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★**Modern methods in the calculus of variations: L^p spaces.**
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This book is intended as a graduate textbook and reference for those who work in the modern calculus of variations. It also prepares the ground for a follow-up volume about calculus of variation methods in Sobolev spaces. The first two parts of the book (on measure theory L^p-spaces and lower semicontinuity-convex analysis) prepare the way for the third and main part which deals with lower semicontinuity and relaxation of functionals defined on L^p-spaces. Here L^p-spaces are always taken over a Borel or Lebesgue measurable subset of Euclidean space, possibly having infinite Lebesgue measure. Following these three parts, an appendix deals with subjects selected from functional analysis and set theory.

First, the main part is a beautiful, extensive exposition of no less than 191 pages. It treats the main subjects of lower semicontinuity, representability and relaxation in three successive stages, namely for integral functionals with integrands of the types $f(z)$, $f(x,z)$ and $f(x,u,z)$. This is done for various modes of convergence of the function variables in L^p, i.e., strong, weak (or weak star if $p=\infty$) and weak star in the sense of measures, with mixtures of these modes for integrands $f(x,u,z)$. As in the other parts of the book, interesting examples and exercises help to keep the reader on track. Several open problems are indicated as well. Except for the relatively small part devoted to Young measures, where relaxation is only discussed for integrands $f(x,z)$ and where the presentation could have taken a somewhat more efficient route (see below), this presentation is admirably complete and solid.

The first two parts of the book fulfill a supporting role. Although alternative sources (admittedly scattered) could have been found, they have received considerable attention and care, and even contain original contributions (two of these refer to unpublished work by the late E. De Giorgi).

Motivated by the applications in the forthcoming second volume, the authors emphasize some finer points that would not be found in most textbooks on measure theory or convex analysis, such as the absence of σ-finiteness assumptions when possible and the Morse covering theorem, or Lipschitz continuity of separately convex functions. However, even though 226 pages are devoted to measure theory and L^p-spaces and 87 pages to lower semicontinuity and convex analysis, the presentation is very much oriented towards part three. For the expert this presents no problems, but some valuable subjects for the graduate student (for instance, monotone classes and related matters) have been suppressed. In the reviewer's opinion it is a pity that some important results for relaxation, such as the Alaoglu-Bourbaki theorem or the Eberlein-Smulian theorem, were stated without accompanying proofs. Section 2.3 is about L^p-spaces of functions with values in a Banach space, a subject that is somewhat exogenous in nature to the remainder of the book (this is more or less admitted in the introduction to that section). It could have been made redundant by making use of Komlos' theorem as a device to extend the weak star convergence of measures (Section

1.3.4) to that of Young measures. Finally, for a graduate textbook of this size, the index is far too limited. But these are small shortcomings in an otherwise excellent presentation.

Reviewed by *Erik J. Balder*

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Modern methods in the calculus of variations. L^p spaces. (English)

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This book is the first of two volumes in the calculus of variations and measure theory. The main objective of this book is to introduce necessary and sufficient conditions for sequential lower semicontinuity of functionals on L^p-spaces.

It is divided into three parts plus an appendix. The first part covers background material on measure theory, integration and L^p-spaces. It contains new proofs and some results are not restricted to the context of σ-finite measures. The authors have in mind the treatment of Hausdorff measures which will play an important role in volume 2.

The second part is devoted to the direct method in the calculus of variations and to some results for convex functions and in convex analysis (regularity of convex functions, recession functions, convex envelopes ...).

The third part is dedicated to functionals defined on L^p-spaces. The analysis of lower semicontinuity is performed for different topologies: strong convergence in L^p, weak and weak star convergence in L^p, weak star convergence in the sense of measures and weak star convergence in the dual of the space of bounded and continuous functions over an open subset of $\mathbb{R}^n$. Various types of functionals are studied, but not functionals depending on gradients which will be studied in volume 2. This part ends with an introduction to relaxation via Young measures.

Finally in the appendix some results in functional analysis, set theory, topological vector spaces are recalled. This book is very nicely written, self-contained and it is an excellent and modern introduction to the calculus of variations. I am waiting impatiently the second volume. Finally, I would like to mention that notes and a list of open problems are included at the end of the volume.

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Keywords : calculus of variations; measure theory; direct method; relaxation; lower semicontinuity

Classification :

∗49-02 Research monographs (calculus of variations)
49J45 Optimal control problems inv. semicontinuity and convergence
28-02 Research monographs (measure and integration)
28B20 Set-valued set functions etc.
52Axx General geometric convexity