THÈSE

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Accélération de la propagation dans les équations de réaction-diffusion par une ligne de diffusion rapide

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À mes parents

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Enfin, je garde cette dernière ligne pour toi Coline, merci pour ton soutien et ta douceur, c'est un trésor très précieux.

[1] cf. [66] pour une liste des membres, version décembre 2014. Pierre, tu es responsable de son exhaustivité, mais en échange tu gagnes une citation !

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Introduction

> *“ This is how you do it: you sit down at the keyboard and you put one word after another until it’s done. It’s that easy, and that hard. ”*
>
> —Neil Gaiman

Objet de la thèse

L’objet général de cette thèse est l’étude de l’accélération de la propagation dans les équations de réaction-diffusion par une ligne de diffusion rapide. On y étudiera un système d’équations paraboliques couplant une équation de réaction diffusion dans un demi-plan ou dans une bande, avec une équation de diffusion sur son bord (supérieur) par l’intermédiaire d’une condition de Robin non homogène. On représentera le modèle par le diagramme suivant :

$$\frac{\partial_t u - D\partial_{xx}^2 u = v - \mu u}{d\partial_y v = \mu u - v}$$

$$\partial_t v - d\Delta v = f(v) \tag{1}$$

On notera que la condition de bord est naturelle au sens où c’est l’unique condition qui permet d’obtenir conservation de la masse totale pour le système en l’absence de réaction (c’est-à-dire quand $f = 0$). Les diffusivités d et D sont des constantes positives, μ est une constante positive indiquant l’intensité des échanges « route vers champ » au bord, et f est une fonction non linéaire telle que $f(0) = f(1) = 0$ qu’on spécifiera plus tard. Les inconnues $u(t, x)$ et $v(t, x, y)$ représentent des densités de population.

Ce système a été proposé par Berestycki, Roquejoffre et Rossi [22] pour étudier mathématiquement l’influence des réseaux de transport sur les invasions biologiques : les données dont on dispose montrent par exemple que le moustique tigre ([5]) ou la chenille processionnaire du pin (cf. le projet ANR URTICLIM) – deux insectes posant des problèmes de santé publique – envahissent leur territoire plus vite que prévu, et l’on pense que les réseaux routiers jouent un rôle dans cette invasion. De manière générale, il existe de nombreuses situations biologiques où des parasites de végétaux sont transportés par des fleuves ou une infrastructure routière, comme par exemple le cas de l'arrivée récente du frelon à pattes jaunes en France. Dans ce contexte, on réferera au demi-plan comme étant « le champ » et à son bord comme étant « la route ».

Figure 1: Processionnaires du pin (Wikipédia, licence CC BY-SA 3.0)

Figure 2: *Vespa velutina* (Wikipédia, licence CC BY-SA 3.0)

La question générale qui sous-tend l'étude du modèle est la suivante : quelle est l'influence de ce couplage et d'une forte diffusivité $D \gg d$ sur la propagation ? Dans [22] il est démontré que si f est de type KPP ($f(v) \leq f'(0)v$) alors quand $D > 2d$, la vitesse de propagation dans la direction x devient supérieure à la vitesse usuelle ($c_* = 2\sqrt{df'(0)}$ cf. ci-dessous) et qu'elle se comporte même en $c\sqrt{D}$ quand $D \to +\infty$. En ce sens dans le régime $D > 2d$ c'est la route qui dirige la propagation.

Il s'avère que la structure particulière de f permet de réduire cette étude à des calculs algébriques. De tels calculs s'avèrent indisponibles pour des non-linéarités plus générales, ce qui pose la question de la robustesse du phénomène découvert dans [22] : c'est cette question qui sous-tend toute la thèse. De fait, certains phénomènes d'accélération sont inhérents à la structure KPP de la non-linéarité. Par exemple, pour l'équation de réaction-diffusion fractionnaire

$$\partial_t u + (-\Delta)^s u = u(1-u)$$

les solutions se propagent exponentiellement vite en temps (Cabré, Coulon, Roquejoffre [27, 28]) mais dans le cas d'une non-linéarité de type ignition ($f(u) = 0$ si $u \leq \theta$ pour un seuil $0 < \theta < 1$), Mellet, Roquejoffre et Sire [64] démontrent que pour $s \geq 1/2$ il existe des ondes progressives. Grâce au principe du maximum (voir [24]) on peut alors montrer qu'elles attirent la dynamique des données initiales à support compact : ces dernières se propagent donc à vitesse finie.

En choisissant f de type ignition (par exemple $f(v) = \mathbf{1}_{v>\theta}(v-\theta)^2(1-v)$), la dégénérescence dans la zone $v \leq \theta$ nous assure l'impossibilité de mener les calculs susmentionnés. D'un point de vue biologique, de telles classes de f sont utilisées pour modéliser l'effet Allee (voir [37]). Enfin, puisqu'on s'intéressera seulement à la propagation dans la direction des x, on remplacera le demi-espace par une bande munie d'une condition de Neumann homogène sur son bord inférieur. D'un point de vue biologique, cette hypothèse signifie que les agents ne peuvent pas traverser cette frontière. Le modèle retenu est donc le suivant :

$$\begin{array}{c} \partial_t u - D\partial_{xx}^2 u = v - \mu u \\ \hline d\partial_y v = \mu u - v \\ \\ \partial_t v - d\Delta v = f(v) \\ \\ -\partial_y v = 0 \\ \hline \end{array} \qquad (2)$$

Nous allons démontrer que l'accélération par la ligne de diffusion rapide est bien une propriété structurelle du système (2) et n'est nullement un effet dû à l'hypothèse KPP. Ceci nous conduira à étudier les ondes progressives de (2), le comportement asymptotique de leur vitesse quand $D \to +\infty$, et un mécanisme d'attraction des données initiales plutôt subtil.

Contexte : propagation et accélération de fronts de réaction-diffusion

Résultats généraux

Dans cette section on rappelle quelques notions classiques concernant la propagation dans les équations de réaction-diffusion. La littérature étant très vaste sur ce sujet, on ne saura être exhaustif et on se contentera de citer des références de base et pertinentes dans le cadre de la thèse. Pour l'équation homogène

$$\partial_t u - d\Delta u = f(u) \qquad x \in \mathbb{R}^N,\ t > 0 \tag{3}$$

où par exemple $f > 0$ et $f(u) \leq f'(0)u$, on sait depuis Kolmogorov-Petrovski-Piskunov [56] qu'il existe des solutions ondes progressives $v(t,x) = \psi(x \cdot e - ct)$ dans toute direction e et pour toute vitesse $c > c_* = 2\sqrt{df'(0)}$, où le profil $\psi(x)$ définit une fonction croissante reliant 0 à 1. En particulier, la vitesse des ondes progressives est proportionnelle à la racine de la diffusivité d, ce que l'on peut obtenir très simplement via le changement d'échelle

$$x \leftarrow \sqrt{d}x, \ c \leftarrow c/\sqrt{d}.$$

Comme expliqué par Aronson-Weinberger [1], dans ce contexte la vitesse minimale des ondes c_* est aussi la vitesse de propagation des données initiales « pas trop grandes » : toute donnée initiale positive à support compact non vide convergera en temps long et dans chaque direction vers l'onde de vitesse minimale. En ce sens, la vitesse de propagation de la masse est c_* : un observateur se déplaçant dans une direction fixée à une vitesse strictement supérieure à c_* ne verra plus rien autour de lui en temps long et s'il se déplace à une vitesse strictement inférieure, il verra l'environnement rempli à sa capacité maximale autour de lui. Ceci indique qu'au premier ordre en temps long, les lignes de niveau de $u(t,x)$ avancent dans chaque direction à vitesse c_*. Ces phénomènes ont par la suite été généralisés à d'autres types de non-linéarités et leur description raffinée (voir par exemple Fife-McLeod [42] où les auteurs prouvent un mécanisme précis de convergence vers les ondes progressives), le changement principal étant que si $f'(0) \leq 0$, il n'existe qu'une seule vitesse admissible pour les ondes. On peut aussi citer les travaux pionniers de Kanel$'$ [53] où l'auteur étudie (3) (en dimension 1) pour des non-linéarités satisfaisant

$$f(v) \leq 0 \text{ pour } v \leq \theta \tag{4}$$

pour un seuil $0 < \theta < 1$. Naturellement – et contrairement au cas KPP où même une masse de Dirac se propage au sens ci-dessus – il est alors nécéssaire que la donnée initiale dépasse le seuil θ quelque part pour qu'il puisse y avoir propagation et non pas convergence uniforme vers 0. Ce phénomène souligne la différence fondamentale entre les non-linéarités de type KPP et celles satisfaisant (4) ; il jouera un rôle important dans cette thèse et on y reviendra. Dans [53] Kanel$'$ étudie la classe de données initiales $v_0(x) = \mathbf{1}_{(-L,L)}(x)$ et démontre l'existence de seuils $L_0, L_1 > 0$ tels que si $L < L_0$ il y a convergence uniforme vers 0, et si $L > L_1$, la solution se propage au sens défini ci-dessus. Récemment, Zlatoš [80] démontre que dans ce contexte, $L_0 = L_1$. Ce résultat a par la suite été généralisé par Du et Matano [40] pour diverses familles de données initiales à un paramètre.

Ce genre de résultat – par exemple l'utilisation du simple changement d'échelle ci-dessus – devient beaucoup plus ardu à démontrer en présence d'hétérogénéités, c'est-à-dire quand (3) n'est pas posée dans $\mathbb{R}^N$ tout entier, ou a des coefficients dépendants de l'espace, ou en présence d'un système avec un couplage spatial comme (1). Les travaux pionniers dans l'étude des équations de réaction-diffusion hétérogènes remontent à Freidlin et Gärtner ([45]), qui étudient une équation de Fisher-KPP où

$$f(x,u) = \mu(x)u - u^2$$

où μ est une fonction 1-periodique en toutes ses variables sur $\mathbb{R}^N$. Ils étudient la vitesse de propagation (au sens défini ci-dessus) à l'aide d'arguments de grandes déviations et prouvent que la vitesse $c_*(e)$ n'est plus isotrope, donnant une formule explicite pour la calculer dans chaque direction. Le traitement des hétérogénéités dans les équations de réaction-diffusion continue d'être grandement étudié depuis. Dans le cadre de l'influence sur la propagation, on peut citer Berestycki-Larrouturou-Lions [17] et Berestycki-Nirenberg [20] qui démontrent l'existence de fronts non planaires dans des cylindres en présence d'un champ d'advection $(\alpha(y), 0, \cdots, 0)$. Les travaux de Roquejoffre [76] adaptent à ce cas multi-dimensionnel hétérogène les résultats de convergence de Fife-McLeod et Kanel$'$. Plus récemment, on citera Berestycki-Hamel [8] et Berestycki-Hamel-Nadirashvili [15, 16] où les auteurs donnent de nouvelles informations sur l'influence de la géométrie du domaine ou des coefficients de l'équation, dans un cadre périodique dans un premier temps puis pour des domaines plus généraux. L'influence de la géométrie sur la propagation a aussi été étudiée dans le cadre des cylindres à section variable par Chapuisat et Grenier [32] ou encore Berestycki-Bouhours-Chapuisat [7].

Enfin, on citera des travaux récents de Liang, Matano et Xiaotao [58, 59] où les auteurs étudient une équation de Fisher-KPP en dimension 1 ou 2 et où

$$f(x, y, u) = b(x)u(1 - u)$$

avec b une mesure de Radon L-périodique, et s'intéressent à maximiser la vitesse de propagation. Sous la contrainte de masse $\int_0^L b(x)\mathrm{d}x = \alpha L$ ($\alpha > 0$ étant une constante), ils démontrent que la mesure qui maximise la vitesse de propagation $c_*(\theta, b)$ dans la direction θ est (pour n'importe quelle direction θ) non pas une fonction lisse mais un peigne de Dirac.

Accélération de fronts de réaction-diffusion

Dans cette section, on poursuit l'étude bibliographique commencée ci-dessus en mettant l'accent sur l'accélération de fronts, ou sur l'augmention de la vitesse de propagation.

Suite aux premiers travaux dans des cylindres hétérogènes cités ci-dessus, la question de l'influence d'un champ d'advection sur la vitesse de propagation a été beaucoup étudiée. Le phénomène est intéressant car il peut être à double tranchant[2] : l'advection peut d'une part augmenter la vitesse de propagation (et quantifier cette effet est déjà une question subtile) mais en présence de non-linéarités à seuil elle peut aussi faire passer la donnée initiale sous le seuil et condamner la solution à converger uniformément vers zéro (*quenching* ou *extinction*) : il s'agit alors de comprendre en termes de taille de la donnée initiale, comment obtenir propagation ou *quenching*. L'influence de l'amplitude du champ d'advection joue

[2] une bonne image donnée dans [41] est celle de l'influence du vent sur le feu : il y a d'une part l'effet « le vent propage le feu » mais il y a aussi l'effet « essayer d'allumer un feu de camp en présence d'un vent fort ... »

ici un rôle fondamental et a été étudiée dans de nombreux articles, à commencer par Audoly-Berestycki-Pomeau [3]. Dans le cas KPP et pour des écoulements parallèles de cisaillement (à moyenne nulle afin de se mettre dans le cas d'un drift nul) dans des cylindres :

$$\partial_t u - \Delta u + A\alpha(y)\partial_x u = f(u) \quad t > 0, x \in \mathbb{R}, y \in \omega \subset R^N$$

Berestycki [6] démontre une augmention linéaire de la vitesse en l'amplitude A du flot

$$c_*(A) \underset{A\to+\infty}{\sim} kA$$

Ce résultat a aussi été obtenu et généralisé par Constantin-Kiselev-Oberman-Ryzhik [33] en introduisant la notion de *bulk burning rate*. Pour les non-linéarités de type ignition, le résultat reste vrai et est récemment démontré par Hamel et Zlatoš [49] (voir la section « Résultats » ci-dessous et le Chapitre 2 pour une description précise de leur résultat). En revanche, Constantin-Kiselev-Ryzhik [34] et Kiselev-Zlatoš [55] démontrent que dans ce cas, le prix à payer pour cette augmention de la vitesse est une augmentation linéaire elle aussi en les seuils L_0 et L_1 introduits ci-dessus

$$L_0 \underset{A\to+\infty}{\sim} k_0 A, \quad L_1 \underset{A\to+\infty}{\sim} k_1 A$$

lorsque l'écoulement n'est pas constant sur des intervalles trop grands. En d'autres termes, on échange une augmention de la vitesse linéaire en A contre une augmention linéaire en A de la taille critique menant au *quenching*.

Le cas des écoulements « cellulaires » (en rouleaux périodiques, voir [3]) a aussi été étudié : le même phénomène intervient, mais l'augmention de la vitesse est de l'ordre de $A^{1/4}$ comme démontré par Novikov et Ryzhik [68] dans le cas KPP et plus récemment par Zlatoš [81] dans le cas ignition. En revanche, Fannjiang-Kiselev-Ryzhik [41] prouvent (pour des écoulements avec des cellules assez petites) que si $L^4 \ln(L) < kA$ où L représente la taille du carré qui supporte la donnée initiale, il y a *quenching*. On citera aussi les simulations numériques de [79].

Enfin, un autre mécanisme intéressant peut être trouvé dans [35] : les auteurs étudient un système couplant une équation de reaction-diffusion et une équation de Burgers. Ils démontrent différents résultats de *quenching* par rapport à un paramètre de gravité, l'un d'entre eux étant un *quenching* indépendant de la taille de la donnée initiale L, lorsque la gravité est suffisamment forte.

Un tout autre mécanisme d'accélération de la propagation réside bien sûr dans la diffusion : Cabré et Roquejoffre [28, 29] démontrent qu'en remplaçant dans l'équation de Fisher-KPP homogène le laplacien par un laplacien fractionnaire $(-\Delta)^\alpha$ [3] :

$$\partial_t u + (-\Delta)^\alpha u = f(u) \quad t > 0, x \in \mathbb{R}^N \tag{5}$$

on passe d'une propagation linéaire en temps à une propagation exponentielle en temps : si $\sigma > \sigma_* := \frac{f'(0)}{N+2\alpha}$, $u(t,x) \to 0$ uniformément dans $\{|x| \geq e^{\sigma t}\}$ lorsque

[3] microscopiquement cela revient à considérer que les agents ne se déplacent plus selon un mouvement brownien, mais selon un processus de Lévy (donc à sauts) α-stable

$t \to +\infty$ et si $\sigma < \sigma_*$, $u(t,x) \to 1$ uniformément dans $\{|x| \leq e^{\sigma t}\}$. Garnier [44] démontre aussi un tel résultat de vitesse asymptotique infinie et donne des bornes inférieures sur l'avancement des lignes de niveaux, pour des noyaux de dispersion généraux à queues lourdes. Cabré-Coulon-Roquejoffre [27] étudient quant à eux (5) en milieu périodique et démontrent qu'en contraste avec la formule de Freidlin-Gärtner [45], la vitesse exponentielle de propagation ne dépend pas de la direction. Inversement, si f est de type ignition dans (5), Mellet-Roquejoffre-Sire [64] démontrent que jusque $\alpha \geq 1/2$ il existe des ondes progressives. Grâce au principe du maximum (cf. [24]) on peut alors adapter le résultat d'Aronson-Weinberger et montrer que la propagation des données initiales à support compact est nécéssairement linéaire en temps.

