Invariants

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ARML Practice 12/09/2012

The classics

Note: the answer to every yes-or-no question in every problem today is "no". I want you to explain why, so no guessing.

1. If you delete two diagonally opposite corners of an 8×8 chessboard, can you tile the remainder with 31 1×2 dominoes?
2. Prove that there is no way to choose the signs in $\pm 1 \pm 2 \pm \cdots \pm 9 \pm 10$ to make the sum equal to 0.
3. The numbers $1, \ldots, 100$ are written on a blackboard. You may choose any two numbers a and b and replace them with the single number $a + b - 1$. When only one number is left, what are all the possible values it can have?

Solutions: the classics

1. Each domino covers one black square and one white square. However, there are 30 squares of one color and 32 of the other.
2. The sum $1 + 2 + \cdots + 10 = 55$ is odd. Changing the sign of any summand leaves the sum odd, so it can never equal 0.
3. After each operation, the sum of all the numbers decreases by 1. Initially, the sum is $1 + 2 + \cdots + 100 = 5050$. After 99 steps, the remaining number must be 4951.

Even or odd?

1. A bag contains 99 red marbles and 99 blue marbles. You take two marbles out of the bag; if they are the same color, you put a red marble in the bag, otherwise you put a blue marble in the bag. If you repeat this until only one marble is left in the bag, what is the color of that marble?
2. The numbers $1, \ldots, n$ are arranged in order. At each step, you can switch two adjacent numbers. Prove that after an odd number of steps, you cannot return to the original position.
3. The numbers $a_1, a_2, \ldots, a_{2013}$ are a permutation of $1, \ldots, 2013$. Prove that the product is even:

$$(a_1 - 1)(a_2 - 2)(\cdots)(a_{2013} - 2013).$$

Solutions: Even or odd?

1. Consider the number of blue marbles mod 2.
2. Consider either:
   2.1 The number of inversions: pairs of numbers a, b with $a > b$ and a written before b, or
   2.2 The number of times two fixed numbers a and b are switched.
3. The sum $(a_1 - 1)(a_2 - 2)(\cdots)(a_{2013} - 2013)$ is 0. 2013 odd numbers cannot sum to an even number.

Numbers besides 2 also exist!

1. A room is initially empty. Every minute, either one person enters or two people leave. After exactly $3^{3^{3^3}}$ minutes, could the room contain $3^{3^3}+1$ people?
2. Starting with the number 8^{2010}, we take the sum of its digits until only one digit remains. What is that digit?
3. Seven vertices of a cube are labeled 0, and the remaining vertex labeled 1. You're allowed to change the labels by picking an edge of the cube, and adding 1 to the labels of both of its endpoints. After repeating this multiple times, can you make all labels divisible by 3?

Solutions: Numbers besides 2 also exist!

1. Consider the number of people in the room mod 3.
2. By the divisibility rule for 9, the sum of the digits of a number is the same mod 9 as the starting number, and $8^{2010} \equiv (-1)^{2010} = 1 \pmod 9$.
3. Divide the vertices into two groups of 4 non-adjacent vertices, and consider the sums of the labels in each group, mod 3.

More complicated invariants

1. You start with four congruent $(3, 4, 5)$ right triangles cut from cardboard. You repeatedly choose a triangle and cut it along the altitude. Prove that you will always have a pair of congruent triangles.
2. Let $a \odot b = \frac{a^2+b^2}{ab}$. Suppose $a, b, c, \ldots, z$ are nonzero integers. Show that if

$$a \odot b \odot c \odot \cdots \odot z$$

is an integer, then it is 2.
3. The numbers $1, \ldots, 20$ are written on a blackboard. You can erase any two numbers a and b and replace them with $ab + a + b$. Show that only one final value can be left on the board after 19 steps, and find it.

Solutions: More complicated invariants

1. If you cut a triangle in half, and then cut both halves in half, two of the four triangles you get will be congruent. Moreover, if you have four congruent triangles and cut them in half as little as possible, you will be forced to get four congruent smaller triangles.

   (If we had thirteen congruent triangles initially, we could use a simpler solution: total area is conserved, and the area of all possible triangles taken exactly once adds up to 12.5 times the area of the original triangle.)

2. If p and q are rational and nonzero, so is $p \odot q$. Moreover, $a \odot b = 1 \odot \frac{b}{a}$. Show that if $1 \odot r$ is an integer, for a rational r, then it must be 2.

3. Since a and b are replaced by $(a+1)(b+1)-1$, the product $(a_1+1)(a_2+1)(\cdots)(a_k+1)$ is unchanged.

Monovariants

1. Cards labeled $1, 2, \ldots, n$ are shuffled in a random order. We apply the following operation: if the top card is k, switch the order of the top k cards. Prove that eventually 1 will be the top card.

2. Some positive numbers are written on a blackboard. At each step, you may erase two numbers a and b such that a does not divide b, replacing them by $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$. Prove that eventually you can no longer pick such a and b, and the set of numbers you get doesn't depend on which numbers you picked at each step.

Solutions: Monovariants

1. After k appears as the top card, it can only appear again if a card bigger than k appears first. So n appears at most once; after which $n-1$ appears at most once, etc.

2. Fix a prime p, and consider for each number only the highest power of p dividing it. At each step, if $p^x | a$ and $p^y | b$, then $p^{\min(x,y)} | \gcd(a, b)$ and $p^{\max(x,y)} | \operatorname{lcm}(a, b)$.

   Eventually these powers will become sorted in the same order for all primes p, at which point you can no longer do anything.