This is not a Power Round

The Clubs of Oddberg: Solutions

ARML Practice 5/26/2013

1. Prove that no two clubs can have the exact same set of members.

Suppose that the conclusion is false; let C_1 and C_2 be two clubs with the same set of members. Then $|C_1| = |C_2| = |C_1 \cap C_2|$; either $|C_1|$ and $|C_2|$ are even, contradicting rule A, or $|C_1 \cap C_2|$ are odd, contradicting rule B.

Therefore the conclusion must be true.

2. Prove that it's possible for Oddberg to have <u>at least</u> 1000 clubs.

To do this, we must demonstrate a configuration of 1000 clubs that follows all the rules. The simplest such configuration is one in which every citizen is in a club containing nobody else. This yields 1000 clubs with 1 citizen each (1 is odd), and any two clubs have 0 citizens in common (0 is even).

A more complicated way to do this is to construct clubs $C_1, \dots, C_{1000}$ where club C_i consists of everybody <u>except</u> the i-th citizen. This yields 1000 clubs with 999 citizens each (999 is odd), and any two clubs have 998 citizens in common (998 is even).

3. Prove that at least one citizen is in an odd number of clubs.

Pick an arbitrary club C. For each member of the club C, count the total number of clubs that citizen is a member of.

This is an odd number, because we can count it in a different way: club C contributes $|C|$ to the count, and every other club C' contributes $|C \cap C'|$ to the count, so we're adding together an odd number and many even numbers.

Therefore at least one of the numbers we add is odd, which means at least one member of C is in an odd number of clubs.

(Incidentally, this proof fails if there are no clubs at all, which is the sole counterexample.)

4. Show that $C_1 \oplus C_2$ contains an even number of citizens.

We can write $|C_1 \oplus C_2|$ as $|C_1 \cup C_2| - |C_1 \cap C_2| = |C_1| + |C_2| - 2|C_1 \cap C_2|$. (The second step here uses the principle of inclusion-exclusion.) This is an odd number, plus an odd number, minus an even number, so the result is even.

We will use the formula $|C_1 \oplus C_2| = |C_1| + |C_2| - 2|C_1 \cap C_2|$ many times later on.

5. Now extend the club addition operation to more than two clubs, by taking

$$\boldsymbol{C_1 \oplus C_2 \oplus \cdots \oplus C_k := ((((C_1 \oplus C_2) \oplus C_3) \oplus \cdots) \oplus C_k).}$$

(a) **Prove that this definition does not depend on the order of $C_1, C_2, \ldots, C_k$.**

We will prove that $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ can be characterized in a different way: a citizen is a member of this sum if and only if the citizen is a member of an odd number of $C_1, \ldots, C_k$.

Let S_i be the partial sum $C_1 \oplus C_2 \oplus \cdots \oplus C_i$. If a citizen is not a member of C_{i+1}, then going from S_i to S_{i+1} does not affect that citizen: he will be in neither or both. If the citizen is a member of C_{i+1}, then going from S_i to S_{i+1} will affect that citizen: either he is in S_i but not S_{i+1}, or in S_{i+1} but not S_i.

We may stat with S_0 being the empty group, so no citizen is a member of it. If a citizen is in m of the clubs $C_1, C_2, \ldots, C_k$, then as we go from S_0 to S_k, the citizen's membership will switch m times. Therefore if m is odd, the citizen will be a member of $S_k = C_1 \oplus C_2 \oplus \cdots \oplus C_k$, and if m is even, the citizen will not be a member.

(b) **For the remainder of these problems, we may assume $C_1, C_2, \ldots, C_k$ are all different. To justify this, explain what happens when the same club occurs multiple times in $C_1, C_2, \ldots, C_k$.**

If a club C occurs multiple times, the only thing that affects the characterization in 6(a) is whether C occurs an even or an odd number of times. If C occurs an even number of times, then it may as well not be included at all, and if C occurs an odd number of times, then including it once will have the same effect.