Cette thèse s'inscrit dans ce contexte général et propose d'étudier un mécanisme d'accélération nouveau, à travers un couplage spatial avec une ligne de diffusion rapide.

Le système (1)

Dans cette sous-section, on présente l'état de l'art concernant le système (1). L'article [22] qui introduit le modèle démontre qu'il est bien posé et qu'il dispose d'un principe de comparaison : une sous-solution et une sur-solution (c'est-à-dire des solutions où les signes = sont remplacés resp. par $\leq$ ou $\geq$) initialement ordonnées restent ordonnées pour tout temps (pour l'ordre usuel composante par composante). Le résultat principal de l'article est le théorème suivant :

Théorème. *([22])*
Soit f de type KPP : $0 < f(v) \leq f'(0)v$ pour tout $v \in]0,1[$. Alors on a les propriétés suivantes.

i) Spreading. Il existe une vitesse $c_* = c_*(\mu, d, D) > 0$ telle que : pour tous $(u_0, v_0) \geq 0$ et $\not\equiv (0,0)$,

- *pour tout $c > c_*$, $\lim_{t\to+\infty} \sup_{|x|\geq ct}(u(x,t), v(x,y,t)) = (0,0)$.*
- *pour tout $c < c_*$, $\lim_{t\to+\infty} \inf_{|x|\geq ct}(u(x,t), v(x,y,t)) = (1/\mu, 1)$.*

ii) Vitesse asymptotique. Si d et μ sont fixes et $D > 0$ on a :

- *Si $D \leq 2d$, alors $c_*(\mu, d, D) = c_{KPP}$*
- *Si $D > 2d$ alors $c_*(\mu, d, D) > c_{KPP}$ et $\lim_{D\to+\infty} c_*(\mu, d, D)/\sqrt{D}$ est finie.*

Comme mentionné ci-dessus, ce théorème utilise l'hypothèse KPP sur la non-linéarité afin de se ramener à des calculs sur le linéarisé en 0. Le seuil $D = 2d$ plutôt que d peut paraître surprenant à première vue. La présence du facteur 2 est en fait due à l'absence de réaction sur la route. Des formules plus précises pour le seuil dans un cadre plus général ont été introduites dans le second article de Berestycki, Roquejoffre et Rossi [21]. Dans cet article, les auteurs ajoutent un terme de transport et de réaction dans l'équation sur la route et étudient l'influence de ces paramètre sur c^*. Le modèle augmenté s'écrit :

$$\frac{\partial_t u - D\partial_{xx}^2 u + q\partial_x u = v - \mu u + g(u)}{d\partial_y v = \mu u - v}$$

$$\partial_t v - d\Delta v = f(v) \tag{6}$$

Ils démontrent le théorème suivant, sous une hypothèse un peu plus forte que KPP (on notera qu'une classe de $g(u)$ convenables est donnée par les mortalités $g(u) = -\rho u)$:

Théorème. *([21])*
Soit f comme ci-dessus avec de plus $s \mapsto f(s)/s$ décroissante. Soit g telle que $g(0) = 0$, $\exists S > 0, g(S) \leq 0$ et $s \mapsto g(s)/s$ soit décroissante. Alors

i) Le système (6) *admet un unique état stationnaire positif borné (U, V), de plus $U \equiv$ constante et $V \equiv V(y)$.*

ii) Il existe deux vitesses de spreading $w_*^\pm$ (dans les directions $\pm e_1$) vers cet état stationnaire.

iii) Si $\frac{D}{d} \leq 2 - \frac{g'(0)}{f'(0)} \mp \frac{q}{\sqrt{df'(0)}}$ alors $w_*^\pm = c_{KPP}$. Sinon $w_*^\pm > c_{KPP}$.

iv) $\lim\limits_{D\to+\infty} \frac{w_*^\pm}{\sqrt{D}} = h > 0$ indépendant de q et $\lim\limits_{q\to+\infty} \frac{w_*^\pm}{|q|} = k$ si $g'(0) \leq \mu$ et 1 sinon, avec $0 < k < 1$ indépendant de D.

Les convergences vers les états stationnaires mentionnées ci-dessus ont lieu à y fixé et dans la direction e_1. Le troisième article de la série [23] étudie précisément le défaut de convergence uniforme en y lors de cette convergence. Les auteurs y montrent qu'en fait la propagation est aussi améliorée dans les directions ν avec $\nu \cdot (-e_2) > 0$, jusqu'à un certain angle limite où la vitesse se met à coller à c_{KPP}. Des estimations quantitatives sur les vitesse dans chaque direction lorsque $D \to +\infty$ y sont aussi données et il y est prouvé que l'angle limite tend vers $\pi/2$ quand $D \to +\infty$: asymptotiquement toutes les directions hormis e_2 voient leur vitesse de spreading améliorées.

Dans le contexte KPP, les papiers de Pauthier [71, 72] étudient quant à eux l'influence des échanges intégraux : il y démontre que l'équation (1) où l'on remplace les échanges localisés au bord par un mécanisme d'échanges intégraux :

$$\begin{cases} \partial_t u - D\partial_{xx}^2 u = -\bar{\mu} u + \int_{-\infty}^{+\infty} \nu(y) v(t,x,y) \mathrm{d}y & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu(y) u(t,x) - \nu(y) v(t,x,y) & (x,y) \in \mathbb{R}^2, t > 0 \end{cases} \tag{7}$$

(où $\bar{\mu} = \int_{\mathbb{R}} \mu(y)\mathrm{d}y$) préserve la propriété d'accélération de la propagation découverte dans [22]. Dans [71], l'auteur étudie la limite de (7) lorsque les fonctions d'échanges $\mu_\varepsilon/\bar{\mu}$ et ν_ε réalisent une approximation de l'unité : il y démontre que les solutions de (7) convergent vers les solutions de (1) (étendues en les $y > 0$ par parité) uniformément en espace et uniformément sur tout compact en temps.

Enfin, toujours dans le contexte KPP, dans la thèse d'A.C. Coulon Chalmin [36] se trouve une étude de (1) où la diffusion sur la route est fractionnaire et en présence d'un terme de mortalité :

$$\partial_t u + (-\partial^2_{xx})^\alpha u = v - \mu u - \rho u$$

$$d\partial_y v = \mu u - v$$

$$\partial_t v - d\Delta v = f(v) \tag{8}$$

L'auteure y démontre que pour $\alpha \in]1/4, 1[$, la propagation est exponentielle en temps sur la route et caractérise l'exposant limite : pour tout $\gamma < \frac{f'(0)}{1+2\alpha}$, $\lim\limits_{t\to+\infty} \inf\limits_{|x|\le e^{\gamma t}} u(t,x) > 0$ et pour tout $\gamma > \frac{f'(0)}{1+2\alpha}$, $\lim\limits_{t\to+\infty} \sup\limits_{|x|\le e^{\gamma t}} u(t,x) = 0$. De plus, il est aussi démontré que pour toute direction non horizontale, la vitesse de propagation dans cette direction dans le champ reste linéaire mais explose quand la direction tend vers e_1.

Résultats de la thèse

En l'absence de calculs directs sur le linéarisé en 0, il est naturel d'attaquer ce problème par le point de vue des ondes progressives. Dans un premier temps l'article [39] étudie l'existence et l'unicité de telles ondes pour le modèle (2). Ce système dispose d'un principe de comparaison, et pour cette raison l'existence des ondes n'est pas une grande surprise. Cependant, j'ai obtenu cette existence grâce à une méthode de continuation reliant le système à l'onde plane classique de combustion. Cette continuation passe par l'intermédiaire d'un problème au bord de type Wentzell et d'une perturbation singulière qui semble nouvelle dans ce contexte. Le résultat est aussi valable (et la démonstration plus simple) pour le cas bistable.

La suite naturelle à ce travail est l'étude de la vitesse de ces ondes lorsque le paramètre D devient grand. C'est l'objet de l'article [38]. Dans cet article, je montre que la vitesse $c(D)$ des ondes ci-dessus se comporte en $c_\infty\sqrt{D}$ lorsque $D \to +\infty$, ce qui généralise en un sens le résultat de Berestycki, Roquejoffre et Rossi dans [22]. J'ai aussi caractérisé le ratio c_∞ comme l'unique vitesse admissible pour des ondes dans un système hypoelliptique a priori dégénéré, où l'espèce de densité v ne diffuserait que verticalement :

$$\begin{array}{ccc} 0 \leftarrow u & -u'' + c_\infty u' = v - \mu u & u \to 1/\mu \\ \hline & d\partial_y v = \mu u - v & \\ & & \\ 0 \leftarrow v & c_\infty \partial_x v - d\partial_{yy}^2 v = f(v) & v \to 1 \\ & & \\ & \partial_y v = 0 & \\ \hline \end{array} \tag{9}$$

Il s'avère que ce système admet bien une unique onde progressive (à translation près) de vitesse c_∞. Ce résultat révèle un parallèle intéressant avec un récent article de Hamel et Zlatoš [49] qui étudie l'augmentation de la vitesse de fronts de combustion par un écoulement de cisaillement. Le modèle étudié est

$$\partial_t v + A\alpha(y)\partial_x v = \Delta v + f(v), \qquad t \in \mathbb{R}, (x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} \tag{10}$$

où l'amplitude du flot $A > 1$ est grande et où $\alpha(y)$ est lisse et $(1, \cdots, 1)$-périodique. Les auteurs démontrent l'existence de $\gamma^*(\alpha, f) \geq \fint_{\mathbb{T}^{N-1}} \alpha(y) \mathrm{d}y$ telle que la vitesse $c^*(A\alpha, f)$ des ondes de (10) satisfait

$$\lim_{A \to +\infty} \frac{c^*(A\alpha, f)}{A} = \gamma^*(\alpha, f)$$

De plus, sous une hypothèse de type Hörmander sur α, ils caractérisent γ^* comme l'unique vitesse admissible pour le système dégénéré suivant :

$$\begin{cases} \Delta_y U + (\gamma - \alpha(y))\partial_x U + f(U) = 0 \text{ dans } D'(\mathbb{R} \times \mathbb{T}^{N-1}) \\ 0 \leq U \leq 1 \text{ p.p. dans } \mathbb{R} \times \mathbb{T}^{N-1} \\ \lim_{x \to +\infty} U(x,y) \equiv 0 \text{ unif. dans } \mathbb{T}^{N-1} \\ \lim_{x \to -\infty} U(x,y) \equiv 1 \text{ unif. dans } \mathbb{T}^{N-1} \end{cases} \tag{11}$$

Dans [38], le rôle de cette condition de Hörmander est joué par une borne inférieure sur la vitesse $c(D)$ lorsque $D \to +\infty$ dans le modèle renormalisé :

$$\begin{array}{ccc} 0 \leftarrow u & -u'' + cu' = v - \mu u & u \to 1/\mu \\ \hline & d\partial_y v = \mu u - v & \\ & & \\ 0 \leftarrow v & -\frac{d}{D}\partial_{xx}^2 v - d\partial_{yy}^2 v + c\partial_x v = f(v) & v \to 1 \\ & & \\ & \partial_y v = 0 & \\ \hline \end{array} \tag{12}$$

J'ai obtenu cette estimée grâce à des identités intégrales nouvelles qui ont dues être menées à partir de zéro, puisque sans cette estimée (12) perd toute hypoellipticité. Une fois cette estimée obtenue, la compacité s'obtient en itérant les identités intégrales. Enfin, des arguments usuels de comparaison comme la méthode de sliding sont adaptés dans un cadre combinant théorie parabolique (dans le champ) et elliptique (sur la route) pour conclure. Cette forme originale du principe de comparaison a motivé l'étude de (9) par une méthode directe, qui a permis de retrouver l'existence de l'onde limite sus-mentionnée, prouvant en un sens que (9) est bien posé.

Pour clore l'étude de ce phénomène d'accélération, il reste à comprendre en quelle mesure les ondes de (2) attirent les données initiales. C'est l'objet du Chapitre 3 du présent manuscrit. Dans un premier temps, en adaptant des résultats classiques dans notre cas hétérogène, on étudie le cas des données « front-like », puis des données à support compact assez grandes sur un intervalle assez grand (de l'ordre de $D^{1/2}$ selon un résultat classique). Il s'avère cependant que ces ondes ne se limitent pas à cette forme d'attraction pour le moins attendue, la taille du support initial étant très liée à la taille de la diffusion. Des simulations numériques m'ont permis de découvrir un mécanisme plus subtil, qu'on qualifiera de « propagation à deux vitesses » : une donnée à support trop petit se propagera d'abord à une vitesse lente – c'est-à-dire indépendante de D – pendant un temps au plus

$$t_D = D^{1/2} \ln D + \underset{D \to +\infty}{O}(1)$$

avant d'atteindre le régime susmentionné. On donne ci-dessous un aperçu de ces simulations.

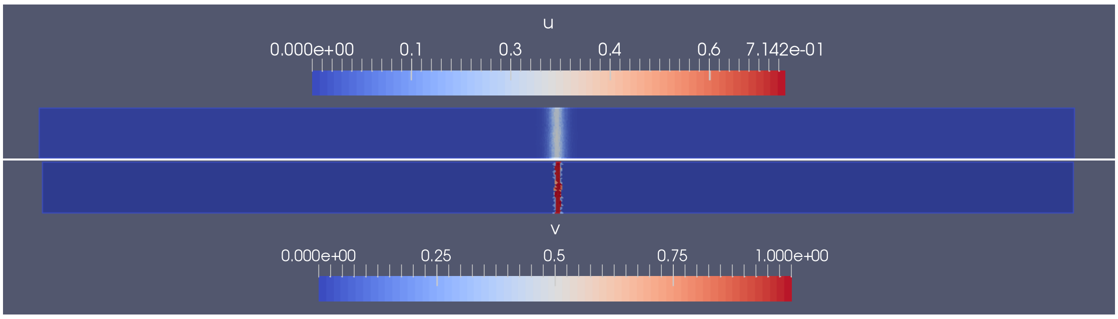

Figure 3: $t = 0$

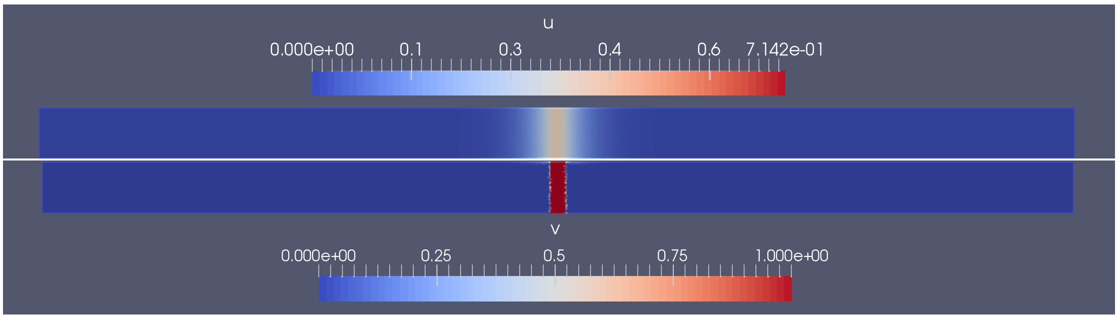

Figure 4: $t = 75\Delta t$

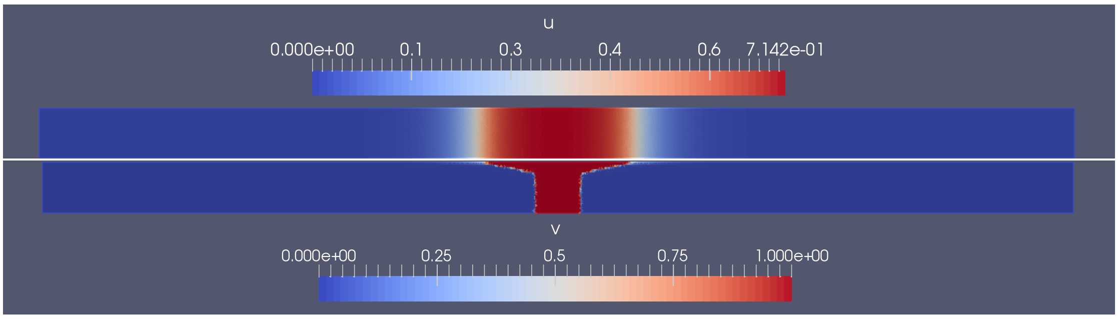

Figure 5: $t = 300\Delta t$

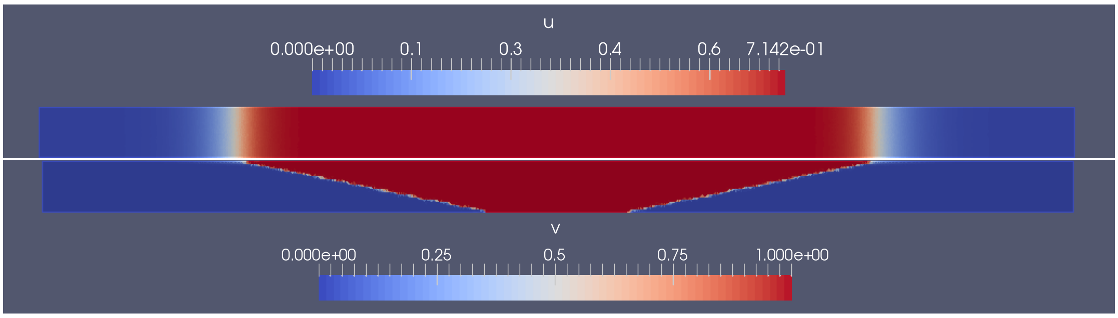

Figure 6: $t = 1000\Delta t$

Ce résultat est surprenant dans le sens où, comme mentionné ci-dessus, les résultats d'accélération de la propagation dans les équations de réaction-diffusion ont tendance à demander en contre-partie un renforcement des hypothèses sur les données initiales pour qu'elles ne mènent pas à l'extinction. Dans le cas présent on démontre un résultat d'accélération à des vitesses de l'ordre $D^{1/2}$ pour des données initiales indépendantes de D. Enfin, la dernière partie du chapitre étudie le comportement du système pour des données initiales supportées uniquement sur la route : contrairement au résultat précédent, on démontre que les données à support trop petit (par rapport à $D^{1/2}$) mènent à l'extinction, et que pour les données à support assez grand, extinction tout comme invasion peuvent apparaître selon la valeur du paramètre d'échange μ.

