Modern Methods in the Calculus of Variations: L^p Spaces, First edition
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For the original text we use the color Red, for corrections the color Green, and for improvements and additions the color Blue. Names in brackets refer to the persons who called the error to our attention (to the best of our recollection) or suggested improvements and additions.[1]

CHAPTER 1:

p. 8 In Exercise 1.10 the sentence "satisfying property (ii) of Proposition 1.7" should be removed. [Pietro Siorpaes]

Actually we have the following proposition. [Michael Klipper, G.L. see also the reference [Coh93]]

Proposition 1 *Let X be a nonempty set, let $\mathfrak{M} \subset \mathcal{P}(X)$ be an algebra, and let $\mu : \mathfrak{M} \to [0, \infty]$ be a finitely additive measure such that*

$$\lim_{n\to\infty} \mu(E_n) = 0 \tag{1}$$

for every decreasing sequence $\{E_n\} \subset \mathfrak{M}$ such that $\bigcap_{n=1}^{\infty} E_n = \emptyset$. Then μ is countably additive.

Proof. Let $\{F_n\} \subset \mathfrak{M}$ be a sequence of mutually disjoint sets such that $\bigcup_{n=1}^{\infty} F_n \in \mathfrak{M}$, and define

$$E_n := \bigcup_{k=n+1}^{\infty} F_k \in \mathfrak{M}.$$

Then by finite additivity we have

$$\mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \mu\left(\bigcup_{k=1}^{n} F_k\right) + \mu\left(\bigcup_{k=n+1}^{\infty} F_k\right) = \sum_{k=1}^{n} \mu(F_k) + \mu(E_n).$$

Since $\{E_n\} \subset \mathfrak{M}$ is a decreasing sequence and $\bigcap_{n=1}^{\infty} E_n = \emptyset$, by (1) we have that $\mu(E_n) \to 0$ as $n \to \infty$, and so letting $n \to \infty$ in the previous identity yields

$$\mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mu(F_k).$$

■

[1] G.L. would like to thank all his students in the class Math 21-720, Measure theory and integration, Fall 2007, for their useful comments. The style of this file is inspired by http://www.hss.caltech.edu/ kcb/IDA-Errata.pdf

p. 13 and 18 In the decomposition of Propositions 1.22 and 1.26 there's no uniqueness in general. Consider the Lebesgue measure $\mathcal{L}^1 : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ and for every Borel set $E \subset \mathbb{R}$ define

$$\mu(E) := \begin{cases} 0 & \text{if } E \text{ is countable,} \\ \infty & \text{otherwise.} \end{cases}$$

Then $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ is a measure, and

$$\mathcal{L}^1 + \mu = 0 + \mu.$$

Since $\mathcal{L}^1$ and the measure identically equal to zero are both nonatomic and σ-finite (and in particular semifinite), we have no uniqueness. [Pietro Siorpaes, Daniel Spector]

p. 58 After formula (1.42), a cleaner argument would be: Define

$$\overline{u}(x) := \begin{cases} v(x) & \text{for } x \in E, \\ u(x) & \text{elsewhere.} \end{cases}$$

Then $\overline{u}$ is admissible for $\nu_{ac}(X)$, and so

$$\nu_{ac}(X) \geq \int_X \overline{u}\, d\mu = \int_X u\, d\mu + \int_E (v - u)\, d\mu > \int_X u\, d\mu,$$

where we have used (1.42). [Hang Yu]

p. 62 To prove uniqueness at the end of Step 1, a simpler argument is the following. Since

$$\infty > \nu(E) = \int_E u\, d\mu = \int_E v\, d\mu$$

for every $E \in \mathfrak{M}$, we have that

$$\int_E (v - u)\, d\mu = 0$$

for every $E \in \mathfrak{M}$, which implies that $u(x) = v(x)$ for μ a.e. $x \in X$ (see, e.g., Remark 1.88).

