Front propagation directed by a line of fast diffusion : existence of travelling waves.

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- Model under study :

$$\partial_t u - D\partial_{xx}^2 u = v(t,x,0) - \mu u$$

$$d\partial_y v = \mu u - v$$

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- Model proposed by Berestycki, Roquejoffre, Rossi to describe the effect of a line of fast diffusion.

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- Ex. 1 : pandemics. The 1347 black plague spread from major roads to inland areas.

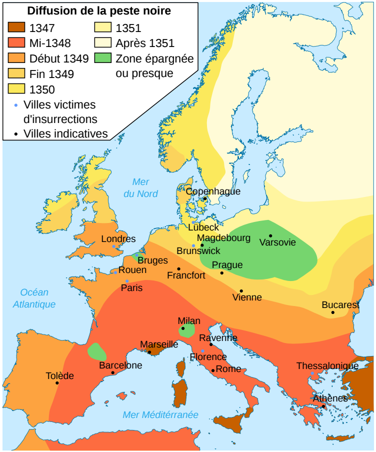
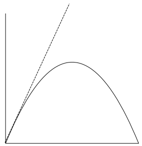


Figure: Source : Wikipédia

- Ex. 2 : the pine processionary moth. Thought to move northwards because of climate change, but roads also thought to play a role.

Figure: Pine processionary from Auray (Britain). Source : Wikipédia

Fisher-KPP propagation



Theorem (Berestycki, Roquejoffre, Rossi 2012)

There is an asymptotic speed of spreading $c^*(D) > 0$ s.t. :

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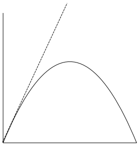

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- If $D \leq 2d$, $c^* = c_{KPP} = 2\sqrt{df'(0)}$.
- If $D > 2d$, $c^* > c_{KPP}$ and $\frac{c^*(D)}{\sqrt{D}}$ has a finite limit as $D \to +\infty$.

Remark : $2\sqrt{df'(0)}$ is the classical spreading speed in $u_t - du_{xx} = f(u)$, $x \in \mathbb{R}^N$.

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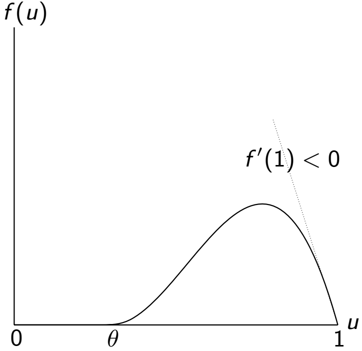

Figure: Example $f = \mathbf{1}_{u>\theta}(u-\theta)^2(1-u)$

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$$d\partial_y \psi = \mu\phi - \psi(x,0)$$

$0 \leftarrow \psi \qquad -d\Delta\psi + c\partial_x\psi = f(\psi) \qquad \psi \rightarrow 1$

$$\partial_y \psi = 0 \tag{1}$$

with uniform limits.

Results

Assumption A

$f \in \mathcal{C}^{1,\alpha}([0,1])$ is a non-negative function, $f = 0$ on $[0,\theta] \cup \{1\}$ for some $\theta > 0$, $f(0) = f(1) = 0$, and $f'(1) < 0$.

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Main result

- There exists $(c,\phi,\psi) \in \mathbb{R}_+^* \times \mathcal{C}^{2,\alpha}(\mathbb{R}) \times \mathcal{C}^{2,\alpha}(\Omega_L)$ a solution of (1) obtained by continuation from the classical 1D problem $-d\psi_0'' + c_0\psi_0' = f(\psi_0)$.

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- If $(\underline{c},\overline{\phi},\overline{\psi})$ is a classical solution of (1), $\underline{c} = c$ and there exists $r \in \mathbb{R}$ such that $\overline{\phi}(\cdot + r) = \phi(\cdot)$ and $\overline{\psi}(\cdot + r) = \psi(\cdot)$.

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Ongoing work : $D \to +\infty$

Probable outcome : $m\sqrt{D} \leq c(D) \leq M\sqrt{D}$

Idea

Continuation from the full model to

$$-d\psi'' + c\psi' = f(\psi),\ \psi(-\infty) = 0, \psi(+\infty) = 1$$

$0 \leftarrow \phi$ $\qquad -D\phi'' + c\phi' = \psi(x,0) - \mu\phi \qquad$ $\phi \to 1/\mu$

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$0 \leftarrow \psi$ $\quad -d\Delta\psi + c\partial_x\psi = f(\psi)$ $\quad \psi \to 1$

$$\partial_y \psi = 0$$

Step 1 : force $\phi = \psi$ on the road with ε, parameter in $(0,1)$.

