Propagation enhancement in reaction-diffusion equations by a line of fast diffusion

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Model and general questions

$$\partial_t u - D\partial_{xx}^2 u = v(t,x,0) - \mu u$$

the road

$$d\partial_y v = \mu u - v$$

$$\partial_t v - d\Delta v = f(v) \qquad (1)$$

the field

- $u(t,x), v(t,x,y)$: population densities.

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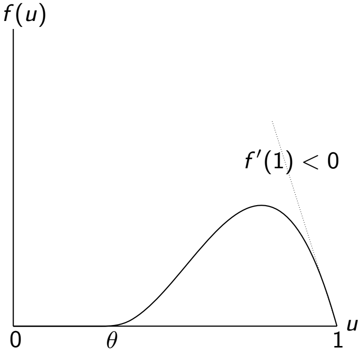
- $u(t,x), v(t,x,y)$: population densities.
- Convention : $\{y=0\}$ is "the road", $\{y<0\}$ is "the field".
  - $d, D > 0$: diffusion coefficients.
  - f : reproduction term.
  - $\mu > 0$: models exchanges between road and field.
- Model proposed by Berestycki, Roquejoffre, Rossi (2012).

The reproduction term f :



General questions :

- How does an initial localized distribution of population (u_0, v_0) evolve ?
- Location of the level sets ?
- Influence of large D ?

Ecological motivation

- Transportation networks increase the speed of biological invasions.

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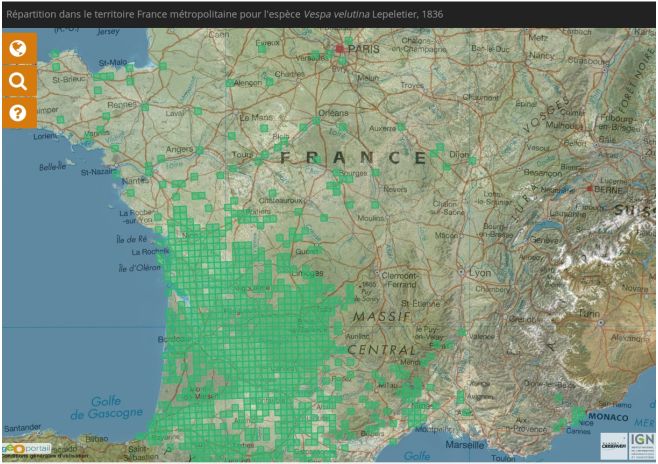
- Ex. 1 : Yellow-legged hornet. Seems to use valleys and watercourses to expand (requires a lot of water to build nests).

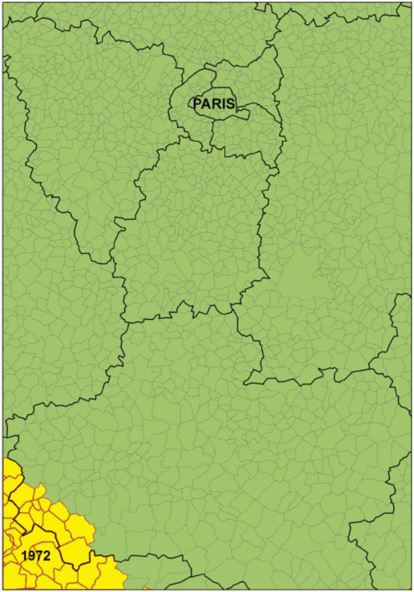
Figure: *Vespa velutina* (Wikipédia, licence CC BY-SA 3.0)



- http://inpn.mnhn.fr/espece/cd_nom/433589/tab/rep/METROP. Green squares indicate observed presence (with dates online).
- Speed of front is 100 km/year.
- Seems to spread along Garonne and then inland.

- Ex. 2 : the pine processionary caterpillar. Thought to move northwards because of climate change, but roads also thought to play a role

Figure: Pine processionary (Wikipédia, licence CC BY-SA 3.0)



Source : ANR Urticlim.

Comparison : the homogeneous KPP case

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- KPP assumption : $f(0) = f(1) = 0$, f concave.
- Define $c_{KPP} := 2\sqrt{df'(0)}$.

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Theorem-definition (Aronson-Weinberger '75)

If $u_0 \in \mathcal{C}_c^\infty$, $0 \leq u_0 \leq 1$, $u_0 \not\equiv 0$. Then

- For all $c > c_{KPP}$, $\lim_{t\to+\infty} \sup_{|x|\geq ct} u(t,x) = 0$.

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Question : what is the influence of D on the **propagation speed** in the direction e_1 in (1) ?

With a line of fast diffusion

Theorem (Berestycki, Roquejoffre, Rossi '12)

Under KPP assumption, propagation speed $c^*(D) > 0$ in the direction e_1 :

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Remark

Thus : propagation enhancement in the direction of the road.

