Combinatorial Game Theory

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There are two kinds of games

Problem (1)

Suppose tic-tac-toe is played on a 4×4 board, but the first player to claim 4 squares on a line loses. Find a strategy that allows the second player to avoid losing.

Problem (2)

In **two-step chess**, *players take turns making two moves at a time: first White moves twice, then Black moves twice, and so on.*

Prove that if both players play optimally, White is guaranteed at least a draw: that is, Black has no foolproof winning strategy.

Misère tic-tac-toe and pairing strategies

- Match the squares of the 4×4 board in pairs:

A	B	C	D
E	F	G	H
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- Whenever the first player claims a square, the second player should claim the matching square.

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- Whenever the first player claims a square, the second player should claim the matching square.
- A line of 4 squares with only 2 different letters on it can't possibly matter in the end: neither player will claim all of it.
- If a line of 4 squares has 4 different letters, the other 4 squares with those letters also form a line. Therefore if the second player ends up claiming the first line, the first player must have already claimed the second line, and lost.

Two-step chess and strategy stealing

- Suppose Black had a winning strategy. White can begin with a “null move” (e.g. Nb1-c3-b1) that doesn’t change the position, and then follow this winning strategy with all the colors reversed. Contradiction!

Two-step chess and strategy stealing

- Suppose Black had a winning strategy. White can begin with a "null move" (e.g. Nb1-c3-b1) that doesn't change the position, and then follow this winning strategy with all the colors reversed. Contradiction!
- This is known as a "strategy stealing" argument. It applies to any game in which a move can be made that can't possibly hurt you (tic-tac-toe is a good example).
- Notably, the strategy stealing argument says nothing about what the strategy actually is.

Problem (Golomb and Hales, *Hypercube Tic-Tac-Toe*, 2002)

Find a strategy allowing the second player to force a draw in (ordinary) 5×5 *tic-tac-toe.*

Problem (USAMO 2004/4)

Alice and Bob play a game on a 6×6 *grid. They take turns writing a number in an empty square of the grid; Alice goes first. When all squares are filled, the square in each row with the largest number is colored black. Alice wins if she can then draw a straight line (possibly diagonal) connecting two opposite sides of the grid that stays entirely in black squares.*

Find, with proof, a winning strategy for one of the players.

Solution: 5×5 tic-tac-toe

- The second player can play according to the following pairing strategy:

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|	—	|	/	—
—	|		|	—
—	/	|	—	|
|	—	—	|	\

- Each row, column, and diagonal contains two paired squares; as soon as the first player claims one of them, the second player claims the other, and therefore the first player cannot claim the whole line.

Solution: USAMO 2004/4

- Bob selects 3 squares in each row as follows:

X				X	X
			X	X	X
		X	X	X	
	X	X	X		
X	X	X			
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- Bob can ensure that no marked square is colored black by following two rules:
  - When Alice writes a number on a marked square, Bob writes a higher number on an unmarked square in the same row.
  - When Alice writes a number on an unmarked square, Bob writes a lower number on a marked square in the same row.

Winning and losing positions

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- Impartial games can be studied by classifying all possible positions into winning and losing positions:
  - A *winning position* is one in which it is either possible to win in one move, or else a move exists that brings it to a losing position.
  - A *losing position* is one in which every move either loses immediately or leads to a winning position.

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- Once all positions are classified, they determine the winning player and provide a strategy.

Problems with impartial games

Problem (“Bachet’s Game”)

There are n tokens on the table. Two players take turns removing any number of tokens between 1 and k from the table. The player that takes the last token wins. Assuming optimal play, for what values of n and k does the first player win?

Problem (2009 Mathcamp Qualifying Quiz, Problem 6)

Two players play a game by starting with the integer 1000, and taking turns replacing the current integer N with either $\left\lfloor \frac{N}{2} \right\rfloor$ or $N - 1$. The player that moves to 0 wins. Assuming optimal play, which player has a winning strategy?

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- From here, we can see that the positions with a multiple of $k + 1$ tokens on the table are the only losing positions. The first player wins provided n is not divisible by $k + 1$, and the winning strategy is to always leave a multiple of $k + 1$ tokens on the table.

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  - We induct on N. When $N = 1$, a single move wins so this is a winning position.
  - For $N = 2k + 1$, we can move to k or $2k$. If k is losing then $2k + 1$ is winning.
  - If k is winning then $2k$ is losing (the only possible moves are to k and $2k - 1$, both of which are winning), so $2k + 1$ is still winning.

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- 125 and 249 are winning, so 250 is losing; therefore 500 is winning. Since 999 is also winning, 1000 is losing.
- In general, if $N = 2^\ell \cdot (2k + 1)$, then N is a winning position if and only if ℓ is even.