Organisation du manuscrit

- Le chapitre 1 est l'article [39] qui s'intéresse aux questions d'existence et d'unicité d'ondes progressives pour le modèle (2).

- Le chapitre 2 est l'article [38] où on démontre l'asymptotique $c(D) \sim c_\infty\sqrt{D}$ de la vitesse de ces ondes et où on caractérise le ratio $c_\infty > 0$ comme l'unique vitesse admissible pour les ondes d'un système hypoelliptique dont on démontre aussi qu'il est bien posé du point de vue des ondes progressives.

- Le chapitre 3 étudie la stabilité des ondes sus-mentionnées : on y met en lumière un mécanisme d'accélération qui permet aux ondes d'attirer les données initiales à support trop petit par rapport à $D^{1/2}$.

- Enfin, dans le chapitre 4 on présente quelques perspectives.

Notations

Operators

Δ	Laplace operator
$(-\Delta)^s$	Fractional laplacian of order $s \in (0,1)$: $(-\Delta)^s h(x) = c_{d,s}\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{h(x)-h(y)}{\|x-y\|^{d+2s}}\mathrm{d}y$
$\dot{q}$	Time-derivative of the function q
$\hat{f}(\xi)$	Fourier transform of the real function $f(x)$
$\mathcal{F}^{-1}g$	Inverse Fourier transform of g
$\hat{u}(\xi, y)$	Partial Fourier transform of the real function $u(x,y)$

Miscellaneous

$\|x\|$	Any norm of the finite-dimensional vector $x \in \mathbb{R}^N$
$[a,b],]a,b[$	Resp. the closed and open intervals between a and b in $\mathbb{R}$
$N(\mathcal{L}), R(\mathcal{L})$	Resp. the kernel and range of the operator $\mathcal{L}$
$\overline{\Omega}$	Closure of the open set $\Omega \subset \mathbb{R}^N$
$\Re z, \Im z$	Resp. real and imaginary parts of the complex number z
$f_{\|[a,b]}$	Restriction on $[a,b]$ of the real function f
$\mathrm{Lip} f$	Lipschitz constant of f
$\cosh, \sinh, \tanh$	Hyperbolic trigonometric functions
$(a,b) \leq (c,d)$	Product order : $a \leq c$ and $b \leq d$
$\begin{pmatrix} a \\ b \end{pmatrix} \leq \begin{pmatrix} c \\ d \end{pmatrix}$	Same as above

Function spaces

$\lvert\cdot\rvert_X$	Norm in the Banach space X
$\lvert\cdot\rvert_\infty$	Abbreviation for $\lvert\cdot\rvert_{L^\infty}$
$\mathcal{C}^k(\Omega)$	Scalar functions on Ω with bounded continuous derivatives up to order k
$\mathcal{C}^k(X;Y)$	The same as above but for functions between the spaces X and Y
$\mathcal{C}^k(\overline{\Omega})$	Scalar functions on Ω with bounded continuous derivatives up to order k and up to the boundary of Ω
$\mathcal{C}(\Omega)$	$\mathcal{C}^0(\Omega)$
$\mathcal{C}^k_{loc}(\Omega)$	Scalar functions with continuous derivatives up to order k
$\mathcal{C}^{k,\alpha}(\Omega)$	Banach space of scalar functions with bounded and α-Hölder continuous derivatives up to order k
$\mathcal{C}^\alpha(\Omega)$	$\mathcal{C}^{0,\alpha}$
$BUC(\Omega)$	Banach space of bounded uniformly continuous scalar functions on Ω
$UC_0(\Omega)$	Subspace of the above of functions decaying to 0 as $\lvert x\rvert \to +\infty$
$\hookrightarrow$	Compact embedding
$BV(a,b)$	Functions $f:(a,b)\to\mathbb{R}$ with bounded variation.
$W^{k,p}(\Omega)$	Sobolev space of functions with distributional derivatives up to order k belonging in $L^p(\Omega)$
$H^k(\Omega)$	The same as above with $p=2$
$H^k_{loc}(\Omega)$	The same as above but derivatives needing only to be in L^2_{loc}.
$H^s(\mathbb{R}^N)$	$\{f\in L^2(\mathbb{R}^N) \mid \mathcal{F}^{-1}\left(1+\lvert\xi\rvert^2\right)^{s/2}\hat{f}\in L^2(\mathbb{R}^N)\}$ (agrees with H^k above when $s\in\mathbb{N}$)
$\mathcal{C}^{\alpha/2,\alpha}$	Functions $v(x,y)$ that are $\mathcal{C}^{\alpha/2}$ in a first variable and $\mathcal{C}^\alpha$ in the others : $\lvert v(x,y)-v(x',y')\rvert\le C\left(\lvert x-x'\rvert^{\alpha/2}+\lvert y-y'\rvert^\alpha\right)$
$\mathcal{C}^{1+\alpha/2,2+\alpha}$	Functions $v(x,y)$ whose derivatives up to order 1 with respect to the x-variable and up to order 2 with respect to the y-variables all lie in $\mathcal{C}^{\alpha/2,\alpha}$.

These Banach spaces are to be endowed with their usual norms, see [26] and [46]. Usually the last two are used in a context where x is replaced by the time-variable t (and y by a general space variable) : this will be the case in Chapter 3. The specific notation above will be used at the end of Chapter 2.

Chapter 1

Continuation of a travelling wave

« Les loups sont entrés dans Paris. »

—Serge Reggiani (1922-2004)

We prove existence and uniqueness of travelling waves for a reaction-diffusion system coupling a classical reaction-diffusion equation in a strip with a diffusion equation on a line. To do this we use a sequence of continuations which leads to further insight into the system. In particular, the transition occurs through a singular perturbation which seems new in this context, connecting the system with a Wentzell type boundary value problem.

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Contents

1.1 Introduction

This paper deals with the following system with unknowns $c > 0, \phi(x), \psi(x, y)$:

$$\begin{cases} -d\Delta\psi + c\partial_x\psi = f(\psi) \text{ for } (x,y) \in \Omega_L \ := \mathbb{R}\times]-L,0[\\ d\partial_y\psi(x,0) = \mu\phi(x) - \psi(x,0) \text{ for } x \in \mathbb{R} \\ -d\partial_y\psi(x,-L) = 0 \text{ for } x \in \mathbb{R} \end{cases}$$
$$-D\phi''(x) + c\phi'(x) = \psi(x,0) - \mu\phi(x) \text{ for } x \in \mathbb{R}$$

along with the uniform in y limiting conditions

$$\mu\phi, \psi \to 0 \text{ as } x \to -\infty$$

$$\mu\phi, \psi \to 1 \text{ as } x \to +\infty$$

and where d, D, μ are positive constants. Those equations will be represented from now on as the following diagram

$$\begin{array}{ccc} 0 \leftarrow \phi & -D\phi'' + c\phi' = \psi - \mu\phi & \phi \to 1/\mu \\ \hline & d\partial_y\psi = \mu\phi - \psi & \\ & & \\ 0 \leftarrow \psi & -d\Delta\psi + c\partial_x\psi = f(\psi) & \psi \to 1 \\ & & \\ & \partial_y\psi = 0 & \\ \hline \end{array} \tag{1.1}$$

If (c, ϕ, ψ) is a solution of (1.1), then $(\phi(x + ct), \psi(x + ct, y))$ is a travelling wave solution connecting the states $(0, 0)$ and $(1/\mu, 1)$ for the following reaction-diffusion system

$$\begin{array}{c} \partial_t u - Du'' = v - \mu u \\ \hline d\partial_y v = \mu u - v \\ \\ \partial_t v - d\Delta v = f(v) \\ \\ \partial_y v = 0 \\ \hline \end{array} \tag{1.2}$$

This system in the whole half-plane $y < 0$ was introduced by Berestycki, Roquejoffre and Rossi in [22] to give a mathematical description of the influence of transportation networks on biological invasions. Under the assumption $f(v) = v(1 - v)$ (which will be referred to from now on as a KPP type non-linearity) the authors showed the following : when $D \leq 2d$, propagation of the initial datum along the x direction occurs at the classical KPP velocity $c_{KPP} = 2\sqrt{df'(0)}$, but when $D > 2d$ it occurs at some velocity $c^*(\mu, d, D) > c_{KPP}$ which satisfies

$$\lim_{D\to+\infty} \frac{c^*}{\sqrt{D}} = c > 0$$

In the present work, we take another viewpoint to extend these results to more general non-linearities. Indeed, the KPP assumption

$$f(v) \leq f'(0)v$$

enables a reduction of the question to algebraic computations. For more general reaction terms (e.g. bistable, or ignition type), propagation is usually governed by the travelling waves. As a consequence, it is necessary to investigate the existence, uniqueness, and stability of solutions of (1.1) in order to generalise this result, which will be seen through the velocity $c(D)$ of the solution.

We assume f to be in a biologically relevant class of nonlinearities that arise in the modelling of Allee effect. Namely f will be of the ignition type :

Assumption A. *$f : [0,1] \to \mathbb{R}$ is a smooth non-negative function, $f = 0$ on $[0,\theta] \cup \{1\}$ with $0 < \theta < 1$, and $f'(1) < 0$. For convenience we will still call f an extension of f on $\mathbb{R}$ by zero at the left of 0 and by its tangent at 1 (so it is negative) at the right of 1.*

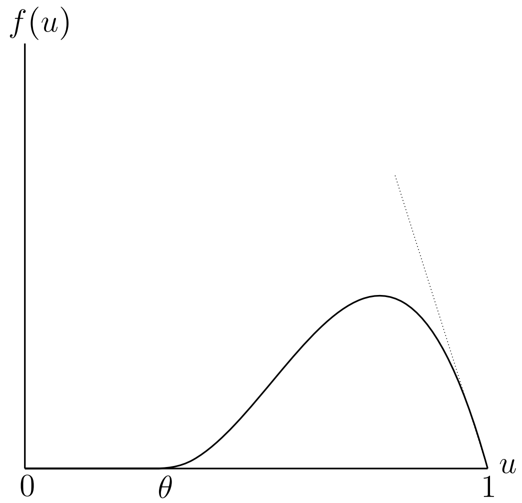


Figure 1.1: Example $f = \mathbf{1}_{u>\theta}(u-\theta)^2(1-u)$

But f could also be of the bistable type. In this case – thanks to the non-degeneracy of f on $(0,\theta)$ – the method described in this paper becomes even simpler. This will be discussed at the end of the paper.

Assumption B. *$f : [0,1] \to \mathbb{R}$ is a smooth function, $f(0) = f(\theta) = f(1)$ for some $0 < \theta < 1$ and $f < 0$ on $(0,\theta)$, $f > 0$ on $(\theta, 1)$ with moreover $f'(0), f'(1) < 0$ and f is of positive total mass : $\int_0^1 f(s)\mathrm{d}s > 0$.*

Our objective is to study (1.1) by a continuation method : we will show that (1.1) can be reduced, through "physical" steps, to the classical one dimensional equation

$$-\psi'' + c\psi' = f(\psi)$$

$$\psi(-\infty) = 0, \psi(+\infty) = 1$$

When f satisfies Assumption A, this is the simplest model in the description of propagation of premixed flames (see the works of Kanel [52]). More precisely, the steps we will follow are :

1) First, a good way of reaching a unique equation is to have $\mu\phi = \psi$ on the boundary $y = 0$. To achieve that, we divide the exchange term by a small $\varepsilon > 0$ and send $\varepsilon \to 0$. Setting $\mu\phi = \psi(x,0) + \varepsilon\phi_1$ we get after a simple computation the limiting model for ψ :

$$-\frac{D}{\mu}\partial^2_{xx}\psi + \frac{c}{\mu}\partial_x\psi = -\phi_1$$

and so this limit is a singular perturbation that sends (S_ε) :

$$\begin{array}{ccc} 0 \leftarrow \phi & -D\phi'' + c\phi' = (\psi(x,0) - \mu\phi)/\varepsilon & \mu\phi \to 1 \\ \hline & d\partial_y\psi = (\mu\phi - \psi(x,0))/\varepsilon & \\ 0 \leftarrow \psi & -d\Delta\psi + c\partial_x\psi = f(\psi) & \psi \to 1 \\ & \text{-}\partial_y\psi = 0 & \\ \hline \end{array} \tag{S_ε}$$

to a unique equation with a Wentzell boundary condition that we call (W_s) (with $s = 1$):

$$\begin{array}{ccc} \hline & d\partial_y\psi = s\left(\frac{D}{\mu}\partial^2_{xx}\psi - \frac{c}{\mu}\partial_x\psi\right) & \\ 0 \leftarrow \psi & -d\Delta\psi + c\partial_x\psi = f(\psi) & \psi \to 1 \\ & \text{-}\partial_y\psi = 0 & \\ \hline \end{array} \tag{W_s}$$

2) By sending the parameter $s \in [0,1]$ to 0, we can pass from (W_s) to (W_0), which is a Neumann problem, which is known to have a unique velocity $c_0 > 0$ and a unique smooth profile $\psi_0(x)$ (up to translations) as solutions. Existence is due to Kanel [52] and uniqueness to Berestycki-Nirenberg [18].

   We will show the following :

Theorem 1.1.1. *(Existence for the Wentzell model)*
There exists $c_w > 0$ and $\psi_w \in \mathcal{C}^{3,\alpha}(\Omega_L)$ for some $0 < \alpha < 1$, solution of (W_1) obtained by continuation from (c_0, ψ_0) that satisfies $0 < \psi_w < 1$, and ψ_w is increasing in the x direction. Moreover, if $(\underline{c}, \overline{\psi})$ is a classical solution of (W_1), we have $\underline{c} = c$ and there exists $r \in \mathbb{R}$ such that $\overline{\psi}(\cdot + r, y) = \psi_w(\cdot, y)$.

Theorem 1.1.2. *(Transition from Wentzell to the system)*
There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, (S_ε) has a solution given by $c_\varepsilon > 0$ and $(\phi_\varepsilon, \psi_\varepsilon) \in \mathcal{C}^{2,\alpha}(\mathbb{R}) \times \mathcal{C}^{2,\alpha}(\Omega_L)$ obtained by continuation from $\left(c_w, \frac{1}{\mu}\psi_w(\cdot,0), \psi_w\right)$ that satisfies $0 < \phi_\varepsilon < \frac{1}{\mu}$, $0 < \psi_\varepsilon < 1$, and ϕ_ε and ψ_ε are increasing in x.

Remark 1.1.1. We wish to emphasise on this result : it consists in a singular perturbation between a system of two unknowns and a scalar boundary value problem of the Wentzell type. This is a non-standard relaxation that appears to be new in this context. Also, observe that we had to pay the loss of one derivative for solving this problem.

Theorem 1.1.3. *(Existence for the full system)*
There exists $c > 0$ and $(\phi, \psi) \in \mathcal{C}^{2,\alpha}(\mathbb{R}) \times \mathcal{C}^{2,\alpha}(\Omega_L)$ a solution of (S_1) obtained by continuation from $(c_{\varepsilon_0}, \phi_{\varepsilon_0}, \psi_{\varepsilon_0})$ that satisfies $0 < \phi < \frac{1}{\mu}$, $0 < \psi < 1$, and ϕ and ψ are increasing in x. Moreover, if $(\underline{c}, \overline{\phi}, \overline{\psi})$ is a classical solution of (S_1), we have $\underline{c} = c$ and there exists $r \in \mathbb{R}$ such that $\overline{\phi}(\cdot + r) = \phi(\cdot)$ and $\overline{\psi}(\cdot + r, y) = \psi(\cdot, y)$.