p. 63 After formula (1.48), "and since $E_\sigma \cup F$ is admissible in the definition of T we arrive at a contradiction." should be replaced by "and since $E_n \cup F$ is admissible in the definition of T, we have that

$$T = \mu(E_\sigma) \geq \mu(E_n \cup F) = \mu(E_n) + \mu(F) \to \mu(E_\sigma) + \mu(F),$$

which yields a contradiction." Similarly, three lines after formula (1.48), "$E_\sigma \cup F$ is admissible" should be replaced by "$E_n \cup F$ is admissible". [Michael Klipper]

p. 109 In the proof of Theorem 1.158 (Lebesgue differentiation theorem) the sentence "by the Besicovitch differentiation theorem" should be replaced by "by the Besicovitch differentiation theorem and Remark 1.154(ii)". [Paolo Piovano]

p. 127 In Exercise 1.199 "$v = u\chi_X + c\chi_{\{\infty\}}$" and "$\|v\|_{C(X^\infty)} = \max\left\{\|u\|_{C_0(X)}, |c|\right\}$" should be replaced by "$(v-c)|_X = u$" and "$\|v-c\|_{C(X^\infty)} = \|u\|_{C_0(X)}$", respectively.

CHAPTER 2:

p. 141 Theorem 2.5 continues to hold for $0 < p < q < \infty$. Note that in (i) by sets of arbitrarily small positive measure we mean that for every $\varepsilon > 0$ there exists $E \in \mathfrak{M}$ such that $0 < \mu(E) \le \varepsilon$, while in (ii) by sets of arbitrarily large finite we mean that for every $M > 0$ there exists $E \in \mathfrak{M}$ such that $M \le \mu(E) < \infty$.

p. 141 Theorem 2.5 can be extended to $q = \infty$. Indeed, we have

Theorem 2 *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $0 < p < \infty$. Then*

(i) $L^p(X)$ is not contained in $L^\infty(X)$ if and only if X contains measurable sets of arbitrarily small positive measure;

(ii) $L^\infty(X)$ is not contained in $L^p(X)$ if and only if $\mu(X) = \infty$.

Proof. (i) Assume that $L^p(X)$ is not contained in $L^\infty(X)$. Then there exists $u \in L^p(X)$ such that

$$\|u\|_{L^\infty} = \infty. \tag{2}$$

For each $n \in \mathbb{N}$ let

$$E_n := \{x \in X : |u(x)| > n\}.$$

Then

$$\mu(E_n) \le \frac{1}{n^p} \int_X |u|^p \, d\mu \to 0$$

as $n \to \infty$. Thus, it suffices to show that $\mu(E_n) > 0$ for all n sufficiently large. If to the contrary, $\mu(E_n) = 0$ for some n, then we would have that $|u(x)| \le n$ for μ a.e. $x \in X$, which would contradict (2). Hence, X contains measurable sets of arbitrarily small positive measure.

Conversely, assume that X contains measurable sets of arbitrarily small positive measure. Then it is possible to construct a sequence of pairwise disjoint sets $\{E_n\} \subset \mathfrak{M}$ such that $\mu(E_n) > 0$ for all $n \in \mathbb{N}$ and

$$\mu(E_n) \searrow 0.$$

Let

$$u := \sum_{n=1}^{\infty} c_n \chi_{E_n},$$

where $c_n \nearrow \infty$ are chosen such that

$$\sum_{n=1}^{\infty} c_n^p \mu(E_n) < \infty. \tag{3}$$

Then $u \in L^p(X) \setminus L^\infty(X)$.

(ii) If $\mu(X) < \infty$, then for any $u \in L^\infty(X)$, we have

$$\int_X |u|^p \, d\mu \leq (\|u\|_{L^\infty})^p \mu(X) < \infty, \tag{4}$$

and so $L^\infty(X) \subset L^p(X)$. Conversely, assume that $\mu(X) = \infty$. Then the function $u :\equiv 1$ belongs to $L^\infty(X)$ but not to $L^p(X)$. ■

[Daniel Spector]

p. 150 In the statement of Vitali's convergence theorem "measurable" should be replaced by "integrable". [Hang Yu]

p. 168 One line above Exercise 2.43, the "no" should be removed. [Daniel Spector]

p. 171 In Exercise 2.45(ii) "$-p(u) \leq L(u) \leq p(u)$" should be replaced by "$L(u) \leq p(u)$". [Pietro Siorpaes]

p. 191 In formula (2.57) "$\frac{\rho}{1+\|\varphi\|_\infty}$" should be replaced by "$\frac{\rho}{\alpha_N(1+\|\varphi\|_\infty)}$". [Rita Gonçalves Ferreira]

p. 196 Line -6 "Theorem 2.99" should be replaced by "Theorem 2.11". [Rita Gonçalves Ferreira]

p. 223 In the definition of the spaces $L^p((X, \mathfrak{M}, \mu); Y)$ and $L^\infty((X, \mathfrak{M}, \mu); Y)$ in Definition 2.109, one should consider, as usual, equivalence classes of functions.