Idea

$$d\partial_y \psi = \frac{D}{\mu}\partial_{xx}\psi - \frac{c}{\mu}\partial_x \psi$$

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$$d\partial_y\psi = \frac{sD}{\mu}\partial_{xx}\psi - \frac{c}{\mu}\partial_x\psi$$

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Step 2 : vary D with s, parameter in $(0, 1)$.

Idea

$$d\partial_y\psi + \frac{c}{\mu}\partial_x\psi = 0$$

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Interpretation : ψ on the road adjusts to ψ in the field with some delay.

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Step 3 : vary $\frac{1}{\mu}$ with t, parameter in $(0,1)$.

Idea

$$d\partial_y \psi = 0$$

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Biological interpretation : the road becomes a fence.

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Biological interpretation : the road becomes a fence.

Theorem : Kanel '69, Berestycki-Nirenberg '90

This problem has a unique solution up to translations, the planar wave.

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  - Openness : relies on a more or less sophisticated application of the implicit function theorem.
- The case $\varepsilon \simeq 0$ is non trivial and is treated separately.

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$$\begin{cases} \psi(x,y) = \dfrac{\mu e^{\lambda x} \cos\left(\sqrt{\lambda\left(\lambda - \frac{c}{d}\right)}(y+L)\right)}{\cos\left(\sqrt{\lambda\left(\lambda - \frac{c}{d}\right)}L\right) - \varepsilon d \sqrt{\lambda\left(\lambda - \frac{c}{d}\right)} \sin\left(\sqrt{\lambda\left(\lambda - \frac{c}{d}\right)}L\right)} \\ \phi(x) = e^{\lambda x} \end{cases}$$

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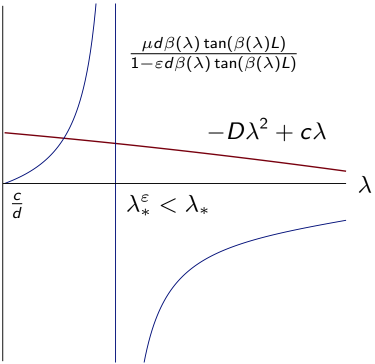
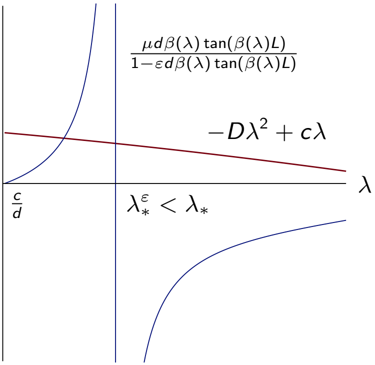


Figure: Equation on λ, $D < d$.



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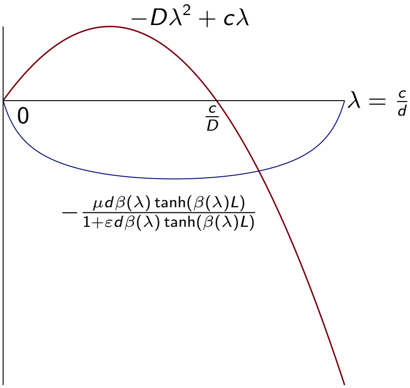

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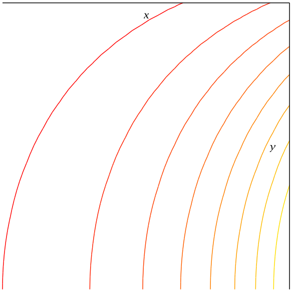

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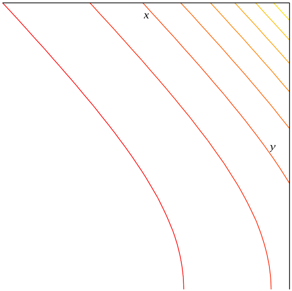

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Interpretation :

- $D < d$: the field drives the road.
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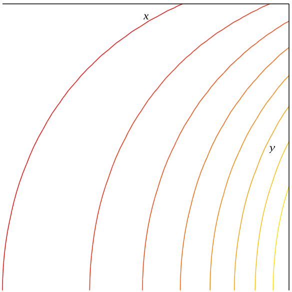

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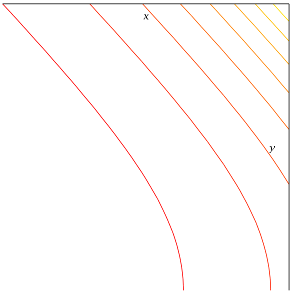

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- $D > d$: the road drives the field.
- In BRR : comparison between $2d$ and D but here $f'(0) = 0$.