The organisation of the paper is as follows :

- In Section 1.2 we show some common a priori properties for solutions of (S_ε) or (W_s). In particular we deal with the uniqueness questions.
- In Section 1.3 we prove Theorem 1.1.1.
- Section 1.4 proves Theorem 1.1.3, provided Theorem 1.1.2, by slight modifications of Section 1.3.
- Finally, we postponed the proof of Theorem 1.1.2 in Section 1.5 because of its particularity (see Remark 1.1.1).

1.2 First properties

1.2.1 A priori bounds, monotonicity, uniqueness

This section is devoted to the proofs of a priori properties. As noticed in [22], the system (1.2) has the structure of a monotone system, which provides a maximum principle. The elliptic counterpart of this maximum principle holds for (S_ε) or (W_s) and it will be our main tool along with the sliding method ([19]) throughout the current section.

Lemma 1.2.1. *Let (c, ϕ, ψ) solve some (S_ε). Then*

$$inf\ \psi \leq \mu\phi \leq sup\ \psi$$

Proof. Because of its uniform limits as $x \to \pm\infty$, necessarily ψ is bounded and $|\psi|_\infty \geq 1$. Because of the limits of ϕ, either $\phi \leq \frac{1}{\mu}$ or $\phi - \frac{1}{\mu}$ reaches a positive maximum. But then at this maximum $\phi' = 0$ and $\phi'' \leq 0$ which means $\psi - \mu\phi \geq 0$, so in every case $\phi \leq \frac{\sup \psi}{\mu}$. The other inequality is similar. □

Proposition 1.2.1. *Let (c, ϕ, ψ) be a solution of some (S_ε), then*

$$0 < \mu\phi, \psi < 1$$

Similarly, if (c, ψ) solves some (W_s), $0 < \psi < 1$.

Proof. Suppose there exists a point (x_0, y_0) where $\psi(x_0, y_0) > 1$. Then, because ψ is assumed to have limits as $x \to \pm\infty$, we see that $\psi - 1$ must reach a positive maximum somewhere. But $\psi - 1$ satisfies locally at this point $-d\Delta(\psi - 1) + c\partial_x(\psi - 1) < 0$. This point cannot be in Ω_L by the strong maximum principle, since $\psi - 1$ would be locally constant which is impossible by looking at the equation. So it has to be on the boundary. It cannot be on $y = -L$ because of the Hopf lemma. So it has to be on $y = 0$ and by the Hopf lemma, $\mu\phi > \psi$ at this point, which is impossible because of Lemma 1.2.1. So $\psi \leq 1$, and then $\phi \leq \frac{1}{\mu}$.

Now knowing these bounds and that the solutions are not constants, comparison with 0 by the strong maximum principle gives $\phi < \frac{1}{\mu}, \psi < 1$.

Finally, since $\psi < 1$, $f(\psi) \geq 0$ and the strong maximum principle along with Lemma 1.2.1 gives $\psi > 0$ and then $\phi > 0$.

The same proof holds for equation (W_s), the case $y_0 = 0$ being treated by the sole Hopf lemma thanks to the sign of $\partial^2_{xx}\psi$ and the nullity of $\partial_x\psi$ on an extremum of ψ.

□

We turn now to the monotonicity of the fronts, using the sliding method of [18] and simplified in [78] in the travelling waves context. We start with a fundamental lemma which asserts that we can slide a supersolution above a subsolution by translating it enough to the left. This lemma is valid for any reaction term such that $f(0) = f(1) = 0$ and $f'(0), f'(1) \leq 0$.

Definition 1. We call a super (resp. sub) solution of some (S_ε) or (W_s) a function which satisfies those equations with the $=$ signs replaced by $\geq$ (resp. $\leq$), and the uniform limits replaced by some constants $\geq 0, \geq 1$ (resp. $\leq 0, \leq 1$).

Lemma 1.2.2. *Let $(c, \underline{\phi}, \underline{\psi})$ be a subsolution of some (S_ε) and $(c, \overline{\phi}, \overline{\psi})$ a supersolution. Then there exists r_0 such that for all $r \geq r_0$, $\overline{\phi}^r := \overline{\phi}(r + \cdot), \overline{\psi}^r := \overline{\psi}(r + \cdot, \cdot)$ satisfy*

$$\overline{\phi}^r > \underline{\phi}, \overline{\psi}^r > \underline{\psi}$$

The same holds with $\underline{\psi}$, $\overline{\psi}$ resp. sub and supersolution of some (W_s).

Proof. We present only the proof for (S_ε), the case (W_s) being simpler. We also assume that $\underline{\psi}, \mu\underline{\phi}, \overline{\psi}, \mu\overline{\phi} \to_{x\to-\infty} 0$ and $\underline{\psi}, \underline{\phi}, \overline{\psi}, \mu\overline{\phi} \to_{x\to+\infty} 1$, the case of different limits being considerably simpler.

First we show that by translating enough, we have the desired order on some $x \geq a$ (which is trivial if the sub and the supersolution have different limits to the right). Fix $\varepsilon > 0$ small enough such that $f' \leq 0$ on $[0, \varepsilon] \cup [1 - \varepsilon, 1]$. Because of the conditions at $\pm\infty$ it is clear that there exists $r_1 > 0$ and $a > 0$ large enough such that

$$\mu\underline{\phi}, \underline{\psi} > 1 - \varepsilon \text{ on } x \geq a \tag{1.3}$$

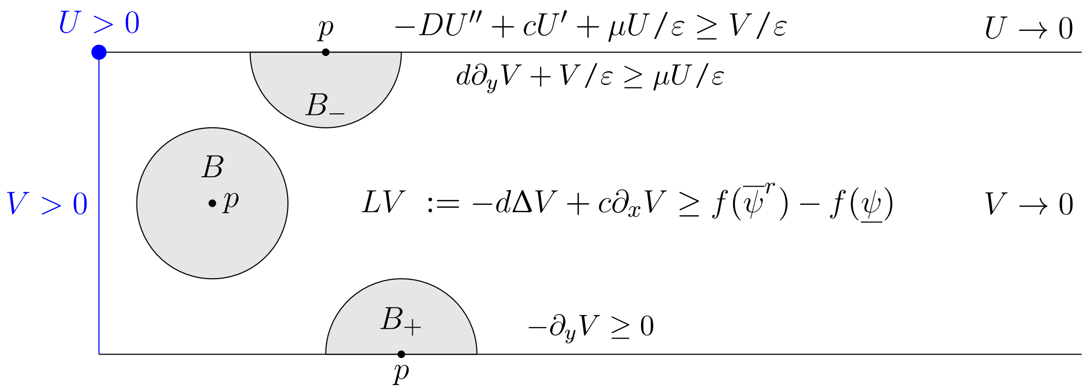
$$\text{For all } r \geq r_1,\ \mu\overline{\phi}^r(a) > \mu\underline{\phi}(a),\ \overline{\psi}^r(a, y) > \underline{\psi}(a, y) \tag{1.4}$$

$$\text{For all } r \geq r_1,\ \mu\overline{\phi}^r > 1 - \varepsilon,\ \overline{\psi}^r > 1 - \varepsilon \text{ on } x \geq a \tag{1.5}$$

Conditions (1.4) and (1.5) are obtained simultaneously by taking r_1 large enough. We assert that this suffices to have

$$\mu\overline{\phi}^r > \mu\underline{\phi}, \overline{\psi}^r > \underline{\psi} \text{ on } x \geq a \text{ for all } r \geq r_1$$

Indeed, call $U = \overline{\phi}^r - \underline{\phi}$ and $V = \overline{\psi}^r - \underline{\psi}$. Then in $[a, +\infty[\times[-L, 0]$ we have the following :



Suppose there is a point where $V < 0$. Then because of its limits, V reaches a minimum $V(p) = m < 0$ at some $p \in]a, +\infty[\times[-L, 0]$.

- Case 1 : $p \in]a, +\infty[\times] - L, 0[$. By continuity, there is a ball B around p such that on B, $1 - \varepsilon < \overline{\psi}^r < \underline{\psi}$. On B, $LV \geq f(\overline{\psi}^r) - f(\underline{\psi}) \geq 0$ since $f(s)$ is decreasing on $s \geq 1 - \varepsilon$. By the strong maximum principle, $V \equiv m$ in B. Thus, $\{V = m\}$ is open. Being trivially closed and being non-void, it is all of $[a, +\infty[\times[-L, 0]$ which is a contradiction.

- Case 2 : p lies on $y = -L$. Again, by continuity there is a half ball B_+ just as B in the previous case. By the Hopf lemma, since $\partial_y V(p) \geq 0$ necessarily $V = m$ is also reached in the interior of B_+, and we fall in case 1.

- Case 3 : p lies on $y = 0$. Taking another half ball B_- as above, either we fall in case 1 or $\mu U < m < 0$ at x_p. But this is impossible also. Indeed, U would reach a negative minimum somewhere, but looking at the equation it satisfies at that minimum, $\mu U \geq V \geq m$.

Every case leading to a contradiction, we conclude that $\overline{\psi}^r - \underline{\psi} \geq 0$ on $x \geq a$. The strong maximum principle applied on U yields now $\overline{\phi}^r - \underline{\phi} \geq 0$ on $x \geq a$ and that the orders are strict.

We do the same thing for $x \leq b$ up to the following subtlety : we can only ask for the following conditions

$$\mu\underline{\phi}, \underline{\psi} < \varepsilon \text{ on } x \leq b \tag{1.6}$$

$$\text{For all } r \geq r_2,\ \mu\overline{\phi}^r(b) > \mu\underline{\phi}(b),\ \overline{\psi}^r(b, y) > \underline{\psi}(b, y) \tag{1.7}$$

Of course an equivalent of condition (1.5) is not available here : we cannot ask to put the supersolution everywhere below ε on $x \leq b$ whereas before it was automatic to put it above $1-\varepsilon$. Nonetheless, the exact same proof as above works, since on an eventual minimum of $\overline{\psi}^r - \underline{\psi}$ we would have this order for free : $\overline{\psi}^r < \underline{\psi} < \varepsilon$.

Finally, taking $r_3 = \max(r_1, r_2)$ we end up with the supersolution above the subsolution on all $x \notin]b,a[$: thanks to the uniform limits of $\overline{\phi}^{r_3}, \overline{\psi}^{r_3}$ to the right, we just have to translate enough again to cover the compact region left.

For the case of equation (W_s), just observe that case 3 is similar to case 2, since on a minimum, $\partial^2_{xx} V \geq 0, \partial_x V = 0$.

□

Remark 1.2.1. The use of the maximum principle in the proof above could be simplified, since on $x \geq a$ we know the sign of

$$k(x,y) = -\frac{f(\overline{\psi}^r) - f(\underline{\psi})}{\overline{\psi}^r - \underline{\psi}} \in L^\infty$$

we could apply it directly in all Ω_L with the operator $-d\Delta + c\partial_x + k$. Nonetheless, this is not true any more on $x \leq b$ and this is the reason why we chose the proof above.

Proposition 1.2.2. *Let (c, ϕ, ψ) be a solution of some (S_ε), then*

$$\phi', \partial_x \psi > 0$$

Similarly, if (c, ψ) solves some (W_s), $\partial_x \psi > 0$.

Proof. Use Lemma 1.2.2 with the solution serving as the sub and the supersolution at the same time : we can translate some $\mu\phi^r, \psi^r$ over $\mu\phi, \psi$. Call r the inf of such r_0, i.e. slide back until the solutions touch (which clearly happens since at $r_0 = 0$ they are the same). Monotonicity will be proved if we show that

$$r = 0$$

Suppose by contradiction that $r > 0$. By continuity

$$\begin{aligned} U &:= \phi^r - \phi \geq 0 \\ V &:= \psi^r - \psi \geq 0 \end{aligned}$$

Moreover, U, V solves

$$0 \leftarrow U \qquad -U'' + cU' + \mu U = V \qquad U \rightarrow 0$$

$$d\partial_y V + V = \mu U$$

$$0 \leftarrow V \qquad -d\Delta V + c\partial_x V + k(x,y)V = 0 \qquad V \rightarrow 0$$

$$-\partial_y V = 0$$

with $k(x,y) = -\frac{f(\psi^r) - f(\psi)}{\psi^r - \psi} \in L^\infty$. The strong maximum principle and Hopf's lemma for comparison with a minimum that is 0 gives that $U, V > 0$ (otherwise

ϕ, ψ would be periodic in x, which is impossible). Then by continuity, for any compact

$$K_a = [-a, a] \times [-L, 0]$$

we can still translate a bit more to the right while keeping the order :

$$\mu\phi^{r-\varepsilon_a} > \mu\phi,\ \psi^{r-\varepsilon_a} > \psi \text{ on } K_a$$

for some small $\varepsilon_a > 0$. Now just do this with a large enough so that on $x \leq -a$, $\psi < \varepsilon$ and on $x \geq a$, $\psi > 1-\varepsilon$ (and so $\psi^{r-\varepsilon_a}$ too, even on $x \geq a - \varepsilon_a$). Then the exact same proof as in Lemma 1.2.2 applies to conclude that

$$\mu\phi^{r-\varepsilon_a} > \mu\phi,\ \psi^{r-\varepsilon_a} > \psi$$

everywhere, which is a contradiction with the minimality of r.

We now know that ϕ and ψ are increasing in x, that is $\phi', \partial_x\psi \geq 0$. To conclude that $\phi', \partial_x\psi > 0$ everywhere, just differentiate (S_ε) or (W_s) with respect to x and apply the strong maximum principle and Hopf's lemma for comparison with a minimum 0. We emphasise on the fact that this result is valid up to the boundary of Ω_L.

□

The proof above gives directly the following rigidity result and its corollaries :

Proposition 1.2.3. *(Uniqueness among sub or supersolutions.)*
Fix $c > 0$. If (S_ε) has a solution, then every supersolution or subsolution is a translate of this solution. The same holds for (W_s).

Proof. Denote (ϕ, ψ) the solution mentioned and $(\overline{\phi}, \overline{\psi})$ an arbitrary supersolution. Let r, U, V be as in the proof of Proposition 1.2.2 (this time r exists thanks to the limit conditions : at some point the supersolution and the solution touch). We end up with either $U, V > 0$ or $U, V \equiv 0$. The first case is impossible for the exact same argument as in Proposition 1.2.2 and this concludes the proof. □

Proposition 1.2.4. *(Uniqueness of the velocity and the profiles up to translation.)*

1. *There is a unique $c_\varepsilon \in \mathbb{R}$ such that (S_ε) can have solutions. The same holds for c_s with (W_s).*

2. *Solutions of (S_ε) are unique up to x-translations, i.e. if (c, ϕ_1, ψ_1) and (c, ϕ_2, ψ_2) are solutions of (S_ε), then there exists $r \in \mathbb{R}$ such that*

$$\phi_2(\cdot + r) = \phi_1(\cdot), \psi_2(\cdot + r, \cdot) = \psi_1(\cdot, \cdot)$$

The same holds for (W_s).

Proof.

1. Call (c, ϕ, ψ) and $(\overline{c}, \underline{\phi}, \underline{\psi})$ two solutions such that $\overline{c} > c$. Observe that thanks to monotonicity :

$$\begin{aligned} &- d\Delta\underline{\psi} + c\partial_x\underline{\psi} = f(\underline{\psi}) + (c-\overline{c})\partial_x\underline{\psi} < f(\underline{\psi}) \\ &- \underline{\phi}'' + c\underline{\phi}' + \mu\underline{\phi}/\varepsilon = \underline{\psi}/\varepsilon + (c-\overline{c})\underline{\phi}' < \underline{\psi}/\varepsilon \end{aligned}$$

   so that $(\underline{\phi}, \underline{\psi})$ is a subsolution of equation (S_ε) with c and is not a solution, which is impossible thanks to Proposition 1.2.3. The case of equations (W_s) is treated in a similar way, just observe that

$$d\partial_y\underline{\psi} = s\left(\frac{D}{\mu}\partial^2_{xx}\underline{\psi} - \frac{\overline{c}}{\mu}\partial_x\underline{\psi}\right) < s\left(\frac{D}{\mu}\partial^2_{xx}\underline{\psi} - \frac{c}{\mu}\partial_x\underline{\psi}\right)$$

2. Apply Proposition 1.2.3, knowing that a solution is also a subsolution.

□

1.2.2 Uniform bounds for c

Our continuation method will need compactness on $c > 0$ if we want to extract a solution from a sequence of solutions. Getting an upper bound will depend on finding supersolutions of (S_ε) or (W_s). Then a lower bound will follow easily via an argument of [17].

Proposition 1.2.5. *There exists $c_{max} > 0$ such that any solution $(c_\varepsilon, \phi_\varepsilon, \psi_\varepsilon)$ of (S_ε) satisfies*

$$c_\varepsilon < c_{max}$$

The same holds for (W_s).