Question : does this phenomenon persist in more general situations ?

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  - KPP assumption : propagates comp. supp. data at exponential speed (Cabré, Roquejoffre '09)
  - $f(u)$ with threshold : propagation linear in time (Mellet, Roquejoffre, Sire).

Back to the homogeneous case, with a threshold

$$\partial_t v - d\Delta v = f(v) \quad t > 0, x \in \mathbb{R}^N$$

Theorem (Kanel '61)

- There exists a unique T.W. profile $\phi \uparrow_0^1$ and a unique speed c such that $u(t,x) = \phi(x \cdot e + ct)$ is a solution.

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- If $|\{v_0 > \delta > \theta\}|$ large enough : propagation at speed c in each direction e.
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- To get the prop. speed, one really needs to study the travelling waves.
- Rescaling and uniqueness gives $c(d) = \sqrt{d}c(1)$.

The travelling waves

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$$c > 0, u(t,x) = \phi(x+ct), v(t,x,y) = \psi(x+ct,y)$$

$$\begin{array}{ccc} 0 \leftarrow \phi & -D\phi'' + c\phi' = \psi(x,0) - \mu\phi & \phi \to 1/\mu \\ \hline & d\partial_y \psi = \mu\phi - \psi(x,0) & \\ & & \\ 0 \leftarrow \psi & -d\Delta\psi + c\partial_x \psi = f(\psi) & \psi \to 1 \\ & & \\ & -\partial_y \psi = 0 & \\ \hline \end{array} \tag{3}$$

Results

Theorem 1 (D., *Appl. Math. Res. Express* 2015)

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Remarks

- (2) enjoys a comparison principle so existence of T.W. is not so surprising.
- Result similar in spirit and related to Berestycki-Larrouturou-Lions '90 :

$$\partial_\nu \psi = 0$$

$$\psi_- \leftarrow \psi \qquad -d\Delta\psi + (c + \alpha(y))\partial_x \psi = f(\psi) \qquad \psi \rightarrow \psi_+$$

$$\partial_\nu \psi = 0$$

Idea of proof

Continuation to

$$-d\psi'' + c\psi' = f(\psi),\ \psi(-\infty) = 0, \psi(+\infty) = 1$$

$0 \leftarrow \phi$ $\qquad -D\phi'' + c\phi' = \psi(x,0) - \mu\phi \qquad$ $\phi \to 1/\mu$

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$$d\partial_y \psi = (\mu\phi - \psi(x,0))/\varepsilon$$

$0 \leftarrow \psi$ $\qquad -d\Delta\psi + c\partial_x\psi = f(\psi)$ $\qquad \psi \to 1$

$$\partial_y \psi = 0$$

Step 1 : impose $\mu\phi = \psi$ on the road via $\varepsilon \in (0,1)$.

Idea of proof

$$d\partial_y \psi = \frac{D}{\mu}\partial_{xx}\psi - \frac{c}{\mu}\partial_x \psi$$

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Step 2 : vary $s \in (0, 1)$.

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One solution : the planar wave (Kanel' 61).

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More than existence : homotopy between sol. and the planar wave through a singular perturb. and a Wentzell BVP.

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- $c = 0$: only discontinuous solutions.

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- Parallel with Hamel-Zlatoš '10 : reaction-diffusion with large shear flow.

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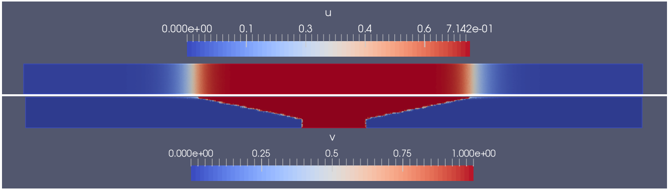

Context : propagation enhancement, the homogeneous case

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Rescaling $x \leftarrow x\sqrt{d}, c \leftarrow c/\sqrt{d}$ gives : $L_\pm(d) = \sqrt{d}L_\pm(1) \quad c(d) = \sqrt{d}c(1)$
Enhancement price : critical size of the initial data that lead to *extinction*.

Context : propagation enhancement, the homogeneous case

$$\begin{cases} \partial_t v - d\partial_{xx}^2 v = f(v) & t > 0, x \in \mathbb{R} \\ v_0(x) = \mathbf{1}_{(-L,L)}(x) \end{cases} \tag{5}$$

Kanel '64 :

- If $L < L_-$, $v \to 0$ as $t \to +\infty$ unif. on $\mathbb{R}$.
- If $L > L_+$, v converges to a pair of T.W. in both directions with speed c.

