Complex Numbers

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ARML Practice 10/7/2012

A short theorem

Theorem (Complex numbers are weird)

$-1 = 1$.

Proof.

The obvious identity $\sqrt{-1} = \sqrt{-1}$ can be rewritten as

$$\sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}}.$$

Distributing the square root, we get

$$\frac{\sqrt{-1}}{\sqrt{1}} = \frac{\sqrt{1}}{\sqrt{-1}}.$$

Finally, we can cross-multiply to get $\sqrt{-1} \cdot \sqrt{-1} = \sqrt{1} \cdot \sqrt{1}$, or $-1 = 1$. □

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  - $|a + bi| = \sqrt{a^2 + b^2}$ (absolute value). Note: $|z| = \sqrt{z \cdot \overline{z}}$.
- We can identify a complex number $a + bi$ with the point (a, b) in the plane.

Complex number facts, continued

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- This has a geometric interpretation: rotation by θ, and scaling by $r = |z|$.

Complex number geometry

Problem (AIME 2000/9.)

A function f is defined on the complex numbers by $f(z) = (a + bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that $b^2 = m/n$, where m and n are positive integers, find m/n.

Problem (AIME 1992/10.)

Consider the region A in the complex plane that consists of all points z such that both $z/40$ and $40/\overline{z}$ have real and imaginary parts between 0 and 1, inclusive. What is the integer that is nearest the area of A?

Solution: AIME 2000/9

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This is true if and only if $a = \frac{1}{2}$. We now need to use $|a + bi| = 8$:

$$8 = |a + bi| = \sqrt{a^2 + b^2} = \sqrt{b^2 + \frac{1}{4}}.$$

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Why is $f(z)$ equidistant from 0 and z for all z, not just $z = 1$?

Write $z = x + yi$. Then $z/40$ has real part $x/40$ and imaginary part $y/40$. If these are between 0 and 1, then $0 \leq x \leq 40$ and $0 \leq y \leq 40$.

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To deal with $40/\overline{z}$, we write it as $40z/(z\overline{z}) = 40z/|z|^2$. So

$$0 \leq \frac{40x}{x^2 + y^2} \leq 1 \quad \text{and} \quad 0 \leq \frac{40y}{x^2 + y^2} \leq 1.$$

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We can rewrite the first as $x^2 + y^2 \geq 40x$, or $(x - 20)^2 + y^2 \geq 20^2$, or $|z - 20| \geq 20$. Similarly, the second becomes $|z - 20i| \geq 20$. The rest is algebra.

Applications

Problem (Basic fact)

Show that given any quadrilateral, the midpoints of its sides form a parallelogram.

Problem (Law of cosines)

Let a, b, and c be the sides of $\triangle ABC$ opposite the vertices A, B, and C respectively. Prove that

$$c^2 = a^2 + b^2 - 2ab\cos\angle C.$$

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It's easiest to show that both pairs of opposite sides are congruent. We have:

$$\left|\frac{a+b}{2} - \frac{b+c}{2}\right| = \frac{|a-c|}{2} = \left|\frac{c+d}{2} - \frac{a+d}{2}\right|.$$

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Write $z = x + yi = r(\cos\theta + i\sin\theta)$. Then $\theta = \angle C$, and $\cos\theta = x/|z|$. Then we have

$$a^2 + b^2 - 2ab\cos\theta = 1 + |z|^2 - 2|z| \cdot \frac{x}{|z|} = 1 + |z|^2 - 2x.$$

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On the other hand,

$$|z - 1|^2 = (x - 1)^2 + y^2 = x^2 - 2x + 1 + y^2 = |z|^2 - 2x + 1.$$

Roots of unity and polynomials

Fact: the equation $z^n = 1$ has n complex roots, which are evenly spaced around the circle $|z| = 1$ and start from $z = 1$. They can be written, for some angle $\theta = \frac{2\pi k}{n}$, k an integer $0 \leq k < n$, as

$$z = \cos\theta + i\sin\theta = \exp(i\theta)$$

We can also think of these as follows. Let $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then the roots of $z^n = 1$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

Problem (AMC 12A 2002/24.)

