Preface

In recent years there has been renewed interest in the calculus of variations, motivated in part by ongoing research in materials science and other disciplines. Often, the study of certain material instabilities such as phase transitions, formation of defects, the onset of microstructures in ordered materials, fracture and damage, leads to the search for equilibria through a minimization problem of the type

$$\min \{ I(v) : v \in \mathcal{V} \},$$

where the class $\mathcal{V}$ of admissible functions v is a subset of some Banach space V.

This is the essence of the calculus of variations: the identification of necessary and sufficient conditions on the functional I that guarantee the existence of minimizers. These rest on certain growth, coercivity, and convexity conditions, which often fail to be satisfied in the context of interesting applications, thus requiring the relaxation of the energy. New ideas were needed, and the introduction of innovative techniques has resulted in remarkable developments in the subject over the past twenty years, somewhat scattered in articles, preprints, books, or available only through oral communication, thus making it difficult to educate young researchers in this area.

This is the first of two books in the calculus of variations and measure theory in which many results, some now classical and others at the forefront of research in the subject, are gathered in a unified, consistent way. A main concern has been to use contemporary arguments throughout the text to revisit and streamline well-known aspects of the theory, while providing novel contributions.

The core of this book is the analysis of necessary and sufficient conditions for sequential lower semicontinuity of functionals on L^p spaces, followed by relaxation techniques. What sets this book apart from existing introductory texts in the calculus of variations is twofold: Instead of laying down the theory in the one-dimensional setting for integrands $f = f(x, u, u')$, we work in N dimensions and no derivatives are present. In addition, it is self-contained in the sense that, with the exception of fundamentally basic results in measure theory that may be found in any textbook on the subject (e.g., Lebesgue dominated convergence theorem), all the statements are fully justified and proved. This renders it accessible to beginning graduate students with basic knowledge of measure theory and functional analysis. Moreover, we believe that this text is unique as a reference book for researchers, since it treats both necessary and sufficient conditions for well-posedness and lower semicontinuity of functionals, while usually only sufficient conditions are addressed.

The central part of this book is Part III, although Parts I and II contain original contributions. Part I covers background material on measure theory, integration, and L^p spaces, and it combines basic results with new approaches to the subject. In particular, in contrast to most texts in the subject, we do not restrict the context to σ-finite measures, therefore laying the basis for the treatment of Hausdorff measures, which will be ubiquitous in the setting of the second volume, in which gradients will be present. Moreover, we call attention to Section 1.1.4, on "comparison between measures", which is completely novel: The Radon–Nikodym theorem and the Lebesgue decomposition theorem are proved for positive measures without our having first to introduce signed measures, as is usual in the literature. The new arguments are based on an unpublished theorem due to De Giorgi treating the case in which the two measures in play are not σ-finite. Here, as De Giorgi's theorem states, a diffuse measure must be added to the absolutely continuous and singular parts of the decomposition. Also, we give a detailed proof of the Morse covering theorem, which does not seem to be available in other books on the subject, and we derive as a corollary the Besicovitch covering theorem instead of proving it directly.

Part II streamlines the study of convex functions, and the treatment of the direct method of the calculus of variations introduces the reader to the close connection between sequential lower semicontinuity properties and existence of minimizers. Again here we present an unpublished theorem of De Giorgi, the approximation theorem for real-valued convex functions, which provides an explicit formula for the affine functions approximating a given convex function f. A major advantage of this characterization is that additional regularity hypotheses on f are reflected immediately on the approximating affine functions.

In Part III we treat sequential lower semicontinuity of functionals defined on L^p, and we separate the cases of inhomogeneous and homogeneous functionals. The latter are studied in Chapter 5, where

$$I(u) := \int_E f(v(x))\,dx$$

with E a Lebesgue measurable subset of the Euclidean space $\mathbb{R}^N$, $f : \mathbb{R}^m \to (-\infty, \infty]$ and $v \in L^p(E; \mathbb{R}^m)$ for $1 \le p \le \infty$. This material is intended for an introductory graduate course in the calculus of variations, since it requires only basic knowledge of measure theory and functional analysis. We treat both bounded and unbounded domains E, and we address most types of strong and weak convergence. In particular, the setting in which the underlying convergence is that of $(C_b(E))'$ is new.

Chapter 6 and Chapter 7 are devoted to integrands $f = f(x, v)$ and $f = f(x, u, v)$, respectively, and are significantly more advanced, since the proofs of the necessity parts are heavily hinged on the concept of multifunctions. An important tool here is selection criteria, and the reader will benefit from a comprehensive and detailed study of this subject.

Finally, Chapter 8 describes basic properties of Young measures and how they may be used in relaxation theory.

The bibliography aims at giving the main references relevant to the contents of the book. It is by no means exhaustive, and many important contributions to the subject may have failed to be listed here.

To conclude, this text is intended as a graduate textbook as well as a reference for more-experienced researchers working in the calculus of variations, and is written with the intention that readers with varied backgrounds may access different parts of the text.

This book prepares the ground for a second volume, since it introduces and develops the basic tools in the calculus of variations and in measure theory needed to address fundamental questions in the treatment of functionals involving derivatives.

Finally, in a book of this length, typos and errors are almost inevitable. The authors will be very grateful to those readers who will write to either fonseca@andrew.cmu.edu or giovanni@andrew.cmu.edu indicating those that they have found. A list of errors and misprints will be maintained and updated at the web page http://www.math.cmu.edu/~leoni/book1.

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