p. 223 Before Definition 2.111, one should add the following definition. "Let $(X, \mathfrak{M}, \mu)$ be a measure space and let Y be a Banach space. Two weakly star measurable functions $v, w : X \to Y'$ are equivalent if for every $y \in Y$, $v(x)(y) = w(x)(y)$ for μ a.e. $x \in X$. Note that the set where the equality fails may depend on y."

p. 226 We would like to thank Nguyen Huy Chieu for pointing out that the proof of the identity

$$\|v(x)\|_{Y'} = \sup_n \frac{|v(x)(y_n)|}{\|y_n\|_Y} = \sup_n \frac{|v_{y_n}(x)|}{\|y_n\|_Y} \tag{5}$$

on Line -7 is far from trivial (see also the file Additional Material). The missing details are as follows:

We begin by proving that if Y is separable, then a weakly measurable function $v : X \to Y'$ belongs to $L_w^\infty(X; Y')$ if and only if there exists a constant $C \geq 0$ such that for every $y \in Y$,

$$|v(x)(y)| \leq C \|y\|_Y \tag{6}$$

for μ a.e. $x \in X$. Note that the set where the equality fails may depend on y. To prove the claim, assume that (6) holds and let $\{y_n\} \subset Y$ be dense in Y. For every $x \in X$, $v(x) \in Y'$, and so by the density of $\{y_n\}$ and the fact that $v(x) : Y \to \mathbb{R}$ is bounded,

$$\|v(x)\|_{Y'} = \sup_{y \in Y \setminus \{0\}} \frac{|v(x)(y)|}{\|y\|_Y} = \sup_n \frac{|v(x)(y_n)|}{\|y_n\|_Y}. \tag{7}$$

Since the functions $x \in X \mapsto v(x)(y_n)$ are measurable, it follows that the function $x \in X \mapsto \|v(x)\|_{Y'}$ is measurable. Moreover, by (6), for every $n \in \mathbb{N}$ there exist $E_n \in \mathfrak{M}$ with $\mu(E_n) = 0$ such that

$$|v(x)(y_n)| \leq C \|y_n\|_Y \quad \text{for all } x \in X \setminus E_n. \tag{8}$$

Let $E_\infty := \bigcup_{n=1}^\infty E_n$. Then $\mu(E_\infty) = 0$ and by (7), for all $x \in X \setminus E_\infty$,

$$\|v(x)\|_{Y'} \leq C.$$

This shows that $v \in L_w^\infty(X; Y')$ with

$$\|v\|_{L_w^\infty(X;Y')} \leq \inf\{C \geq 0 : \text{ property (6) holds}\} =: M_\infty(v).$$

Since $\|v\|_{L_w^\infty(X;Y')}$ is an admissible constant in (6), it follows that $\|v\|_{L_w^\infty(X;Y')} = M_\infty(v)$.

To prove (5), define

$$g(x) := \sup_n \frac{|v_{y_n}(x)|}{\|y_n\|_Y}, \quad x \in X.$$

Fix $y \in Y \setminus \{0\}$. Since $\{y_n\}$ is dense in Y, we may find a subsequence $\{y_{n_j}\}$ converging to y. Using the fact that $v_{y_{n_j}} \to v_y$ in $L^p(X)$, there exist a subsequence, not relabeled, and a set $F_y \in \mathfrak{M}$ with $\mu(F_y) = 0$, such that $v_{y_{n_j}}(x) \to v_y(x)$ for all $x \in X \setminus F_y$ as $j \to \infty$. Hence,

$$\frac{|v(x)(y)|}{\|y\|_Y} = \lim_{j \to \infty} \frac{\left|v_{y_{n_j}}(x)\right|}{\left\|y_{n_j}\right\|_Y} \leq g(x),$$

or, equivalently,

$$|v(x)(y)| \leq g(x) \|y\|_Y \tag{9}$$

for all $x \in X \setminus F_y$.