6. **It turns out that when k is odd, $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ satisfies almost all the requirements of being a club:**

(a) **Prove that for any club D which is not one of $C_1, C_2, \ldots, C_k$, the sum $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ and D have an even number of citizens in common.**

We induct on k. For $k = 1$, we need to prove that when $D \neq C_1$, C_1 and D have an even number of citizens in common, which follows from rule B.

For the inductive step, let $S_k = C_1 \oplus C_2 \oplus \cdots \oplus C_k$. Then (by the inductive hypothesis) $|S_k \cap D|$ is even, and (by rule B) $|C_{k+1} \cap D|$ is also even. We want to show that if S_{k+1} is defined as $S_k \oplus C_{k+1}$, then $|S_{k+1} \cap D|$ is even.

We can write $|S_{k+1} \cap D|$ as $|S_k \cap D| + |C_{k+1} \cap D| - 2|S_k \cap C_{k+1} \cap D|$. This is an even number, plus an even number, minus twice some number, so it's even (compare this to the solution to problem 4).

By induction, $|S_k \cap D|$ is even for all k.

(b) **Prove that $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ contains an even number of citizens when k is even, and an odd number of citizens when k is odd.**

We need to show that $|S_k|$ and $|S_{k+1}|$ have different parity: if S_k is even, then S_{k+1} is odd, and vice versa.

By rule A, C_{k+1} is odd. So $|S_k \oplus C_{k+1}| = |S_k| + |C_{k+1}| - 2|S_k \cap C_{k+1}|$, which is $|S_k|$, plus an odd number, minus an even number. This changes the parity of $|S_k|$.

(The result we want follows by induction on k, since we have verified it already for $k = 2$.)

7. **Nevertheless, $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ can never be a club in Oddberg, even when k is odd (unless k is 1). Why not?**

   $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ always has an odd number of citizens in common with any of the clubs $C_1, C_2, \ldots, C_k$, violating rule B if it were to be made a club. Here's why.

   The order of $C_1, C_2, \ldots, C_k$ does not matter so we will just show that rule A would be violated for C_1. We induct on k: when $k = 1$, $S_1 = C_1$, so $|S_1 \cap C_1| = |C_1|$ is odd. Going from S_k to S_{k+1}, each member of $|C_1 \cap C_{k+1}|$ either changes from being included in the group to not being included, or vice versa (which contributes 1 or -1 to the difference between $|S_k \cap C_1$ and $|S_{k+1} \cap C_1|$), and there is an even number of citizens in $|C_1 \cap C_{k+1}|$, by rule B. So $|S_k \cap C_1|$ continues to be odd for all k.

   An equivalent statement is that $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ must always contain at least one citizen (unless k is 0). Why is this equivalent?

   If $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ were the empty group, then we would also have $C_1 \oplus C_2 \oplus \cdots \oplus C_{k-1} = C_k$ (check this!), which is a club in Oddberg, so it must satisfy all the conditions, contrary to the previous statement. So counterexamples to the two statements can easily be transformed into each other.

8. **Suppose Oddberg has n clubs. Show that it must have 2^n distinct gatherings.**

   The way to get 2^n gatherings is to form all possible sums $C_1 \oplus C_2 \oplus \cdots \oplus C_k$: there are 2^n of these, because for each of n clubs, we have the choice to include it in the sum, or not to include it.

   To see that these gatherings are all distinct, we use the previous problem. Suppose they were not distinct: then we would have a counterexample of the form

   $$C_1 \oplus C_2 \oplus \cdots \oplus C_k = C_{k+1} \oplus C_{k+1} \oplus \cdots \oplus C_m.$$

   But then $C_1 \oplus C_2 \oplus \cdots \oplus C_m$ would be the empty group, which is impossible by problem 7.

9. **Conclude that Oddberg can have at most 1000 clubs.**

   If Oddberg had 1001 or more clubs, then it would have 2^{1001} or more distinct gatherings. But this is impossible: there are only 2^{1000} possible groups of citizens.