Proof. Observe that if

$$\begin{cases} -Dr^2 + cr \geq 0 \\ -dr^2 + cr \geq \mathrm{Lip} f \end{cases}$$

Then $(e^{rx}, \mu e^{rx})$ is a supersolution of (S_ε) which is not a solution. The first inequation gives $r = \alpha c/D$ with $\alpha \in [0,1]$ and the best choice for minimising c in the second one is $\alpha = D/(2d)$ or $\alpha = 1$ depending on $D \lesseqgtr 2d$. More precisely,

$$c_{max} = \begin{cases} 2\sqrt{d\mathrm{Lip}f} \text{ if } D \leq 2d \\ \sqrt{\dfrac{D^2}{D-d}\mathrm{Lip}f} \text{ if } D \geq 2d \end{cases} \tag{1.8}$$

The exact same computation holds for (W_s). □

Remark 1.2.2. Note that at $s = 0$, D does not appear in the equation so the lowest c_{max} is valid but exhibits a discontinuity as soon as $s > 0$ (if $D > 2d$). Of course, since this is only an upper bound it is not a problem. Actually, this is just technical : if we had done the continuation from Neumann to oblique and from oblique to Wentzell in two steps, this discontinuity would not be since the comparison would occur between $sD \lesseqgtr 2d$.

Proposition 1.2.6. *There exists $c_{min} > 0$ such that any solution of (S_ε) satisfies*

$$c_\varepsilon \geq c_{min} > 0$$

The same holds for equation (W_s).

Proof. In this proof we get rid of the ε for the sake of notations. We integrate the equation for ψ on $\Omega_{L,M} := [-M, M] \times [-L, 0]$ using integration by parts. For the first term, we have

$$\begin{aligned}
\int_{\Omega_{L,M}} -d\Delta\psi &= \int_{\partial\Omega_{L,M}} -d\partial_\nu\psi \\
&= \int_{[-L,0]} d\partial_x\psi(-M, y)dy - \int_{[-L,0]} d\partial_x\psi(M, y)dy + \int_{-M}^{M} -d\partial_y\psi(x, 0)dx \\
&= \int_{[-L,0]} d\partial_x\psi(-M, y)dy - \int_{[-L,0]} d\partial_x\psi(M, y)dy + \int_{-M}^{M} (\psi(x, 0) - \mu\phi(x))dx
\end{aligned}$$

Using elliptic estimates and dominated convergence, we see that the first two terms go to zero as $M \to \infty$, which gives

$$\int_{\Omega_L} -d\Delta\psi = \int_{\mathbb{R}} (\psi(x, 0) - \mu\phi(x))dx = \int_{\mathbb{R}} (-D\phi'' + c\phi')dx = \frac{c}{\mu}$$

thanks to elliptic estimates on ψ.

For the second term, we have

$$\int_{\Omega_{L,M}} c\partial_x\psi = \int_{[-L,0]} c\psi(M, y)dy - \int_{[-L,0]} c\psi(-M, y)dy \to cL$$

by dominated convergence. We thus have

$$c = \frac{1}{L + 1/\mu} \int_{\Omega_L} f(\psi) \tag{1.9}$$

Now, any solution satisfies $-d\Delta\psi + c\partial_x\psi = f(\psi)$ in Ω_L with c and $f(\psi)$ bounded independently of c by the constant $M_0 = \max(d, c_{max}, \sup f)$. Thus on the ball B of centre $(0, -L/2)$ and radius $L/4$, standard L^2 elliptic estimates and the Sobolev embedding give for any $0 < \beta < 1$

$$|\psi|_{C^\beta(B)} \leq C_1 \left(|\psi|_{L^2(2B)} + |f(\psi)|_{L^2(2B)} \right) \leq C_1 |2B| (1 + \sup f^2) \leq C_2$$

with C_2 independent of ε and where $2B$ denotes B with doubled radius, and $|2B|$ its measure. We just proved that all solutions share a modulus of continuity independent of ε on the ball B. Since f is Lipschitz, the same holds for $f(\psi)$.

Now normalise the solutions by translation so that

$$\psi\left(0, \frac{-L}{2}\right) = \frac{1+\theta}{2}$$

The previous estimate enables us to choose a radius $r_0 > 0$ small enough that depends only on C_2 and Lip$f$ such that $f(\psi) \geq \frac{1}{2} f(\frac{1+\theta}{2})$ on the ball $r_0 B$. This implies the lower bound

$$\int_{\Omega_L} f(\psi) \geq |r_0 B| \frac{1}{2} f\left(\frac{1+\theta}{2}\right) > 0$$

that gives the existence of

$$c_{min} = \frac{|r_0 B|}{2(L + 1/\mu)} f\left(\frac{1+\theta}{2}\right)$$

that depends only on $d, \mu, L, c_{max}, \sup f, \mathrm{Lip} f$.

For equation (W_s), the exact same proof holds since (1.9) is replaced by

$$c = \frac{1}{L + s/\mu} \int_{\Omega_L} f(\psi) \geq \frac{1}{L + 1/\mu} \int_{\Omega_L} f(\psi)$$

for $s \in [0, 1]$. □

1.3 From Neumann to Wentzell

Set

$$P_W = \{s \in [0,1] \quad | \quad (W_s) \text{ has a solution}\}$$

The main goal of this section is to prove that P_W is open and closed in $[0, 1]$, as in [4]. We will proceed as follows :

- We already know that $0 \in P_W$ so that $P_W \neq \varnothing$.

- In Section 1.3.1 we prove that P_W is closed, using the bounds on c from Section 1.2 and a regularity result up to the boundary for (S_ε) or (W_s).

  We emphasise on a small but interesting technical difficulty : in the context of (W_s), no standard L^p estimates up to the boundary appear to be in the literature. As a consequence, we had to use a weak Harnack inequality up to the Wentzell boundary to prove the Hölder regularity of $f(\psi)$, which is needed to use the Schauder estimates of [63].

- In Section 1.3.2 we prove that P_W is open, by perturbing (W_s) for s close to some $s^0 \in P_W$ in a weighted space where we can apply the implicit function theorem.

Together with the uniqueness properties of Section 1.2, this will prove Theorem 1.1.1.

1.3.1 P_W is closed

In this section we consider a sequence $(s^n) \subset P_W$ that converges to $s^\infty \in]0,1]$ and we want to show that (W_{s^∞}) has a solution thanks to the compactness results we already obtained. Denote (c_n, ψ_n) a solution of (W_{s^n}). Throughout all this section we break the translation invariance by making the normalisation

$$\max_{x \le 0} \psi_n = \theta \tag{1.10}$$

We also drop a finite number of terms of the sequence (s^n) so that for all $n \ge 0$, $s^n > \frac{s_\infty}{2} > 0$, which will be needed to ensure the uniform ellipticity of the boundary operator in (W_s) so that we can use the elliptic estimates up to the Wentzell boundary.

By Section 1.2.2 we can extract from c_n some subsequence still denoted c_n that satisfies

$$\lim_{n \to +\infty} c_n = c^\infty \in [c_{min}, c_{max}] \tag{1.11}$$

We now derive global Schauder estimates for (W_s) from the standard local ones of [63]. We describe the argument exhaustively for once because we will refer to it later for the more complicated case of (S_ε). We chose deliberately to use that $|\psi_s| \le 1$ only at the end to give the inequality in its full generality, since the proof will serve later purposes.

Proposition 1.3.1. *There exists $\alpha > 0$ and a constant $C_{Sch} = C(D, d, c_{max}, Lipf, L, \mu)$ such that for all $n \ge 0$*

$$|\psi_n|_{\mathcal{C}^{2,\alpha}(\Omega_L)} \le C_{Sch} \left(|\psi_n|_{L^\infty(\Omega_L)} \right) \le C_{Sch} \tag{1.12}$$

Proof. We only prove a local estimate near $y = 0$, the rest of the strip being treated similarly but with classical interior Schauder estimates or up to the Neumann boundary (see [46], Theorem 6.29).

Schauder estimates up to the Wentzell boundary are already proved in [63], but of course they need a bound on the $\mathcal{C}^\alpha$ norm of the data $f(\psi)$ (or on the bounded coefficient $-f(\psi)/\psi$ after rewriting the equation). Usually, this not a problem and, for example, can be derived from $W^{2,p}$ estimates up to the boundary.

Nonetheless, no such L^p estimates appear to be in the literature concerning Wentzell boundary conditions. We overcome this technical difficulty by using directly a $\mathcal{C}^\alpha$ (for some small $\alpha > 0$) estimate up to the boundary (see [61], Theorem 2) which relies on a weak Harnack inequality up to the boundary (see [62]) : in other words, the Krylov-Safonov inequality of [57] is valid up to a Wentzell boundary.

Call $B_- \subset \Omega_L$ (resp. $2B_-$) some half-ball of centre $(x, 0)$ and radius $\varepsilon > 0$ (resp. 2ε) small. By the references above there exists $\alpha > 0$ and some C_α depending only on $c_{max}, d, D, \mu, \varepsilon$ such that

$$|\psi_n|_{\mathcal{C}^\alpha(B_-)} \le C_\alpha \mathrm{Lip} f |\psi_n|_{L^\infty(2B_-)}$$

since f is Lipschitz and $f(0) = 0$. This yields

$$|f(\psi_n)|_{\mathcal{C}^\alpha(B_-)} \le C_\alpha \left(\mathrm{Lip} f\right)^2 |\psi_n|_{L^\infty(2B_-)}$$

and then by plugging this in the Schauder estimates up to the Wentzell boundary ([63], Theorem 1.5) :

$$|\psi_n|_{\mathcal{C}^{2,\alpha}(B_-)} \le C_W \left(|\psi_n|_{L^\infty(2B_-)}\right)$$

for some $C_W = C(d, D, c_{max}, \mu, \mathrm{Lip} f)$.

To obtain the global estimate, just use the global L^∞ bound and observe that the above estimate does not depend on the position of B_-.

□

Remark 1.3.1. Of course we can now iterate the Schauder estimate for any $\mathcal{C}^{k,\alpha}$ provided enough regularity on f. Namely, if f has k Lipschitz derivatives, ψ_n is uniformly in $\mathcal{C}^{k+2,\alpha}$ for every $0 < \alpha < 1$.

Using (1.12) with Ascoli's theorem and the process of diagonal extraction for every $[-N, N] \times [-L, 0]$, we get a subsequence still denoted ψ_n that converges in $\mathcal{C}^2_{loc}(\overline{\Omega_L})$ to a function $\psi^\infty \in \mathcal{C}^2(\overline{\Omega_L})$. Remembering (1.11) we can pass to the limit in (W_{s^n}) to get that (c^∞, ψ^∞) solves (W_{s^∞}) apart from the limiting conditions. This is the aim of the following lemmas.

Proposition 1.3.2. $\psi^\infty(x, \cdot)$ *converges uniformly to* 0 *as* $x \to -\infty$.

Proof. This relies on a comparison with the exponential supersolution already computed in Proposition 1.2.5. Observe that thanks to (1.10), any solution of (W_s)+(1.10) satisfies $f(\psi_s) \equiv 0$ on $x \le 0$. As a consequence

$$\overline{p}_s := \theta e^{r_s x}$$

where

$$r_s = \frac{c_s}{\max(d, D)} \ge \frac{c_{min}}{\max(d, D)} =: r$$

is a supersolution of (W_s) on $x \le 0$.

Since $\overline{p}_s - \psi_s$ is non-negative on $x = 0$, goes uniformly to 0 as $x \to -\infty$, satisfies a Neumann boundary condition on $y = -L$, a Wentzell boundary condition on $y = 0$ and $-d\Delta u + c\partial_x u \ge 0$ inside $x < 0$, the strong maximum principle and Hopf's lemma give for all $x \le 0$:

$$\psi_s(x, y) \le \theta e^{r_s x} \le \theta e^{r x} \tag{1.13}$$

The result is obtained by taking $s = s^n$ and making $n \to +\infty$ in the above inequality. □

The right limit condition is obtained by simple computations already done in [17] in the Neumann case, we adapt them here.

Proposition 1.3.3. $\psi^\infty(x, \cdot)$ *converges uniformly to* 1 *as* $x \to +\infty$.

Since bounds and monotonicity pass to the $\mathcal{C}^2$ limit, we have $0 \leq \psi^\infty \leq 1$, as well as $\psi^\infty_x \geq 0$. As a consequence there exists $\beta(y) \leq 1$ such that $\psi^\infty(x,y) \to \beta(y)$ as $x \to +\infty$. Let us define the functions $\psi^\infty_j(x,y) = \psi^\infty(x+j,y)$ in $[0,1]\times[-L,0]$ for every integer j. Elliptic estimates and Ascoli's theorem tell us that up to extraction, $\psi^\infty_j \to \delta$ in the $\mathcal{C}^1$ sense for a $\mathcal{C}^1$ function δ. By uniqueness of the simple limit, $\beta = \delta \in \mathcal{C}^1$. So ψ^∞_j lies in a compact set of $\mathcal{C}^1([0,1]\times[-L,0])$ and has a unique limit point $\beta \in \mathcal{C}^1$: then it converges to it in the $\mathcal{C}^1$ topology.

Lemma 1.3.1. *$\int_{\Omega_L} f(\psi^\infty) < +\infty$ and $\int_{\Omega_L} |\nabla\psi^\infty|^2 < +\infty$*

Proof. For the first integral we integrate (W_{s^∞}) on $Q_M := [0,M]\times[-L,0]$ using integration by parts. We obtain

$$\begin{aligned}\int_{Q_M} f(\psi^\infty) = &\int_{-L}^0 -d\partial_x\psi^\infty(M,y)dy + \int_{-L}^0 d\partial_x\psi^\infty(0,y)dy \\ &+ \frac{s^\infty}{\mu}\left(c^\infty(\psi^\infty(M,0)-\psi^\infty(0,0)) - D(\partial_x\psi^\infty(M,0)-\partial_x\psi^\infty(0,0))\right) \\ &+ \int_{-L}^0 c^\infty\psi^\infty(M,y)dy - \int_{-L}^0 c^\infty\psi^\infty(0,y)dy\end{aligned}$$

which can be written as

$$\int_{Q_M} f(\psi^\infty) = A(M) - A(0)$$

with

$$A(m) = c^\infty\int_{-L}^0 \psi^\infty(m,y)dy + \frac{s^\infty}{\mu}\left(c^\infty\psi^\infty(m,0) - D\partial_x\psi^\infty(m,0)\right) - d\int_{-L}^0 \partial_x\psi^\infty(m,y)dy$$

Since the first two terms in $A(m)$ are bounded (thanks to $\psi^\infty \leq 1$ and elliptic estimates) and the last one is non-positive, the boundedness of the positive integral $\int_{Q_M} f(\psi^\infty)$ as $M \to +\infty$ follows.

For the second integral, we proceed in the same manner, but integrating the equation multiplied by ψ^∞ and integrating by parts on $Q_M = [-M,M]\times[-L,0]$ we get

$$d\int_{Q_M}|\nabla\psi^\infty|^2 = B(M) - B(-M) + \int_{Q_M} f(\psi^\infty)\psi^\infty - \frac{s_\infty D}{\mu}\int_{-M}^M(\partial_x\psi^\infty)^2(x,0)dx \tag{1.14}$$

with

$$\begin{aligned}B(m) = &-\frac{c^\infty}{2}\int_{-L}^0\psi^\infty(m,y)^2dy - \frac{c^\infty s^\infty}{2\mu}\psi^\infty(m,0)^2 + \frac{s_\infty D}{\mu}(\partial_x\psi^\infty\psi^\infty)(m,0) \\ &+ d\int_{-L}^0(\psi^\infty\partial_x\psi^\infty)(m,y)dy\end{aligned}$$

The third term in (1.14) is bounded thanks to $0 \leq \psi^\infty \leq 1$ and what we just saw. The last one is non-positive. The first two terms in $B(m)$ are bounded, the third one also by elliptic estimates, so $\int_{Q_M} |\nabla \psi^\infty|^2 \to +\infty$ would mean

$$d\int_{-L}^{0} (\psi^\infty \partial_x \psi^\infty)(m,y)dy \underset{m\to+\infty}{\longrightarrow} +\infty$$

which is impossible since it is the derivative of the bounded function

$$m \mapsto d\int_{-L}^{0} \frac{1}{2}\psi^\infty(m,y)^2 dy.$$

□

End of the proof of Prop. 1.3.3. We now turn back to the study of the right limit. The second integral in Lemma 1.3.1 being finite, necessarily $\nabla\beta = 0$ [1]. So β is a constant. Moreover,

$$0 \leq \theta \leq \max_{[-L,0]} \psi^\infty(0,y) \leq \beta \leq 1$$

We also have $f(\beta) = 0$ because of the finiteness of the first integral[2] so $\beta = \theta$ or $\beta = 1$. Suppose by contradiction that $\beta = \theta$. Then $f(\psi^\infty) \equiv 0$, and integrating the equation satisfied by ψ^∞ on $[-m,m] \times [-L,0]$ just as above and making $m \to +\infty$ yields

$$0 = A(\infty) - A(-\infty) = c^\infty \left(L + \frac{s^\infty}{\mu}\right)\theta$$

since $\partial_x \psi^\infty(m,y) \to 0$ uniformly in y as $x \to \pm\infty$. This is of course, impossible, since $c^\infty > c_{min} > 0$ and $\theta > 0$. □

As a conclusion, ψ^∞ satisfies all the desired properties, and we have proved that P_W is closed.

