Geometry

Tangent Circles

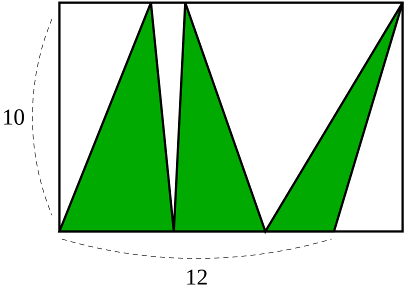
Misha Lavrov

ARML Practice 4/14/2013

Finding areas

Source: fivetriangles.blogspot.com

Find the area of the shaded region:



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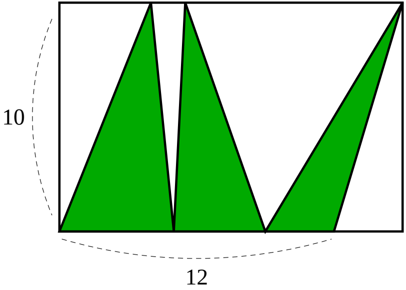

Solution: The area of each triangle is $\frac{1}{2}bh = 5b$, so the total shaded area is $5b_1 + 5b_2 + 5b_3 = 5(b_1 + b_2 + b_3) + 5 \cdot 12 = 60$.

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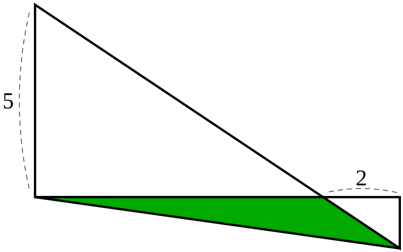

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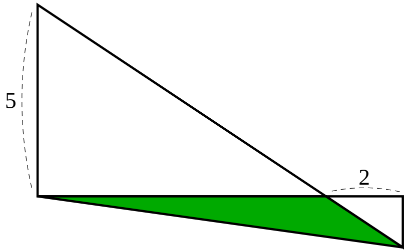

Solution: Let the base of the large right triangle be x, and the height of the small right triangle be y. By similar triangles, $5 : x = y : 2$, so $y = \frac{10}{x}$. The shaded region is a triangle with height y and base x, so its area is $\frac{1}{2}xy = 5$.

Finding areas

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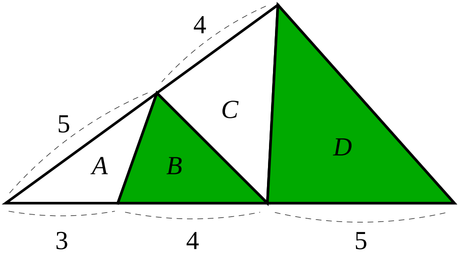
Find the ratio of the area of triangle B to the area of triangle D.



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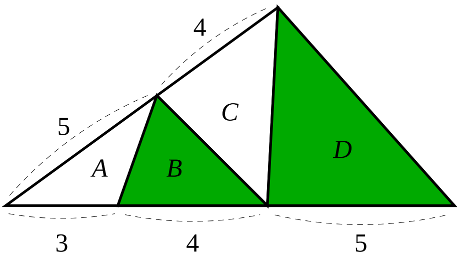
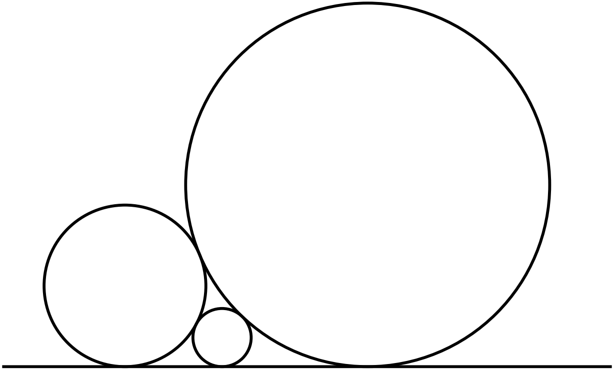


Solution: If two triangles have the same height, then their areas are in the same ratio as their bases. Therefore

$[A] : [B] = 3 : 4 \quad [A]+[B] : [C] = 5 : 4 \quad [A]+[B]+[C] : [D] = 7 : 5.$

So $[B] = \frac{4}{3}[A]$, $[C] = \frac{28}{15}[A]$, $[D] = 3[A]$, and $[B] : [D] = 4 : 9$.

Kissing Circles

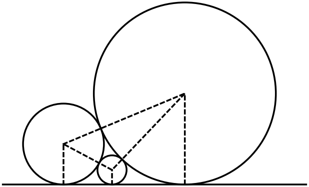


If the two larger circles have radius 4 and 9, then what is the radius of the smallest circle?