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- As $\varepsilon \to 0$, everything goes smoothly to the Wentzell case.
- Provided uniform (on t, s or ε) bounds on c, we have a uniform positive lower bound for λ and uniform bounds on ϕ (or h). This will be necessary for comparison purposes.

Closedness : some hints

These properties are valid for the three models.

Lemma : a priori bounds

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Prop. : c_{min} provided c_{max}

IBP : $c = \frac{1}{L+t/\mu} \int_{\Omega_L} f(\psi)$. This gives that if there exists $c_{max} > 0$ s.t. any sol. satisfies $c < c_{max}$, then there exists $c_{min} = \frac{M_0}{L+1/\mu} f(\frac{1+\theta}{2})$.

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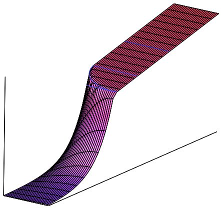

Figure: Graph of $\overline{\psi}$ for some arbitrary values of the parameters

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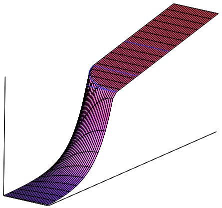

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Proof.

Glue a positive exponential solution with a linear solution of the problem with f replaced by $\|f\|_\infty$. If c is large enough, contact points only occur at salient angles, contradiction. □

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- Classical computations from Berestycki, Larrouturou, Lions gives the uniform limit to the right.

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- Limiting conditions are obtained thanks to exponential solutions.

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&W\psi_1 + \varepsilon\frac{c_1}{\mu}\partial_x\psi_1 + \left(-\frac{\varepsilon D}{\mu}\partial_{xx} + \varepsilon\frac{c_0 + c_1\varepsilon}{\mu}\partial_x\right) d\partial_y\psi_1 \\
&= -\frac{c_1}{\mu}\partial_x\psi_0 - \left(-\frac{D}{\mu}\partial_{xx} + \frac{c_0 + c_1\varepsilon}{\mu}\partial_x\right) d\partial_y\psi_0
\end{aligned}
\tag{2}
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Key lemma : construction of an auxiliary function

Compute $\tilde{\psi}$ not for the linearised operator but for the simpler problem $-d\Delta u + u = 0$ by a partial Fourier transform.

By taking a small enough exponent in the weight, there exists $u = \tilde{\psi}(\varepsilon, c_1, v)$ in $\mathcal{C}^{2,\alpha}$ that solves for $v \in \mathcal{C}^{3,\alpha}$

$$Wu + \varepsilon \frac{c_1}{\mu} \partial_x u + \left(-\frac{\varepsilon D}{\mu} \partial_{xx} + \varepsilon \frac{c_0 + c_1 \varepsilon}{\mu} \partial_x\right) d \partial_y u = h_0 - \varepsilon \frac{c_1}{\mu} \partial_x v - \left(-\frac{\varepsilon D}{\mu} \partial_{xx} + \varepsilon \frac{c_0 + c_1 \varepsilon}{\mu} \partial_x\right) d \partial_y v$$

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$$-d\Delta u + u = 0$$

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and $|u|_{\mathcal{C}^{2,\alpha}} \leq C_1 |h_0|_\infty + C_2 \left| \frac{1}{\varepsilon} K_0 \left(\frac{|x|}{d\varepsilon}\right) * (h_0 + \varepsilon h(v)) \right|_\alpha + C_3 |h_0 + \varepsilon h(v)|_\alpha$

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Proof.

Uses explicit (heavy) computations by partial Fourier transform, and a Paley-Wiener type theorem. □

Rescaled problem

Set $u_D(x) = \phi(\sqrt{D}x)$, $v_D(x,y) = \psi(\sqrt{D}x, y)$, $c_D = \frac{c}{\sqrt{D}}$. Equation on (c, u, v)

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- Lower bound : needs a uniform local $\mathcal{C}^{1,\alpha}$ estimate on v_D (true when $D = +\infty$).

Thank you for your attention.

Oblique case

Search for $p(x, y) = e^{\lambda x}\phi(y)$ zero of $-d\Delta + c\partial_x$ with $(Ob)_s$ boundary condition.