Find the number of ordered pairs of real numbers (a, b) such that $(a + bi)^{2002} = a - bi$.

Solution: AMC 12A 2002/24

Multiplying by $a + bi$ again, we get $(a + bi)^{2003} = a^2 + b^2$, or $z^{2003} = |z|^2$.

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If $|z| = 0$, then $z = 0$ is one solution. If $|z| = 1$, then $z^{2003} = 1$, which has 2003 solutions.

Roots of unity, continued

Problem (HMMT 2010 Algebra/4.)

Suppose that there exist nonzero complex numbers a, b, c, d *such that* z *satisfies* $az^3 + bz^2 + cz + d = 0$ *and* $bz^3 + cz^2 + dz + a = 0$. *Find all possible (complex) values of* z.

Problem (ARML 1995/T5.)

Determine all integer values of θ *with* $0 \leq \theta \leq 90$ *for which* $(\cos \theta^\circ + i \sin \theta^\circ)^{75}$ *is a real number.*

Solution: HMMT 2010 Algebra/4

Multiplying the first equation by z gives us $az^4 + bz^3 + cz^2 + dz = 0$. Now we subtract the second equation to get $az^4 - a = 0$. Since $a \neq 0$, we must have $z^4 = 1$.

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The fourth roots of unity are 1, i, -1, and $-i$. We can get all of these except 1 by setting $a = b = c = d = 1$, so that both equations become $z^3 + z^2 + z + 1 = 0$.

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If $z = 1$, then we must have $a + b + c + d = 1 = 0$, but fortunately it's not too hard to find examples of such a, b, c, and d. So all four of the values we found are possible values of z.

Solution: ARML 1995/T5

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So we could try to solve $(\cos\theta + i\sin\theta)^{75} = 1$ and $(\cos\theta + i\sin\theta)^{75} = -1$ separately. But we can also combine these two into $(\cos\theta + i\sin\theta)^{150} = 1$.

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There are 150 roots, but we want ones for which $0 \leq \theta \leq 90^\circ$. There are $150/4 = 38$ of these. We could write down what they are, but that's boring.

Even more roots of unity

Problem (AIME 1997/14, modified.)

Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Find the probability that $|v + w| \geq 1$.

Problem (AIME 1996/11, modified.)

Let P be the product of the roots of $z^4 + z^3 + z^2 + z + 1 = 0$ that have a positive imaginary part, and suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $r > 0$ and $0 \leq \theta < 360$. Find θ.

Solution: AIME 1997/14

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Exercise: in the original AIME problem, you were asked to find the probability that $|v + w| \geq \sqrt{2 + \sqrt{3}}$. What is the answer then?

Solution: AIME 1996/11

We recognize $z^4 + z^3 + z^2 + z + 1$ as $\frac{z^5-1}{z-1}$. So $z^4 + z^3 + z^2 + z + 1 = 0$ if $z^5 = 1$ and yet $z \neq 1$.

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There are four roots: $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, ω^2, ω^3, and ω^4.

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The first two have positive imaginary part, and their product is ω^3, which is $\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$. So $\theta = \frac{6\pi}{5} = 216^\circ$.

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Exercise: in the original problem, we instead had $z^6 + z^4 + z^3 + z^2 + 1 = 0$. How does this change the answer?

Hard problems

Problem (AIME 1994/13.)

The equation

$$z^{10} + (13z - 1)^{10} = 0$$

has ten complex real roots $r_1, \overline{r_1}, \ldots, r_5, \overline{r_5}$. *Find the value of*

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

Problem (AIME 1998/13.)

If $a_1 < a_2 < \cdots < a_n$ *is a sequence of real numbers, we define its complex power sum to be* $a_1 i + a_2 i^2 + \cdots + a_n i^n$. *Let* S_n *be the sum of all complex power sums of all nonempty subsequences of* $1, 2, \ldots, n$.*Given that* $S_8 = -176 - 64i$, *find* S_9.