Let $E := \{x \in X : g(x) = 0\}$. Define the function

$$f(x) := \begin{cases} 0 & \text{if } x \in E, \\ \frac{v(x)}{g(x)} & \text{if } x \in X \setminus E. \end{cases}$$

It follows from (9) and the previous remark that $f \in L_w^\infty(X; Y')$ with $\|f\|_{L_w^\infty(X;Y')} \leq 1$. Since

$$\|f\|_{L_w^\infty(X;Y')} = \operatorname*{esssup}_{x \in X} \|f(x)\|_{Y'},$$

there exists $E_0 \in \mathfrak{M}$ with $\mu(E_0) = 0$ such that $\|f(x)\|_{Y'} \leq 1$ for all $x \in X \setminus E_0$. Define

$$v_0(x) := g(x) f(x), \quad x \in X.$$

Let $y \in Y$. If $x \in E \setminus F_y$, then $v_{y_n}(x) = 0$ for all $n \in \mathbb{N}$, and so $v(x)(y) = 0$, while $v_0(x)(y) = g(x) f(x)(y) = 0$. On the other hand, if $x \in X \setminus E$, then $v_0(x)(y) = g(x) f(x) = v(x)(y)$. Hence, v_0 is equivalent to v because $v_0(x) = v(x)$ for every $x \in X \setminus F_y$. Since $\|f(x)\|_{Y'} \leq 1$ for all $x \in X \setminus E_0$, we have that

$$\begin{aligned} \sup_{y \in Y \setminus \{0\}} \frac{|v_0(x)(y)|}{\|y\|_Y} &= \sup_{y \in Y \setminus \{0\}} \frac{|g(x) f(x)|}{\|y\|_Y} = g(x) \sup_{y \in Y \setminus \{0\}} \frac{|f(x)|}{\|y\|_Y} \\ &= g(x) \|f(x)\|_{Y'} \leq g(x) \end{aligned}$$

for every $X \setminus E_0$. This shows that $\|v_0(x)\|_{Y'} \leq g(x)$ for all $X \setminus E_0$. To prove the converse inequality, note that if $x \in E$, then there is nothing to prove since $g(x) = 0$ and $v_0(x) = 0$, while if $x \in X \setminus E$, then $v_0(x) = v(x)$, and so

$$\|v_0(x)\|_{Y'} = \|v(x)\|_{Y'} = \sup_{y \in Y \setminus \{0\}} \frac{|v(x)(y)|}{\|y\|_Y} \geq \sup_n \frac{|v(x)(y_n)|}{\|y_n\|_Y} = g(x).$$

Finally, we note that (2.103) continues to hold with $v_0(x)(y)$ in place of $v_y(x)$, precisely

$$L(uy) = \int_X u(x) v_0(x)(y)\, d\mu(x),$$

because $v_0(x)(y) = v(x)(y)$ for all $x \in X \setminus F_y$ and $\mu(F_y) = 0$.

p. 226 Line -3 "$L_w^p(X; Y')$" should be replaced by "$L_w^q(X; Y')$".

p. 227 Lines 10 and 12 "$L_w^p(X; Y')$" should be replaced by "$L_w^q(X; Y')$".

CHAPTER 4:

p. 256 Line 20 "$z_1, \ldots, z_k \in \operatorname{ri}_{\text{aff}}(C)$" should be replaced by "$z_1, \ldots, z_k \in C$".

p. 259 Line 4 "$t \leq f(z)$" should be replaced by "$t \leq f(v_2)$". [Daniel Spector]

p. 259 Line -1 "b^q" should be replaced by "b^p". [Daniel Spector]

p. 279 In Exercise 4.55, "$|z_1|$ should be replaced by "$|z_2|$".

