizens of Oddberg form clubs according to the following rules:

A. Each club has an odd number of members.

B. Every pair of clubs has an even number of members in common.

The citizens of Oddberg have found that no matter how the clubs are formed, they have never in the course of the town's history had more clubs than citizens. Our goal is to prove that this is always the case.

If you have lots of experience with proofs and would like a challenge, ignore the next section and just:

0. Prove that Oddberg can have <u>at most</u> 1000 clubs.

2 Problems

2.1 Observations

1. Prove that no two clubs can have the exact same set of members.
2. Prove that it's possible for Oddberg to have <u>at least</u> 1000 clubs.
3. Prove that at least one citizen is in an odd number of clubs.[1] (Hard)

2.2 Club addition

Define the *club addition operation* as follows. If we have two clubs C_1 and C_2, we define their sum $C_1 \oplus C_2$ to be the group of citizens that are in C_1 or C_2 but not both. (Note that $C_1 \oplus C_2$ is *not* necessarily a club: it does not have to satisfy any of Oddberg's ordinances!)

Of course, this operation can be applied to any two groups of citizens, even if they are not clubs.

[1] Actually, this is not entirely true: there is exactly one configuration of clubs in which every citizen is a member of an even number of clubs. Find it and you will see why I missed it when proving this result.

4. First, show that $C_1 \oplus C_2$ contains an even number of citizens.

5. Now extend the club addition operation to more than two clubs, by taking

$$C_1 \oplus C_2 \oplus \cdots \oplus C_k := ((((C_1 \oplus C_2) \oplus C_3) \oplus \cdots) \oplus C_k).$$

   (a) Prove that this definition does not depend on the order of $C_1, C_2, \ldots, C_k$.

   (b) For the remainder of these problems, we may assume $C_1, C_2, \ldots, C_k$ are all different. To justify this, explain what happens when the same club occurs multiple times in $C_1, C_2, \ldots, C_k$.

6. It turns out that when k is odd, $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ satisfies almost all the requirements of being a club:

   (a) Prove that for any club D which is not one of $C_1, C_2, \ldots, C_k$, the sum $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ and D have an even number of citizens in common. (Hint: induct on k.)

   (b) Prove that $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ contains an even number of citizens when k is even, and an odd number of citizens when k is odd.

7. Nevertheless, $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ can never be a club in Oddberg, even when k is odd (unless k is 1). Why not?

   An equivalent statement is that $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ must always contain at least one citizen (unless k is 0). Why is this equivalent?

2.3 The proof

Define a *gathering* of citizens to be one of the sums $C_1 \oplus C_2 \oplus \cdots \oplus C_k$, when $C_1, C_2, \ldots, C_k$ are all clubs. As a special case, we say that the group of no citizens at all is also considered a gathering: this is what we get by taking $k = 0$.

8. Suppose Oddberg has n clubs. Show that it must have 2^n distinct gatherings. (Two gatherings are considered the same if they contain the same citizens as members.)

9. Conclude that Oddberg can have at most 1000 clubs.