Combinatorics

Misha Lavrov

ARML Practice 10/21/2012

Unrelated Review Problem

Problem (AMC 200 12B/16.)

A function f is defined by $f(z) = i \cdot \overline{z}$*, where* $i = \sqrt{-1}$ *and* $\overline{z}$ *is the complex conjugate of z. How many values of z satisfy both* $|z| = 5$ *and* $f(z) = z$*?*

Also, what are all these values?

Fibonacci Numbers

- How many ways are there to tile a 1×10 rectangle with 1×1 and 1×2 tiles?
- How many 10-letter strings can be made using the letters M and O without having two M's in a row?
- (Hard) How many of these have an even number of M's? *(Hint: find $E_n + O_n$ and $E_n - O_n$; then solve for E_n.)*
- A fair coin is flipped 10 times. What is the probability that no outcome (heads or tails) ever comes up 3 times in a row?
- Say the first and second Fibonacci numbers are both 1. What is the next year N after 2012 such that the N^{th} Fibonacci number is divisible by 3?

Stars and Bars

Theorem (Stars and Bars)

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There are $7 - 1$ places to insert $3 - 1$ separators:

$$\star \vdots \star \vdots \star \vdots \star \vdots \star \vdots \star \vdots \star$$

Stars and Bars – Exercises

Problem (Easy corollary)

What is the number of ways to write n as a sum of k **non-negative** *integers?*

Problem (Application)

How many ways are there to choose 7 letters from the alphabet, with repetition? (a.k.a. 7-letter words distinct up to anagrams).

Problem (From my research)

What is the number of different ways to write n as a sum of k positive integers, if each integer q can be colored one of q different colors?

Stars and Bars – Solutions

- If n is written as a sum of k non-negative integers, just add 1 to each integer in the sum. We get $n+k$ written as a sum of k positive integers, so there are $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ such sums.

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- We can think of this as writing 7 as a sum of 26 nonnegative integers, counting the number of times each letter is used (e.g. $7 = 2 + 3 + 0 + 2 + 0 + \cdots + 0$ corresponds to AABBBDD).

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$$16 = (1+3) + (4+1) + (1+2) + (2+2).$$
- This is a partition of $n+k$ into a sum of $2k$ positive integers, so there are $\binom{n+k-1}{2k-1}$ ways to do so.

Solving Linear Recurrences

Theorem

To solve the recurrence $f(n) = af(n-1) + bf(n-2)$*, solve the quadratic equation* $x^2 = ax + b$*. If there are two roots* r_1 *and* r_2*, then* $f(n) = C_1 r_1^n + C_2 r_2^n$*, for some* C_1, C_2*. If there is one root* r*, then* $f(n) = (C_1 + C_2 n) r^n$*.*

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Example: suppose $f(1) = 1$, $f(2) = 2$, and

$$f(n) = 4f(n-1) - 4f(n-2).$$

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Since $f(1) = 2(C_1 + C_2) = 1$ and $f(2) = 4(C_1 + 2C_2) = 2$, we get $C_1 = \frac{1}{2}$ and $C_2 = 0$, so $f(n) = 2^{n-1}$.

Linear Recurrences – Exercises

Problem (Classic result)

Find a formula for the n^{th} *Fibonacci number* F_n*, if* $F_1 = F_2 = 1$*.*

Problem (Corollary – Nonhomogeneous linear recurrences)

How can we solve a recurrence such as

$$f(n) = af(n-1) + bf(n-2) + c?$$

Example: $f(n) = f(n-1) + f(n-2) + 1$.

Linear Recurrences – Solutions

- The quadratic equation is $x^2 - x - 1 = 0$, which has roots $\phi_1, \phi_2 = \frac{1 \pm \sqrt{5}}{2}$. Solving for the constants, we get

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- We can eliminate the constant by adding or subtracting the right thing from both sides. For example:

$$\begin{aligned} f(n) &= f(n-1) + f(n-2) + 1 \\ f(n) + 1 &= (f(n-1) + 1) + (f(n+2) + 1) \\ f(n) + 1 &= F_n. \end{aligned}$$