p. 294 The proof of Proposition 4.75 only shows that $f = g$ in $\operatorname{dom}_e f$. To prove that $f = g$ outside $\operatorname{dom}_e f$, fix $v_0 \in V \setminus \operatorname{dom}_e f$ and $t_0 \in \mathbb{R}$ and find L as in (i). Then

$$\langle v', v\rangle_{V',V} + \alpha_0 t \geq \alpha + \varepsilon \quad \text{for all } (v,t) \in \operatorname{epi} f \text{ and } \langle v', v_0\rangle_{V',V} + \alpha_0 t_0 \leq \alpha - \varepsilon,$$

and so letting $t \to \infty$, we obtain $\alpha_0 \geq 0$. If $\alpha_0 > 0$, then we proceed as before to conclude that $g(v_0) \geq t_0$. If $\alpha_0 = 0$, we have

$$\langle v', v\rangle_{V',V} \geq \alpha + \varepsilon \quad \text{for all } v \in \operatorname{dom}_e f \text{ and } \langle v', v_0\rangle_{V',V} \leq \alpha - \varepsilon,$$

so that

$$0 \geq 2\varepsilon + \langle v', v_0\rangle_{V',V} - \langle v', v\rangle_{V',V} \quad \text{for all } v \in \operatorname{dom}_e f.$$

By part (i) there exist $w' \in V'$ and $c \in \mathbb{R}$ such that

$$f(v) \geq \langle w', v\rangle_{V',V} + c \quad \text{for all } v \in V.$$

Using the last two inequalities we obtain that for all $t > 0$ and for all $v \in \operatorname{dom}_e f$,

$$\begin{aligned} f(v) &\geq \langle w', v\rangle_{V',V} + c \\ &\geq \langle w', v\rangle_{V',V} + c + t\left(2\varepsilon + \langle v', v_0\rangle_{V',V} - \langle v', v\rangle_{V',V}\right) =: g_t(v). \end{aligned}$$

Since $f = \infty$ outside $\operatorname{dom}_e f$, we have that $f(v) \geq g_t(v)$ for all $t > 0$ and for all $v \in V$. Thus g_t is an admissible affine function. Hence

$$\begin{aligned} \infty = f(v_0) &\geq \sup\{g(v_0) : g \text{ affine continuous, } g \leq f\} \\ &\geq g_t(v_0) = \langle w', v_0\rangle_{V',V} + c + t2\varepsilon \to \infty \end{aligned}$$

as $t \to \infty$.

CHAPTER 5:

p. 327 A simpler proof of Lemma 5.2. Without loss of generality we may assume that $L = 1$. Since $a_n \to \infty$, we may construct an increasing sequence $n_k \nearrow \infty$ such that $a_{n_k} \geq 4^k$ for all $k \in \mathbb{N}$. Define

$$b_n := \begin{cases} \frac{1}{2^k} & \text{if } n = n_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=1}^{\infty} b_n = \sum_{k=1}^{\infty} \frac{1}{2^k} \leq 1,$$

while

$$\sum_{n=1}^{\infty} a_n b_n \geq \sum_{k=1}^{\infty} a_{n_k} b_{n_k} \geq \sum_{k=1}^{\infty} \frac{4^k}{2^k} = \infty.$$

[Pietro Siorpaes]

CHAPTER 6:

p. 396 In Step 3 of the proof of Theorem 6.18 the space X is taken to be $\mathbb{R}^N$, which is the case used in the remainder of the book. For a general metric space, one should use the fact that metric spaces are paracompact (see, e.g., Rudin, Mary Ellen, A new proof that metric spaces are paracompact. Proc. Amer. Math. Soc. 20 (1969), 603) to find a locally finite refinement and then construct a partition of unity subordinated to the refinement (see Michael, Ernest, A note on paracompact spaces. Proc. Amer. Math. Soc. 4 (1953), 831–838). [Giovanni Leoni]

p. 429 The function g_v introduced in (6.50) belongs to $L^1(E)$ and so in Step 3 on page 431, in place of points of approximate continuity of g_{v_n} we can take Lebesgue points. Hence, (6.59) is not needed and (6.61) should be replaced by

$$\frac{1}{|B(x,r) \cap E|} \int_{B(x,r)\cap E} \left| g_{v_{n_x}}(y) - g_{v_{n_x}}(x) \right| \, dy \leq \varepsilon,$$

and (6.53) now reads

$$\int_{B(x,r)\cap E} g_{v_{n_x}}(y) \, dy \leq |B(x,r) \cap E| \left(g_{v_{n_x}}(x) + \varepsilon \right).$$