1.3.2 P_W is open

This part is about applying the implicit function theorem to some function $F(s,c,\psi)$ in order to get a solution for $s > s^0$ close to a value s^0 of the parameter for which we already have a solution c^0, ψ^0. In this section we take $\mu = 1$ without loss of generality to clarify the diagrams. We also suppose $s^0 > 0$, the case $s^0 = 0$ is simpler and will be discussed at the end of the subsection. We set

$$\psi = \psi^0 + (s-s^0)\psi^1,\ c = c^0 + (s-s^0)c^1$$

where $s \in [s^0, s^0+\delta]$, $\delta > 0$ small to be fixed later. After a simple but tedious computation, we get that the corresponding equation for ψ^1, c^1 is :

[1] or else we have uniformly $|\nabla u_j|^2 \to g$, with $g > \delta > 0$ in $[y_0 - \varepsilon, y_0 + \varepsilon]$ and the second integral would be greater than $\int_{[R,\infty)\times[y_0-\varepsilon,y_0+\varepsilon]} \delta/2 = +\infty$ for R large enough.

[2] $\int_{Q_\infty} f(\psi^\infty) = \sum_{j=0}^\infty \int_{Q_1} f(\psi^\infty_{2j})$ so $\int_{Q_1} f(\psi^\infty_{2j}) \to 0$, but this $\to \int_{-L}^0 f(\beta(y))dy$ too.

$$\mathcal{W}\psi^1 = -(c^0 + c^1 s)\partial_x \psi^0 + D\partial^2_{xx}\psi^0 - (s - s^0)(c^0 + c^1 s)\partial_x \psi^1 + (s - s^0)D\partial^2_{xx}\psi^1$$

$$\mathcal{L}\psi^1 + c^1\partial_x\psi^0 = R(s - s^0, c^1, \psi^1)$$

$$\partial_y \psi^1 = 0$$

where

$$\mathcal{W} = d\partial_y + c^0 s^0 \partial_x - s^0 D \partial^2_{xx}$$

and

$$\mathcal{L} = -d\Delta + c^0\partial_x - f'(\psi^0)$$

and R being a function that goes to 0 as $s \to s^0$ and decays quadratically in the variables ψ^1, c^1 in a setting that will be defined later[3].

We will solve the order 1 problem, i.e. the one obtained by taking $s = s^0$ and then we will apply the implicit function theorem in a good functional setting to obtain the existence of a solution to the above problem for s close to s^0. The upper boundary condition should be seen as close to a fixed non-homogeneous Wentzell boundary condition. That is why we first need some information about the operator $\mathcal{L}$ with Wentzell condition.

$$\mathcal{W}g = 0$$

$$\mathcal{L}g = 0$$

$$-\partial_y g = 0$$

It is well known (see [77],[74]) that this operator is not Fredholm in the usual spaces of bounded uniformly continuous functions due to the degeneracy of f in the range $[0, \theta]$. The way to circumvent this difficulty is to endow the space with a weight that sees the exponential decay of the solutions as $x \to -\infty$.

Definition 2. Let

$$r = \frac{c_{min}}{4\max(d, D)} \tag{1.15}$$

so that $-dr^2 + c^0 r \geq 0$ and $-Dr^2 + c^0 r \geq 0$ (the 4 will serve later purposes, see Lemma 1.3.4). Define w to be a $\mathcal{C}^2$ function such that

$$w(x) = \begin{cases} e^{rx} \text{ for } x < 0 \\ \text{constant for } x > 1 \end{cases} \tag{1.16}$$

and such that

$$w''(x) \leq \frac{c^0}{D} w' \tag{1.17}$$

for $x \in (0, 1)$. Define also $w_1 = 1/w$.

[3] $R(s - s^0, c^1, \psi^1) = -(s - s^0)c^1\partial_x\psi^1 + (s - s^0)\frac{f''(\psi^0)}{2}(\psi^1)^2 + (s - s^0)^2\frac{f'''(\psi^0)}{6}(\psi^1)^3 + \cdots = (s - s^0)\mathcal{O}(c^1, \psi^1)$, the $\mathcal{O}$ being in $\mathbb{R} \times \mathcal{C}^{1,\alpha}$ norm.

Proof of the exitence of w. Satisfying the differential inequation (1.17) will be necessary because of the Wentzell condition. Take for instance

$$w(x) = \int_0^x r e^{r(s-s^2)} \mathrm{d}s + 1$$

for $x \in (0, 1/2)$ so that

$$\begin{aligned} w(0) &= 1 \\ w'(0) &= r \\ w''(0) &= r^2 \end{aligned}$$

fits with the definition of w as $x < 0$. Since $r < c^0/D$, (1.17) is satisfied. Moreover

$$\begin{aligned} w(1/2) &= a > 0 \\ w'(1/2) &= a' > 0 \\ w''(1/2) &= 0 \end{aligned}$$

so now we can just continue w in a concave increasing way (so that (1.17) is also satisfied) :

$$\begin{aligned} w''(x) &= \frac{a'}{48}\left(x - \frac{1}{2}\right)(x-1) \\ w'(x) &= \int_{1/2}^x w''(s)\mathrm{d}s + a' \\ w(x) &= \int_{1/2}^x w'(s)\mathrm{d}s + a \end{aligned}$$

and choose the constant in Def. 2 to be $\int_{1/2}^1 w'(s)\mathrm{d}s + a$. The author wishes to thank an anonymous referee for detecting a problem with the initial definition of w. □

Definition 3. Let

$$\mathcal{C}^\alpha_w(\Omega_L) = \{u \in \mathcal{C}^\alpha(\Omega_L) \mid w_1 u \in \mathcal{C}^\alpha(\Omega_L)\}$$

and

$$X = \mathcal{C}^{2,\alpha}_w(\Omega_L)$$

the set of $\mathcal{C}^2$ functions on Ω_L whose derivatives up to order 2 are in $\mathcal{C}^\alpha_w(\Omega_L)$. We endow X with the norm

$$|u|_X = |w_1 u|_{\mathcal{C}^{2,\alpha}}$$

X is clearly a Banach space, which contains ψ^0. Indeed, at the left of Ω_L, $w_1\psi^0$ satisfies a linear homogeneous Wentzell problem. Thus, the $\mathcal{C}^{2,\alpha}_w$ estimate directly comes from the Schauder estimates of [63] by the L^∞ estimate for $w_1\psi^0$, which was already proved to get the left-limit condition, in Proposition 1.3.2. At the right of Ω_L, w_1 is a bounded smooth function and being $\mathcal{C}^{2,\alpha}_w$ here is equivalent to being $\mathcal{C}^{2,\alpha}$. Now we have :

Lemma 1.3.2. *$\mathcal{L}$ has closed range and there exists $X_1 \simeq R(\mathcal{L})$ a closed subspace of X and $Y_2 \simeq N(\mathcal{L})$ such that*

$$X = N(\mathcal{L}) \oplus X_1$$

$$Y = R(\mathcal{L}) \oplus Y_2$$

Moreover $N(\mathcal{L}) = N(\mathcal{L}^2) = \mathbb{R}\partial_x \psi_0$. Finally, denote $\mathcal{L}^*$ the adjoint of $\mathcal{L}$. Then $N(\mathcal{L}^*)$ is one dimensional too. Calling e^* the unique generator that satisfies

$$< e^*, \partial_x \psi^0 >= 1$$

we get that e^* is a positive measure that happens to be a smooth positive function, solving

$$\mathcal{L}^* e^* = \left(-d\Delta - c^0 \partial_x - f'(\psi^0)\right) e^* = 0$$

endowed with the dual boundary conditions

$$d\partial_y e^* - c^0 s^0 \partial_x e^* - s^0 D \partial^2_{xx} e^* = 0 \text{ on } y = 0$$
$$\partial_y e^* = 0 \text{ on } y = -L$$

Moreover e^* is bounded on $x > 0$ and has at most Ce^{-rx} growth as $x \to -\infty$.

Proof. The proof will be postponed to the last paragraph of this section. It all relies on the fact that $\mathcal{L}$ is a Fredholm operator of index 0 on the weighted space X. □

Now we want to transform the problem into a fixed Wentzell homogeneous problem. We do this by creating an auxiliary function $\tilde{\psi}(s, c^1, v)$ such that we search for ψ^1 as

$$\psi^1 = \tilde{\psi}(s, c^1, v) + v$$

where $\tilde{\psi}(s, c^1, v)$ solves for $A > 0$ large enough

---

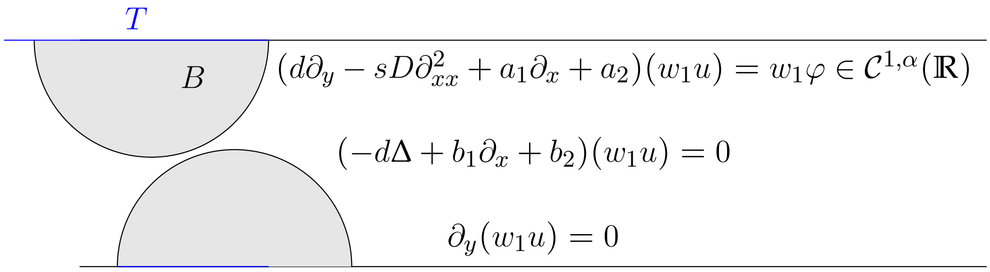
$$\mathcal{W}u = D\partial^2_{xx}\psi^0 - (c^0 + c^1 s)\partial_x \psi^0 - (c^0 + c^1 s)(s - s^0)\partial_x(u + v) + D(s - s^0)\partial^2_{xx}(u + v)$$

$$\mathcal{L}u + Au = 0$$

$$\partial_y u = 0$$

Lemma 1.3.3. *Such a function exists and satisfies $\tilde{\psi} \in \mathcal{C}^1([0,1] \times \mathbb{R} \times \mathcal{C}^{2,\alpha}_w(\overline{\Omega_L}); \mathcal{C}^{2,\alpha}_w(\overline{\Omega_L}))$.*

Proof. For $A > |f'(\psi^0)|_\infty$ it is known that the above problem has a unique solution that lies in $\mathcal{C}^{2,\alpha}(\overline{\Omega_L})$ provided $v \in \mathcal{C}^{2,\alpha}(\overline{\Omega_L})$, since this gives that the data for the Wentzell condition lies in $\mathcal{C}^\alpha(\mathbb{R})$ (see theorem 1.6 in [63] along with the remark at its end). What is important to show is that if v lies in the weighted space, the solution u is in it too. On $x \geq 0$, $w_1 u$ is trivially $\mathcal{C}^{2,\alpha}$ as the product of a smooth bounded function and a $\mathcal{C}^{2,\alpha}$ function. The only problem might come from unboundedness at $x \to -\infty$. In other words, we need to show that u decays like Ce^{rx} as $x \to -\infty$. We see that $w_1 u$ satisfies an elliptic problem too, so conversely by Schauder estimates the problem is reduced to showing this L^∞ bound for $w_1 u$ on $x < 0$. More precisely : we see that $w_1 u$ solves :



where on $x < 0$

$$
\begin{aligned}
a_1 &= c^0 s + c^1 s(s - s^0) - 2sDr \\
a_2 &= s\left(-3Dr^2 + c^0 r + c^1 r(s - s^0)\right) \geq 0 \\
b_1 &= c^0 - 2dr \\
b_2 &= A - f'(\psi^0) + c^0 r - dr^2 \geq 0
\end{aligned}
$$

and where

$$\varphi(c^1, s, v) = -(c^0 + c^1 s)\partial_x\psi^0 + D\partial^2_{xx}\psi^0 - (c^0 + c^1 s)(s - s^0)\partial_x v + D(s - s^0)\partial^2_{xx} v$$

Using Schauder estimates up to the boundary for the Wentzell problem, we see that provided a global L^∞ estimate for $w_1 u$, $w_1 u$ is in $\mathcal{C}^{2,\alpha}(B \cup T)$ with constant independent of the position of the closed half balls $B \cup T$ depicted on the diagram above. Since we can cover all $\overline{\Omega_L}$ with translations of $B \cup T$, this gives $w_1 u \in \mathcal{C}^{2,\alpha}(\overline{\Omega_L})$. This weighted L^∞ global estimate is the object of the next lemma. It simply relies on the maximum principle. $\square$

Lemma 1.3.4. *Let $u = \tilde{\psi}(s, c^1, v)$. There exists two constants $K' < 0, K > 0$ such that*

$$K' \leq w_1 u \leq K$$

Proof. We already have $u \leq Kw$ for $K > \max(0, \sup u)$ on Ω_L^+. We now want to show that this is also (eventually with a larger constant) true in Ω_L^- by using the maximum principle. Suppose there exists a point where $u > Kw$. That means that $Kw - u$ reaches a negative minimum somewhere in $\overline{\Omega_L^-}$ or tends to a negative infimum as $x \to -\infty$. First, let us see that this minimum cannot be reached. $Kw - u$ satisfies :

$$(d\partial_y - sD\partial^2_{xx} + (c^0 s + c^1 s(s - s^0)\partial_x)(Kw - u) = rs(-Dr + c^0 + c^1(s - s^0))Kw - \varphi$$

$$(\mathcal{L} + A)(Kw - u) = (-dr^2 + c^0 r + A - f'(\psi^0))Kw > 0$$

$$\partial_y(Kw - u) = 0$$

In order to conclude to a contradiction thanks to Hopf's lemma, we only need to ensure

$$rs(-Dr + c^0 + c^1(s - s^0))Kw - \varphi > 0$$

so that

$$(-Dr + c^0 + c^1(s - s^0))K > \frac{\text{sup} w_1 \varphi}{rs^0}$$

suffices. Now observe that thanks to (1.15) and since $s \in [s^0, s^0 + \delta]$ then provided

$$c^1 > \frac{Dr - c_{min}}{2\delta} \tag{1.18}$$

we have that $K > \max\left(0, \dfrac{2\text{sup}(w_1\varphi)}{rs^0(c_{min} - Dr)}\right)$ suffices and in the end we have the desired result with

$$K = \max\left(\max\left(0, \sup(u)\right), \max\left(0, \frac{2\text{sup}(w_1\varphi)}{rs^0(c_{min} - Dr)}\right)\right)$$

From now on, we assume condition (1.18) and we will see that this is not restrictive.

Now if the minimum is obtained at infinity, let us denote (x_n, y_n) a minimizing sequence. Since y_n is bounded we can extract a subsequence that converges to $y_\infty \in [-L, 0]$. Let us set

$$(Kw - u)^n(x, y) \ := (Kw - u)(x + x_n, y + y_\infty) \tag{1.19}$$

We have two subcases :

i) $y_\infty \in]-L, 0[$. Then (1.19) defines a sequence of uniformly bounded functions in some small ball B in the interior of Ω_L. By standard elliptic estimates, we can extract from it a subsequence that converges in $\mathcal{C}^2(B)$ to some $(Kw - u)^\infty$ that satisfies $(-d\Delta + c^0\partial_x + A)(Kw - u)^\infty \geq 0$ in B but reaches its negative infimum $m < 0$ inside B : as a consequence, $(Kw - u)^\infty \equiv m$ in B, but this is impossible since $Am < 0$.

ii) $y_\infty = 0$ or $-L$: the exact same analysis applies, replacing the ball B by a half-ball $B_\pm$ supported on $y = 0$ or $y = -L$ and using elliptic estimates up to the boundary, and Hopf's lemma.

For the other bound, we proceed in the same way by looking at $u - K'w$ with $K' < \min(0, \inf u)$ and using the existence of $\inf(w\varphi)$, we get

$$K' = \min\left(\min\left(0, \inf u\right), \min\left(0, \frac{2\inf(w_1\varphi)}{rs^0(c_{min} - Dr)}\right)\right)$$

that works. $\square$

Thanks to this auxiliary function, we are now left with the following equivalent problem, on v :

$$\mathcal{W}v = 0$$

$$\mathcal{L}v + c^1\partial_x\psi^0 = R(s - s^0, c^1, v) - \mathcal{L}\tilde{\psi}(s, c^1, v)$$

$$\partial_y v = 0$$

Calling $\mathcal{P} =< e^*, \cdot > \partial_x\psi^0$ and $\mathcal{Q} = Id - \mathcal{P}$ the projections onto Y_2 and $R(\mathcal{L})$ we are now able to apply these projections onto the equation to get a set of two equations that are equivalent to this one. Nonetheless, since $\tilde{\psi}$ on the boundary $y = 0$ depends on c^1 even when $s = s^0$, we should be careful and try to make this dependence explicit. For this, we need to have an explicit representation of e^* to be able to compute the projections. This technical difficulty only comes from the fact that the unknown c appears in the boundary condition of (W_s).