Rescaling $x \leftarrow x\sqrt{d}, c \leftarrow c/\sqrt{d}$ gives : $L_\pm(d) = \sqrt{d}L_\pm(1) \quad c(d) = \sqrt{d}c(1)$
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Remarks

- Zlatoš '06 : $L_- = L_+$.
- Du-Matano '10 : generalisation to continuous monotone 1-parameter families of comp. supp. initial data.

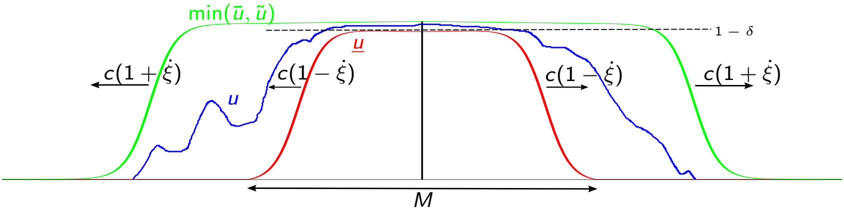
Large support w.r.t. $\sqrt{D}$

Theorem 4 (D., Roquejoffre, 2015)

Let (u_0, v_0) be ≥ 0 and compactly supported. There exists $\delta > 0$ and $M > 0$ indep. of D such that if

$$\mu u_0, v_0 > 1 - \delta \text{ for } x \in (-M\sqrt{D}, M\sqrt{D})$$

then $\mu u, v$ stays trapped (up to exponential error) between two shifts of a pair of travelling waves evolving in both directions.



What about small initial data when D is large ?

Theorem 5 (D., Roquejoffre, 2015)

There exists $M', \delta' > 0$ independent of $D > d$ such that if

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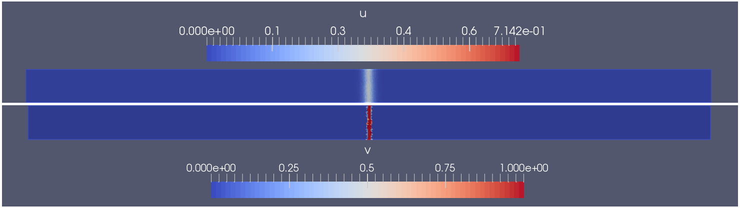

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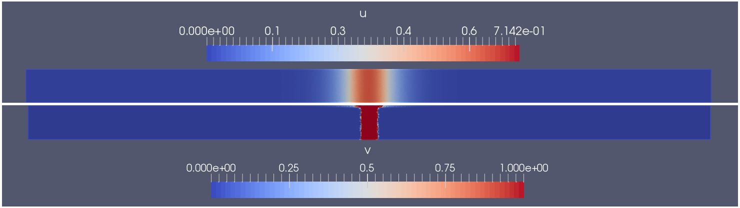

Additional information

Ongoing work : asymptotic lower bound

For all $\nu > 0$,

$$\liminf_{D\to+\infty} \frac{t_D}{D^{1/7-\nu}} = +\infty.$$

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- If $L > a_1\sqrt{D}$, invasion occurs if $\mu < \mu^-$ and extinction if $\mu > \mu^+$.

Some perspectives

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Merci pour votre attention !

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Theorem ? (D., Roquejoffre, 2015)

Define $\varepsilon := (1/D)^{1/2}$ and

$$T_{\alpha,\varepsilon} := \sup\{T > 0 \mid |v - \underline{v}| < \varepsilon^\alpha \text{ for all } 0 < t < T\}.$$

Let $\alpha \in (0,1)$, then for all $\delta < \min\left(\alpha, 2/7, \frac{5}{2}(1-\alpha)\right)$ one has

$$\left(\frac{1}{\varepsilon}\right)^\delta = \underset{\varepsilon \to 0}{o}(T_{\alpha,\varepsilon})$$

Limiting case is $\delta = \alpha = 2/7$.

A parallel : speed-up of combustion front by a shear flow

Model :

$$\partial_t v + A\alpha(y)\partial_x v = \Delta v + f(v), \qquad t \in \mathbb{R}, (x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} \tag{7}$$

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γ^* is the unique admissible velocity for :

$$\begin{cases} \Delta_y U + (\gamma - \alpha(y))\partial_x U + f(U) = 0 \text{ in } D'(\mathbb{R} \times \mathbb{T}^{N-1}) \\ 0 \leq U \leq 1 \text{ a.e. in } \mathbb{R} \times \mathbb{T}^{N-1} \\ \lim_{x \to +\infty} U(x,y) \equiv 0 \text{ uniformly in } \mathbb{T}^{N-1} \\ \lim_{x \to -\infty} U(x,y) \equiv 1 \text{ uniformly in } \mathbb{T}^{N-1} \end{cases}$$