In turn the first six lines on page 432 are not needed and in lines -1 up to -5 the factor $(1-\varepsilon)$ is not needed. [Irene Fonseca]

p. 433 The definition of the function ψ_k in (6.65) should be changed. For $x \in E$ we set

$$\psi_k(x) := \begin{cases} \sup\{n \in \mathbb{N} : \, h(x,n) \leq -kn^p\} & \text{if there is } n \in \mathbb{N} \text{ such that } h(x,n) \leq -kn^p, \\ 1 & \text{otherwise,} \end{cases} \tag{10}$$

Then after (6.66), we claim that without loss of generality, we may assume that for all $x \in B_k$ there is $n \in \mathbb{N}$ such that $h(x,n) \leq -kn^p$. To see this, we introduce the set

$$B_k' := \{x \in B_k : \text{there is } n \in \mathbb{N} \text{ such that } h(x,n) \leq -kn^p\}.$$

Since

$$\infty = \int_{B_k} (\psi_k)^p \, dx = \int_{B'_k} (\psi_k)^p \, dx + 1 \left| B_k \setminus B'_k \right|,$$

we have that $\int_{B'_k} (\psi_k)^p \, dx = \infty$, and so, by replacing B_k with B'_k, the claim is proved. [Irene Fonseca]

p. 434 Line 5 (and identifying B_k with B'_k as in the previous item), to find the sets E_k we proceed as follows. Write

$$B_{k,m} := \{x \in B_k : \psi_k(x) = m\}.$$

Then

$$\sum_{m=1}^{\infty} m^p \left| B_{k,m} \right| = \sum_{m=1}^{\infty} \int_{B_{k,m}} (\psi_k)^p \, dx = \int_{B_k} (\psi_k)^p \, dx = \infty,$$

and so there exists $m_k \in \mathbb{N}$ such that

$$\infty > \int_{\bigcup_{i=1}^{m_k} B_{k,i}} (\psi_k)^p \, dx > \frac{1}{k^2}.$$

Hence, there exists a measurable set $E_k \subset \bigcup_{i=1}^{m_k} B_{k,i}$ such that

$$\int_{E_k} (\psi_k)^p \, dx = \frac{1}{k^2}.$$

[Irene Fonseca]

p. 438 Lines -2 and -5 The $\leq$ should be $<$. [Irene Fonseca]

CHAPTER 8:

p. 520 In the definition of the function φ_t, "$s \leq t+1$" should be replaced by "$s \geq t+1$".

p. 521 We observe that property (8.6) in Definition 8.3 is redundant. Indeed, it follows from the latter part of the proof of Theorem 8.2 that (8.6) is a consequence of having

$$\lim_{n\to\infty} \int_E h(x)\, \varphi(v_n(x)) \, dx = \int_E h(x) \int_{\mathbb{R}^m} \varphi(z)\, d\nu_x(z)\, dx$$

for every $h \in L^1(E)$ and $\varphi \in C_0(\mathbb{R}^m)$. Actually it suffices to take $h = 1$ and $\varphi = \varphi_t$, where φ_t is introduced below (8.5).

p. 523 In part (iii) of Theorem 8.6 the compact set K can be replaced by a closed set as in the reference [Ba89]. The proof in the book continues to work without changes. [Marco Barchiesi]

p. 528 Line 18 "$\bigcup_{j=1}^{i-i}$" should be replaced by "$\bigcup_{j=1}^{i-1}$". [Daniel Spector]

APPENDIX A1:

p. 554 In the statement of Corollary A.21(ii), "base" should be replaced by "local base".

p. 555 In Definition A.25 "a neighborhood that is convex" should be replaced by "a local base of convex neighborhoods".

p. 555 Line 17 "$p_E : X \to \mathbb{R}$" should be replaced by "$p_E : X \to [0, \infty]$".

p. 557 In the statement of the analytic form of the Hahn–Banach theorem, "$L(x)$" in the last line should be replaced by "$L_1(x)$". [Daniel Spector]

p. 557 "In the statement of Theorem A.36 the functional L has the additional property that $E \cup F$ is not contained in the hyperplane $\{x \in X : L(x) = \alpha\}$.".

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