Thanks to the smoothness and decay properties of e^*, v and $\partial_x\psi_0$, all the integration by parts make sense and we find

$$\int_{\Omega_L} e^*\mathcal{L}v = \int_{y=0}(vd\partial_y e^* - e^* d\partial_y v) = 0$$

$$\begin{aligned}\int_{\Omega_L} e^*\mathcal{L}\tilde{\psi} = &\int_{y=0} e^*\left(\left(c^0 + c^1 s\right)\partial_x\psi^0 - D\partial^2_{xx}\psi^0\right) \\ &+ (s - s^0)\int_{y=0} e^*\left(\left(c^0 + c^1 s\right)\partial_x(\tilde{\psi} + v) - D\partial^2_{xx}(\tilde{\psi} + v)\right)\end{aligned}$$

and we get the first equation [4]:

$$\begin{aligned}c^1\left(1 + s\int_{y=0} e^*\partial_x\psi^0\right) = &-\int_{y=0} e^*\left(c^0\partial_x\psi^0 - D\partial^2_{xx}\psi^0\right) \\ &+ \int_{\Omega_L} e^* R \\ &- (s - s^0)\int_{y=0} e^*\left(\left(c^0 + c^1 s\right)\partial_x(\tilde{\psi} + v) - D\partial^2_{xx}(\tilde{\psi} + v)\right)\end{aligned} \tag{1.20}$$

The second equation should be seen as an equation on $v_R \in X_1$ with the decomposition

$$v = v_N\partial_x\psi^0 + v_R$$

and $v_N \in \mathbb{R}$ being free : this is, of course, due to the x-translation invariance of (W_s). From now on, we fix $v_N \in \mathbb{R}$.

[4] where $\tilde{\psi}$ means $\tilde{\psi}(s, c^1, v)$ and R means $R(s - s^0, c^1, v + \tilde{\psi}(s, c^1, v))$

$$
\begin{aligned}
\mathcal{L}v_R = R - &\left(\int_{\Omega_L} e^* R\right) \partial_x \psi^0 - \mathcal{L}\tilde{\psi} \\
&+ (c^0 + c^1 s) \left(\int_{y=0} e^* \partial_x \psi^0 + (s - s^0) \int_{y=0} e^* \partial_x (\tilde{\psi} + v)\right) \partial_x \psi^0 \\
&- (s - s^0) D \int_{y=0} e^* \partial^2_{xx} (\tilde{\psi} + v) \\
&- \left(\int_{y=0} e^* D \partial^2_{xx} \psi^0\right) \partial_x \psi^0
\end{aligned}
\tag{1.21}
$$

The system of equations (1.20), (1.21) is non-linear and coupled but in the case $s = s^0$ it is much simpler. It becomes

$$
c^1 \left(1 + s^0 \int_{y=0} e^* \partial_x \psi^0\right) = - \int_{y=0} e^* \left(c^0 \partial_x \psi^0 - D \partial^2_{xx} \psi^0\right) \tag{1.22}
$$

$$
\mathcal{L}v_R = -\mathcal{L}\tilde{\psi} + (c^0 + c^1 s^0) \left(\int_{y=0} e^* \partial_x \psi^0\right) \partial_x \psi^0 - \left(\int_{y=0} e^* D \partial^2_{xx} \psi^0\right) \partial_x \psi^0 \tag{1.23}
$$

which has clearly a unique solution : since $\int_{-\infty}^{+\infty} (e^* \partial_x \psi^0)(x, 0) dx > 0$, (1.22) has a solution c^1_* that satisfies condition (1.18) provided we take δ small enough. (1.23) is automatically uniquely solvable with a solution v^*_R since its right hand side lies in $R(\mathcal{L})$ and does not depend on v.

Now for $s > s^0$ we said that this system was non-linear and coupled, but this is when the implicit function theorem does all the work. Since X_1 is closed in X and $\mathcal{L}$ is Fredholm so image-closed, we have the right Banach setting to apply it. We may see this system of equations as $F(s, c^1, v_N, v_R) = 0$ with

$$
F \ : \left[s^0, s^0 + \delta\right] \times \left[\frac{Dr - c_{min}}{2\delta}, +\infty\right[\times X_1 \to \mathbb{R} \times R(\mathcal{L})
$$

associating to its parameters the equations (1.20), (1.21) in this order. F is a $\mathcal{C}^1$ function because it consists in affine bounded operators composed with usual and $\mathcal{C}^1$ functions. Moreover, we can compute the differential of F at (s^0, c^1_*, v^*_R) with respect to (c^1, v_R). In matrix representation, it is

$$
\begin{pmatrix} 1 + s^0 \int_{x=0} e^* \partial_x \psi^0 & 0 \\ * & \mathcal{L} \end{pmatrix}
$$

which is invertible since $1 + s^0 \int_{x=0} e^* \partial_x \psi^0 > 0$, and $\mathcal{L}$ is invertible on X_1. That being, the implicit function theorem says that there exists $\delta' > 0$ and a neighbourhood $\mathcal{V}$ of (c^1_*, v^*_R) such that for each $s \in [s^0, s^0 + \delta'[$, the system of equations has a unique solution $(c^1_s, v^s_R) \in \mathcal{V}$. Then we can construct back ψ from c^1_s, v^s_R, v_N and it will clearly satisfy the original equation. The left limit condition for it is obtained directly because of the structure of X. The only thing left to show is that the right limit condition holds. This is the case provided δ is taken small enough, and it is the object of the next proposition.

Remark 1.3.2. Note that this is valid for every $v_N \in \mathbb{R}$, which will provide us with a whole 1-dimensional manifold of solutions in the end. Of course, thanks to Proposition 1.2.4 all of these solutions will be x-translates of each other.

Proposition 1.3.4. *Let*

$$c = c^0 + (s - s^0)c^1, \ \psi = \psi^0 + (s - s^0)\psi^1$$

If $\delta > 0$ is small enough, we have uniformly in y :

$$\lim_{x \to +\infty} \psi(x, y) = 1$$

Proof. First, we show that $\psi < 1$, by contradiction. We know that $\psi \in \mathcal{C}^{2,\alpha}_w$ is bounded. Suppose there exists a point where $\psi > 1$. Then either $\psi - 1$ reaches a positive maximum somewhere, or it tends to a positive maximum as $x \to \infty$. These two cases are both impossible, because of respectively the argument given in theorem 1.2.1 and the compactness argument given in the proof of Lemma 1.3.4 (take B or $B_\pm$ small enough so that $f(\psi) < 0$ on it). So $\psi \leq 1$ and the strong maximum principle and Hopf's lemma and the fact that ψ cannot be constant give

$$\psi < 1$$

Now we fix $\varepsilon > 0$. For a large enough we have $\psi^0 > 1 - \frac{\varepsilon}{2}$ on $x \geq a$. Moreover, we can take δ small enough such that $|(s - s^0)\psi^1|_\infty < \frac{\varepsilon}{2}$, what gives $1 - \varepsilon < \psi < 1$ for $x \geq a$. We assert that this property suffices to have $\psi \to 1$ for δ small enough, and we will show that by a maximum principle argument using an exponential solution to the right for the linearised problem near 1.

On $x \geq a$, by Taylor's formula applied on f, we have

$$-d\Delta(1 - \psi) + c\partial_x(1 - \psi) = f'(1)(1 - \psi) + o(\varepsilon)$$

So by choosing $\varepsilon > 0$ small enough, we have

$$L_1(1 - \psi) \ := -d\Delta(1 - \psi) + c\partial_x(1 - \psi) - \frac{1}{2}f'(1)(1 - \psi) \leq 0$$

We now look for a positive solution p of $L_1 p = 0$ endowed with the boundary condition of (W_s) that has exponential decay as $x \to +\infty$, for comparison purposes. Unlike the proof of Proposition 2.2.2, we cannot expect a supersolution with the form $p(x, y) = e^{-\gamma x}$ with $\gamma > 0$, since inequations

$$\begin{aligned} -d\gamma^2 - c\gamma - 1/2f'(1) \geq 0 \\ s(-D\gamma^2 - c\gamma) \geq 0 \end{aligned}$$

cannot be solved simultaneously. This motivates the research for a $p(x, y) = e^{-\gamma x}\phi(y)$, $\phi > 0$. For p to be a solution of $L_1 p = 0$ endowed with the boundary condition of (W_s), the equations are

$$\begin{cases} -\phi'' + \left(-\frac{1}{2d} f'(1) - \gamma \left(\gamma + \frac{c}{d}\right)\right) \phi = 0 \\ d\phi'(0) - s(-D\gamma^2 - c\gamma)\phi(0) = 0 \\ \phi'(-L) = 0 \end{cases} \tag{1.24}$$

Since $f'(1) < 0$, this can be solved by

$$\phi(y) = \cosh\left(\beta(\gamma)\,(y + L)\right)$$

where

$$\beta(\gamma) = \sqrt{-f'(1)/(2d) - \gamma(\gamma + c/d)}$$

and $0 < \gamma < \gamma_{lim} = \frac{\sqrt{c^2 - 2df'(1)} - c}{2d}$ solving

$$s(D\gamma^2 + c\gamma) = d\beta(\gamma)\tanh(\beta(\gamma)L)$$

as pictured in Figure (1.3.2).

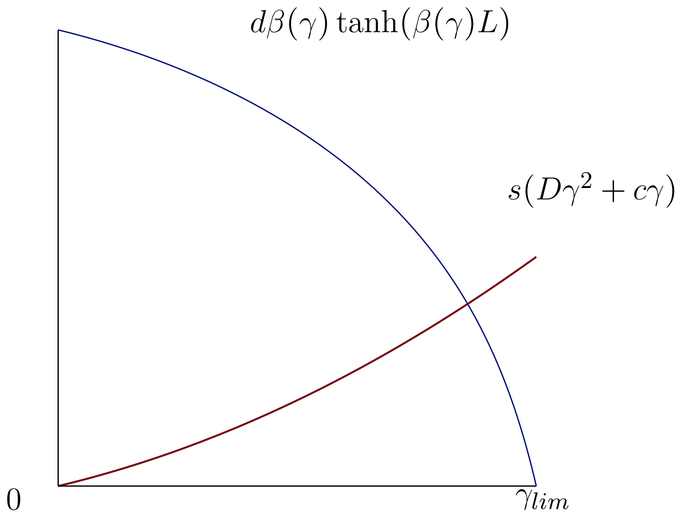


Figure 1.2: Equation (1.24) on γ

Now chose $C > 0$ such that $1 - \psi < Cp$ on $x = a$ and observe that $U = Cp - (1 - \psi)$ solves on $x \geq a$

$$U > 0 \quad \boxed{\begin{array}{c} d\partial_y U + s\left(-D\partial^2_{xx} U + c\partial_x U\right) = 0 \\ \\ L_1 U \geq 0 \\ \\ -d\partial_y U = 0 \end{array}}$$

Now suppose that there is a point where $U < 0$. Then either U reaches a negative minimum or tends to a negative infimum $m < 0$ as $x \to +\infty$. The first case is impossible thanks to the strong maximum principle and Hopf's lemma. The second is impossible also thanks to the compactness argument already given in Proposition 1.3.4, since $L_1(m) = -1/2f'(1)m < 0$. As a consequence, for all $x \geq a$:

$$0 < 1 - \psi \leq Ce^{-\gamma x}\phi(y) \leq C\max(\phi)e^{-\gamma x}$$

which gives the desired result by sending $x \to +\infty$.

□

This section is now finished and Theorem 1.1.1 is proved.

Remark 1.3.3. Note that the above subsection does not apply exactly when $s^0 = 0$. Indeed, in this case the estimates up to the Wentzell boundary do not hold. Nonetheless, this situation is way simpler : just apply the standard estimates up to the Neumann boundary. We leave it to the reader to check that everything holds, all the other computations being simpler (for instance, no information on e^* is required).

1.3.3 Proof of lemma 1.3.2

Proof of the Fredholm property

$\mathcal{L}$ is Fredholm of index 0 as an operator $\mathcal{C}^{2,\alpha}_w \to \mathcal{C}^{\alpha}_w$ if and only if

$$\tilde{\mathcal{L}}u := \frac{1}{w}\mathcal{L}(wu)$$

defines a Fredholm operator of index 0 as an operator $\mathcal{C}^{2,\alpha} \to \mathcal{C}^{\alpha}$ endowed with the boundary condition $\partial_y u = 0$ on $y = -L$ and $d\partial_y u + \frac{1}{w}c^0 s^0 \partial_x(wu) - \frac{1}{w}s^0 D\partial^2_{xx}(wu) = 0$ on $y = 0$.

We do not have any closed formula for the coefficients of $\tilde{\mathcal{L}}$, but we know that

$$\tilde{\mathcal{L}}u = -d\Delta u + (c^0 - 2dr)\partial_x u + (c^0 r - dr^2 - f'(\psi^0))u \text{ on } x < 0, \tilde{\mathcal{L}} = \mathcal{L} \text{ on } x > 1$$

Moreover the 0-order term of $\tilde{\mathcal{L}}$ is $c^0 r - dr^2 > 0$ on $x < 0$, and tends to $-f'(1) > 0$ uniformly in y as $x \to \infty$; thus it is greater than some positive constant, away from a compact set : this indicates a decomposition invertible + compact for $\mathcal{L}$.

The boundary condition $\tilde{\mathcal{L}}$ is endowed with is unchanged on $y = -L$ and is

$$\tilde{\mathcal{W}}u := d\partial_y u + \left(c^0 s^0 - 2\frac{w'}{w}s^0 D\right)\partial_x u - s^0 D\partial^2_{xx}u + \frac{s^0}{w}\left(c^0 w' - Dw''\right)u = 0$$

on $y = 0$, and we know that the zero order term satisfies

$$c^0 w' - Dw'' > 0$$

thanks to the definition of r and the properties of w asked in Definition 2.

Now call $\gamma(x)$ a positive function that smoothly connects $c^0 r - dr^2 > 0$ on $x < 0$ with $-f'(1)$ on $x > 1$ such that $\gamma \geq \min(c^0 r - dr^2 > 0, -f(1)) := \gamma_0 > 0$.

We now call $\tilde{\mathcal{T}}$ the operator $\tilde{\mathcal{L}}$ with its 0-order term replaced by $\gamma(x)$, and we want to show that $\tilde{\mathcal{T}}$ is invertible, and that $\tilde{\mathcal{S}} := \tilde{\mathcal{L}} - \tilde{\mathcal{T}}$ satisfies $\tilde{\mathcal{S}}\tilde{\mathcal{T}}^{-1}$ is compact on $\mathcal{C}^{\alpha}$ and $\tilde{\mathcal{T}}^{-1}\tilde{\mathcal{S}}$ on $\mathcal{C}^{2,\alpha}$, in order to have

$$\tilde{\mathcal{L}} = (Id + \tilde{\mathcal{S}}\tilde{\mathcal{T}}^{-1})\tilde{\mathcal{T}} = \tilde{\mathcal{T}}(Id + \tilde{\mathcal{T}}^{-1}\tilde{\mathcal{S}})$$

which is the Fredholm property with index 0 we want.

First suppose that $\tilde{\mathcal{T}}$ is indeed invertible : then the compactness of the perturbation is easy to obtain. Indeed, $\tilde{\mathcal{S}}$ is no more than the multiplication by a function that is $\equiv 0$ on $x \leq 0$ and that tends uniformly in y to 0 as $x \to \infty$. So, taking (u_n) a bounded sequence in $\mathcal{C}^{\alpha}(\Omega_L)$, we have that $(\tilde{\mathcal{T}}^{-1}u_n)$ is bounded in $\mathcal{C}^{2,\alpha}$, so by applying a chain of Ascoli theorems and the process of diagonal extraction we can extract from $(\tilde{\mathcal{T}}^{-1}u_n)$ a sequence we note (v_n) that converges in $\mathcal{C}^2_{loc}$ to v. We now want to extract from $(\tilde{\mathcal{S}}v_n)$ a sequence that converges in $\mathcal{C}^{\alpha}$. But this is easy since $\tilde{\mathcal{S}}v_n = 0$ on $x < 0$ and $\tilde{\mathcal{S}}v_n \to 0$ uniformly in y as $x \to \infty$, so in fact the $\mathcal{C}^2_{loc}$ convergence of v_n suffices to have $\tilde{\mathcal{S}}v_n \to \tilde{\mathcal{S}}v$ in whole $\mathcal{C}^2$, so in $\mathcal{C}^{\alpha}$. For $\tilde{\mathcal{T}}^{-1}\tilde{\mathcal{S}}$ on $\mathcal{C}^{2,\alpha}$ we apply the same argument : we just have to see that $\tilde{\mathcal{T}}^{-1}(\tilde{\mathcal{S}}(u_n))$ is bounded in $\mathcal{C}^{4,\alpha}$ since u_n is bounded in $\mathcal{C}^{2,\alpha}$. Then we extract from it something that converges in $\mathcal{C}^3_{loc}$, but in fact, in whole $\mathcal{C}^3$ so in $\mathcal{C}^{2,\alpha}$.