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We get $\phi(y) = \cos\left(\sqrt{\lambda(\lambda - \frac{c}{d})}(y+L)\right)$, and we are left with $\lambda > \frac{c}{d}$ solving

$$\tan\left(\sqrt{\lambda\left(\lambda - \frac{c}{d}\right)}L\right) = \frac{cs\lambda}{\mu d\sqrt{\lambda\left(\lambda - \frac{c}{d}\right)}} \qquad (3)$$

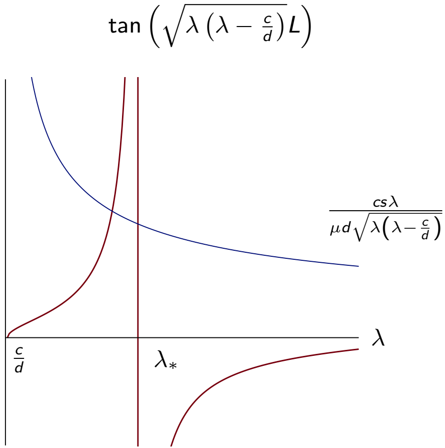

Figure: Equation (3) on λ, oblique case.

Wentzell case

Comparison between d and sD (in [?] : $2d$ and D but here $f'(0) = 0$) :

- $sD < d$: $\phi(y) = \cos\left(\sqrt{\lambda(\lambda - \frac{c}{d})}(y + L)\right)$ and $\lambda > \frac{c}{d}$ solves

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- $sD = d$: $\phi \equiv 1$ and $\lambda = \frac{c}{d}$

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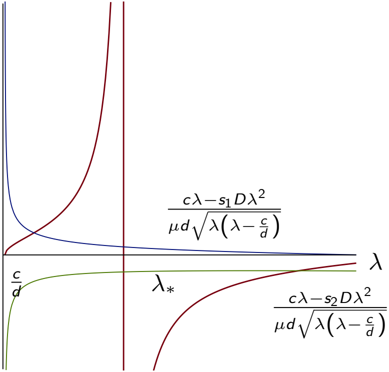

Figure: Equation (4) on λ, Wentzell case with $s_1 D < d, s_2 D > d$ and $\lambda > \frac{c}{d}$.

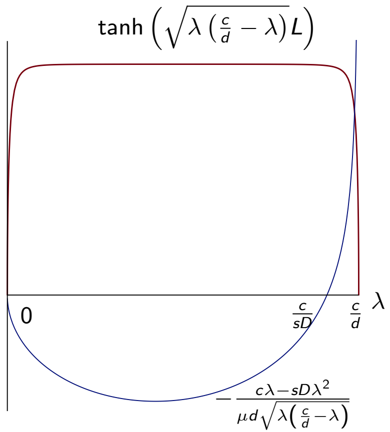

Figure: Equation (5) on λ, Wentzell case with $sD > d$ and $\lambda < \frac{c}{d}$.

P_{Ob} is open

Starting with a solution c_0, ψ_0 of some $(Ob)_{s_0}$ we set $\psi = \psi_0 + (s - s_0)\psi_1, c = c_0 + (s - s_0)c_1$ and solve the problem in c_1, ψ_1 :

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$$\mathcal{L}\psi^1 + c^1 \partial_x \psi^0 = R(s - s^0, c^1, \psi^1)$$

$$\partial_y \psi^1 = 0$$

with $\mathcal{L}g = -d\Delta g + c^0 \partial_x g - f'(\psi^0)$ and R being a quadratic remainder in ψ_1, c_1 that goes to 0 as $s \to s_0$.

P_{Ob} is open : functional setting

In a suitable weighted Hölder space inspired from works of [**?**, **?**], the linearised $\mathcal{L}$ has the Fredholm property of index 0 so that we can apply a Lyapunov-Schmidt reduction :

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$$X = \{u \in \mathcal{C}^{2}(\Omega_L) \mid D^{\beta} u \in \mathcal{C}^{\alpha}(\Omega_L), |\beta| = 2\}$$

We endow X with the norm $\|u\|_X = \|w_1 u\|_{\mathcal{C}^{2,\alpha}}$.

Prop. : reduction to a homogeneous boundary problem

There exists $\tilde{\psi}(s, c^1, \cdot) : \mathcal{C}_w^{2,\alpha}(\overline{\Omega_L}) \to \mathcal{C}_w^{2,\alpha}(\overline{\Omega_L})$ a $\mathcal{C}^1$ function such that by writing $\psi_1 = \tilde{\psi}(s, c_1, v) + v$ we have the equivalent equation

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$\tilde{\psi}(s, c^1, v)$ solves for $A > 0$ fixed large enough

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$$\mathcal{L}u + Au = 0$$

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