Kissing Circles

Solution I

Rule of thumb when solving problems about circles: start by drawing all the possible radii to every point of interest.



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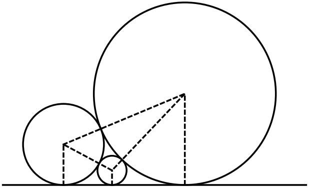


Let r be the unknown radius, x and y the distances between the tangent points on the line. Then:

$$\begin{cases} (9+4)^2 = (9-4)^2 + (x+y)^2 & \Leftrightarrow x+y=12 \\ (4+r)^2 = (4-r)^2 + x^2 & \Leftrightarrow x = 4\sqrt{r} \\ (9+r)^2 = (9-r)^2 + y^2 & \Leftrightarrow y = 6\sqrt{r} \end{cases}$$

Kissing Circles

Solution II

Definition

The **curvature** of a circle at a point of tangency is $1/R$ (where R is the radius), negated for internal tangency.

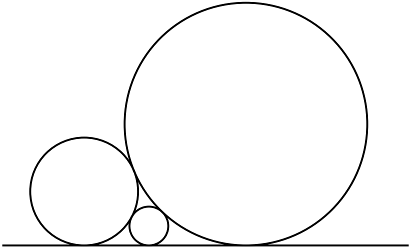
Theorem (The Descartes Circle Theorem)

If 4 *circles are pairwise tangent, with curvatures* c_1, c_2, c_3, c_4, *then*

$$c_1^2 + c_2^2 + c_3^2 + c_4^2 = \frac{1}{2}(c_1 + c_2 + c_3 + c_4)^2.$$

Kissing Circles

Solution II



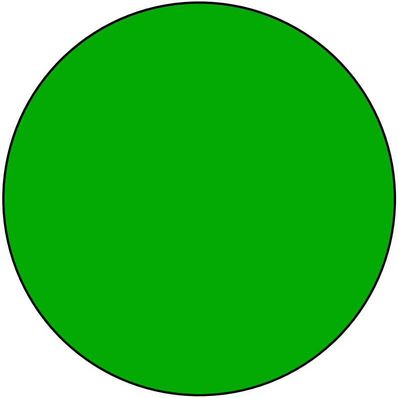
In our case, the line is a circle with curvature 0, so we have

$$c^2 + \left(\frac{1}{9}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{1}{2}\left(c + \frac{1}{9} + \frac{1}{4}\right)^2$$

where c is the curvature of the circle with unknown radius. This is a quadratic equation with solutions $c = \frac{1}{36}$ and $c = \frac{25}{36}$.

ARML 2010 Power Round

- A king rules over a (small) kingdom shaped like a circle with radius 1 mile.



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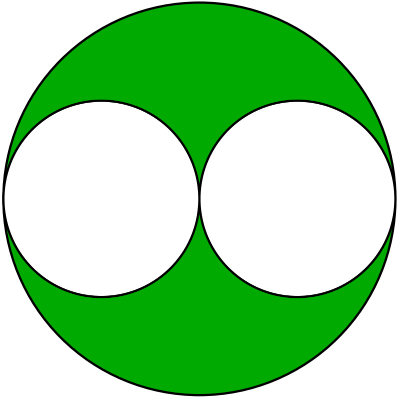

ARML 2010 Power Round

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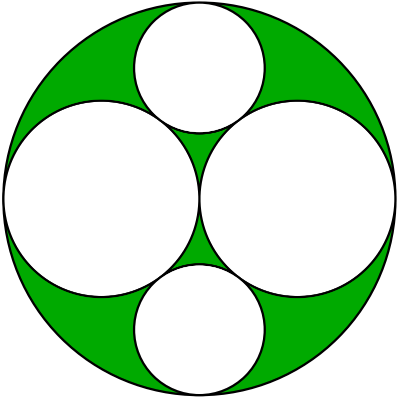
- To pay off his debts, he is forced to sell off two smaller circular regions of his kingdom.
- In future years, he continues this process, always selling off circles tangent to his kingdom's current boundaries.
- How much land does he sell each year?



A harder circles problem

Problem (USAMO 2007/2.)

The plane is covered by non-overlapping discs of various sizes, each with radius at least 5. *Prove that at least one point* (m, n) *where* m *and* n *are integers remains uncovered.*

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Hint: Show that between any 3 circles of radius ≥ 5, there is room for a fairly large circle.