It remains to show that $\tilde{\mathcal{T}} : \mathcal{C}^{2,\alpha} \to \mathcal{C}^{\alpha}$ is indeed invertible, that is, to show that the following problem is uniquely solvable

$$\tilde{\mathcal{W}}u = 0$$

$$\tilde{\mathcal{T}}u = f \in \mathcal{C}^{\alpha}(\Omega_L)$$

$$\partial_y u = 0$$

but this is the case, since the 0-order terms of $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{W}}$ are > 0 (see theorem 6.31 in [46] and more precisely the remark at the end of its proof).

Computation of the kernel

Suppose $\mathcal{L}u = 0$. We will show that

$$P := \{\Lambda \in \mathbb{R} \, |\forall \lambda < \Lambda, \; u > \lambda \partial_x \psi^0\}$$

has a supremum λ_0, and that $u = \lambda_0 \partial_x \psi^0$. First, we show that this set is non-void : for every truncated (compact) rectangle K, we can find $\lambda \in \mathbb{R}$ such that $u > \lambda \partial_x \psi^0$ on K. Now just chose K big enough such that outside K we have $f'(\psi^0) \leq 0$, so we have the strong maximum principle, and since $\mathcal{L}(u - \lambda \partial_x \psi^0) = 0$, the comparison $u - \lambda \partial_x \psi^0 > 0$ is inherited in all Ω_L [5]. Now P being non-void and trivially bounded

[5] a point where it is ≤ 0 at the left of K means that a non-positive minimum is reached at the left of K, which is impossible ; the right of K is treated in the same way but with the compactness argument given in Lemma 1.3.4 since we do not know a priori that $u - \lambda \partial_x \psi^0 \to 0$ as $x \to \infty$ even if it is the case.

from above, it has the supremum we announced. By continuity, $u - \lambda_0 \partial_x \psi^0 \geq 0$, and moreover we have $\mathcal{L}(u - \lambda_0 \partial_x \psi^0) = 0$. Now suppose $u - \lambda_0 \partial_x \psi^0 \not\equiv 0$ by contradiction : because of the strong maximum principle, we have $u - \lambda_0 \partial_x \psi^0 > 0$, and again on any truncated rectangle K we can find $\varepsilon > 0$ small enough such that $u > (\lambda_0 + \varepsilon) \partial_x \psi^0$ on K, and choosing K large enough and proceeding as above, we have a contradiction regarding the maximality of λ_0.

Now, suppose $\mathcal{L}^2 u = 0$. Then $\mathcal{L} u = \alpha \partial_x \psi^0$ for some $\alpha \in \mathbb{R}$. We suppose $\alpha \neq 0$ and we will obtain a contradiction. By linearity we can suppose $\alpha = 1$, i.e. $\mathcal{L} u = \partial_x \psi^0 > 0$. Now, the fact that for every $\lambda \in \mathbb{R}$, $\mathcal{L}(u - \lambda \partial_x \psi^0) = \partial_x \psi^0$ is positive, enables to do the exact same proof as above to have a contradiction too (we will necessarily have $u > \lambda_0 \partial_x \psi^0$ and the contradiction, since $\mathcal{L} u \neq 0$).

Properties of e^*

Finally, let e^* generate the kernel of the adjoint of $\mathcal{L}$. Let us normalise e^* by the condition $< e^*, \partial_x \psi_0 >= 1$, and show that e^* is a positive measure. Similarly to [74][6] we infer that $\mathcal{L}$ is sectorial on $BW_0 := \{u \in UC_0(\Omega_L) \mid w_1 u \in UC_0(\Omega_L)\}$ and that 0 is the bottom of its spectrum. As a consequence we have the following realisation of e^* on BW_0 :

$$\forall u_0 \in BW_0, \lim_{t \to +\infty} e^{-tL} u_0 =< e^*, u_0 > \partial_x \psi_0$$

Indeed, decomposing u_0 on $N(\mathcal{L}) \subset BW_0$ and its orthogonal complement we get $e^{-tL} u_0 = e^{-tL}(< e^*, u_0 > \partial_x \psi_0 + b_0)$, the first term being constantly $< e^*, u_0 > \partial_x \psi_0$ and the second one decaying exponentially fast to zero as $t \to +\infty$. Knowing that $\partial_x \psi_0 > 0$ and applying this on every non-negative u_0 in $\mathcal{D}(\Omega_L) \subset BW_0$, since non-negativity is preserved over time for $e^{-tL} u_0$, we get that e^* is a non-negative distribution, that is a positive measure.

Moreover, e^* satisfies $\mathcal{L}^* e^* = 0$ in the sense of distributions along with its dual boundary condition, which is an hypoelliptic problem (see [31] Thm 4.2 or [70] Thm 3)(iii)). As a consequence, e^* is a smooth non-negative function up to the boundary of Ω_L.

Then, the strong maximum principle gives $e^* > 0$. Finally, using the weak Harnack inequality up to the boundary of [62] and the classical subsolution estimate up to the boundary of [46], Theorem 9.20, we obtain a full Harnack inequality up to the boundary for e^*. Using it in x large (where $-f'(\psi_0) > 0$)) on half-balls touching the boundaries we get that e^* is bounded : indeed, if its supremum were blowing up, its infimum would also : but this is impossible since e^* is integrable on $x > 0$ (note that $w \in X$).

The same argument for $-x$ large gives that e^* has at most a Ce^{-rx} growth.

[6] where the author treats this exact problem with a Neumann condition instead of a Wentzell one, but this does not change his proof.

1.4 Continuation from small $\varepsilon > 0$ to $\varepsilon = 1$

First we avoid the singularity near $\varepsilon = 0$: it will be studied later since it deals with a very unusual boundary condition. Let us set for $\varepsilon_0 > 0$,

$$P_{\varepsilon_0} = \{\varepsilon \in [\varepsilon_0, 1] \quad | \quad (S_\varepsilon) \text{ has a solution}\}$$

We now adapt the proofs of the previous section, following the same steps. The main differences are technical : all the computations are adapted easily, the counterpart of the regularity result in Proposition 1.3.1 has no technical difficulty any more, but the weight function in Section 1.4.2 changes a bit. For technical reasons we had to chose e^{rx} everywhere, so we will need to be careful about the boundedness of solutions.

1.4.1 P_{ε_0} is closed

This subsection follows exactly subsection 1.3.1. Consider a sequence $\varepsilon_n \to \varepsilon_\infty \in [\varepsilon_0, 1]$ and call (c_n, ϕ_n, ψ_n) the associated sequence of solutions of (S_{ε_n}) normalised in translation by

$$\max_{x \leq 0, y \in [-L,0]} (\mu\phi_n(x), \psi_n(x,y)) = \theta \tag{1.25}$$

Thanks to Propositions 1.2.5 and 1.2.6 we can extract from c_n a subsequence such that

$$\lim_{n \to +\infty} c_n = c^\infty > 0 \tag{1.26}$$

We now state a regularity result which is the counterpart of Proposition 1.3.1 in the case of (S_ε) :

Proposition 1.4.1. *There exists $\alpha > 0$ and constants $C_{Sch1,2} = C(D, d, c_{max}, Lipf, L, \mu)$ such that for all $n \geq 0$*

$$|\psi_n|_{\mathcal{C}^{2,\alpha}(\Omega_L)} \leq C_{Sch1}\left(|\psi_n|_{L^\infty(\Omega_L)} + |\mu\phi_n|_{L^\infty(\mathbb{R})}\right) \leq 2C_{Sch1}$$
$$|\mu\phi_n|_{\mathcal{C}^{2,\alpha}(\mathbb{R})} \leq C_{Sch2}\left(|\psi_n|_{L^\infty(\Omega_L)} + |\mu\phi_n|_{L^\infty(\mathbb{R})}\right) \leq 2C_{Sch2}$$

Proof. We adapt the proof of Proposition 1.3.1. By classical ODE theory (use Fourier transform or the variation of constants), there exists $C_{ode} = C(D, \mu, c_{max})$ such that

$$|\mu\phi_n|_{\mathcal{C}^{1,\alpha}} \leq |\mu\phi_n|_{W^{2,\infty}} \leq C_{ode}\left(|\mu\phi_n|_\infty + |\psi_n|_\infty\right) \leq 2C_{ode}$$

Seeing the right-hand side $f(\psi_n)$ in (S_ε) as $-\frac{f(\psi_n)}{\psi_n}\psi_n$ in the left-hand side, which yields a bounded 0-order term since f is Lipschitz, we can use the Hölder continuity estimate up to the mixed boundary of [60] and iterate with the classical Schauder estimate up to the Robin boundary (see [46], Lemma 6.29) so that on half-balls B_- supported on $y = 0$ on a segment T :

$$|\psi_n|_{\mathcal{C}^{2,\alpha}(B_-)} \leq C_R\left((1 + C_{ode})|\psi_n|_{L^\infty(2B_-)} + C_{ode}|\mu\phi_n|_\infty\right)$$

for some constant C_R. Finally we obtain the desired result by plugging the above estimate in the standard Schauder estimates for ϕ :

$$|\phi_n|_{\mathcal{C}^{2,\alpha}(T)} \leq C_{Sch}\left(|\phi_n|_{L^\infty(2T)} + C_R\left((1+C_{ode})|\psi_n|_{L^\infty(2T)} + C_{ode}|\mu\phi|_\infty\right)\right)$$

As before, we obtain the global estimate by covering $\mathbb{R}\times\Omega_L$ with such T and B_- using that the above estimate holds independently of the position of B_-, and other half-balls where standard Schauder estimates up to the Neumann boundary hold. □

Thanks to the previous estimate, as before we extract from (ϕ_n, ψ_n) a subsequence still denoted (ϕ_n, ψ_n) that converges in $\mathcal{C}^2_{loc}$ to ϕ^∞, ψ^∞ satisfying (S_{ε_∞}) except the limiting conditions. We now conclude just as in Propositions 1.3.2 and 1.3.3 :

Proposition 1.4.2. *$\mu\phi^\infty$ and ψ^∞ satisfy uniformly in y*

$$\lim_{x\to-\infty} \mu\phi^\infty(x), \psi^\infty(x,y) = 0$$

$$\lim_{x\to+\infty} \mu\phi^\infty(x), \psi^\infty(x,y) = 1$$

Proof. For the left limit, just observe that thanks to condition (1.25), (1.13) still holds for both $\mu\phi_\varepsilon$ and ψ_ε. For the right limit, the computations of Proposition 1.3.3 still hold : the only difference is that the boundary term

$$\frac{s^\infty}{\mu}\int_0^M \left(-D\partial^2_{xx}\psi^\infty(x,0) + c^\infty\partial_x\psi^\infty(x,0)\right)dx$$

should be replaced here by

$$\int_0^M \left(-D\partial^2_{xx}\phi^\infty(x) + c^\infty\partial_x\phi^\infty(x)\right)dx$$

which is treated in the exact same way. □

1.4.2 P_{ε_0} is open

To simplify the notations, we note $M = 1/\varepsilon$ and we search around a solution (c^0, ϕ_0, ψ_0) for $M = M_0$, a solution $c = c^0 + (M-M_0)c^1, \phi = \phi_0 + (M - M_0)\phi_1, \psi = \psi_0 + (M-M_0)\psi_1$. The equations on c^1, ϕ_1, ψ_1 are

$$\frac{l(\phi_1,\psi_1) = (M-M_0)(\psi_1-\mu\phi_1) - c^1\phi_0' - (M-M_0)c^1\phi_1' + \psi_0 - \mu\phi_0}{E(\phi_1,\psi_1) = (M-M_0)(\mu\phi_1-\psi_1) + \mu\phi_0 - \psi_0}$$

$$\frac{\begin{array}{c}\mathcal{L}\psi_1 + c^1\partial_x\psi_0 = R(M-M_0, c^1, \psi_1)\\[2ex] \partial_y\psi = 0\end{array}}{}\qquad (1.27)$$

where

$$l(\phi,\psi) = -D\phi'' + c^0\phi' - M_0(\psi - \mu\phi)$$
$$\mathcal{L}\psi = -d\Delta\psi + c^0\partial_x\psi - f'(\psi^0)\psi$$
$$E(\phi,\psi) = d\partial_y\psi - M_0(\mu\phi - \psi)$$

The functional setting will be $r = \min(\frac{c_{min}}{D}, \frac{c_{min}}{d})$, $w(x) = e^{rx}$ on the whole real line (we will see later why we need to take the exponential everywhere instead of connecting it with a constant like before)

$$X = \mathcal{C}^{2,\alpha}_w(\mathbb{R}) \times \mathcal{C}^{2,\alpha}_w(\Omega_L)$$

and we will work with the operator from X to $Y = \mathcal{C}^{\alpha}_w(\mathbb{R}) \times \mathcal{C}^{\alpha}_w(\Omega_L)$

$$\mathscr{L}(\phi,\psi) \ := (l(\phi,\psi), \mathcal{L}\psi)$$

endowed with the exchange condition $E(\phi,\psi) = 0$ on $y = 0$ and the Neumann condition $-d\partial_y\psi = 0$ on $y = -L$.

Treating the boundary as usual in the system case, we can obtain the same properties as in Lemma 1.3.2, with this time $N(\mathscr{L})$ generated by $(\phi_0', \partial_x\psi^0)$:

Lemma 1.4.1. *$\mathscr{L} : X \to Y$ is a Fredholm operator of index 0. As a consequence, the following decompositions hold*

$$X = N(\mathscr{L}) \oplus X_1$$

$$Y = R(\mathscr{L}) \oplus Y_2$$

where $X_1 \simeq R(\mathcal{L})$ is a closed subspace of X and $Y_2 \simeq N(\mathcal{L})$. Moreover

$$N(\mathscr{L}) = N(\mathscr{L}^2) = \mathbb{R}(\phi_0', \partial_x\psi_0)$$

Proof. This proof is postponed in section 1.4.3 to lighten this section. It relies on the same arguments as Lemma 1.3.2, up to the subtlety of the system case. These technicalities are the reason why we chose $w(x) = e^{rx}$ everywhere. Observe that the exponential growth of w as $x \to +\infty$ adds a difficulty in proving the Fredholm property : we have to prove that the invertible part of $\mathscr{L}$ yields bounded solutions. □

In order to work with this fixed problem, we have to kill the non-homogeneities and the small terms, so as before we look for solutions with form

$$\phi_1 = \tilde{\phi}(M, c^1, \phi, \psi) + \phi$$

$$\psi_1 = \tilde{\psi}(M, c^1, \phi, \psi) + \psi$$

where $\tilde{\phi}, \tilde{\psi}$ solves, for A large enough,

$$
\begin{array}{c}
-D\tilde{\phi}'' + c^0\tilde{\phi}' - M(\tilde{\psi} - \mu\tilde{\phi}) + A\tilde{\phi} = (M - M_0)(\psi - \mu\phi) - c^1\phi_0' - (M - M_0)c^1\phi' + \psi_0 - \mu\phi_0 \\
\hline
d\partial_y\tilde{\psi} - M(\mu\tilde{\phi} - \tilde{\psi}) = (M - M_0)(\mu\phi - \psi) + \mu\phi_0 - \psi_0 \\
\\
(-d\Delta + c^0\partial_x - f'(\psi^0) + A)\tilde{\psi} = 0 \\
\\
\partial_y\psi = 0 \\
\hline
\end{array}
\tag{1.28}
$$

Lemma 1.4.2. *Such a function $(\tilde{\phi}, \tilde{\psi})$ exists and satisfies*

$$(\tilde{\phi}, \tilde{\psi}) \in \mathcal{C}^1\left(\mathbb{R} \times \mathbb{R} \times X; X\right)$$

Proof. See Section 1.4.3 for the solvability of this equation provided A large enough. The fact that $\tilde{\phi}, \tilde{\psi}$ are not only $\mathcal{C}^{2,\alpha}$ but $\mathcal{C}^{2,\alpha}_w$ is shown just as before : thanks to the Schauder type estimate as in Proposition 1.4.1, it suffices to show that $w_1\tilde{\psi}$ and $w_1\tilde{\phi}$ are bounded. For this, repeat the proof of Lemma 1.3.4 but treating the boundary as usual in the system case. □

Thus we are left with the following problem to solve in $c^1 \in \mathbb{R}$, $(\phi, \psi) \in X$:

$$\mathscr{L}(\phi, \psi) + c^1(\phi_0', \partial_x\psi_0) = (R_1, R_2) - \mathscr{L}(\tilde{\phi}, \tilde{\psi}) \tag{1.29}$$

As before, applying the projection $\mathscr{P}$ onto Y_2 on (1.29) yields an equation on c^1, and applying $\mathscr{Q} = Id - \mathscr{P}$ yields an equation on the image part of the decomposition of $(\phi, \psi) = \Lambda(\phi_0', \partial_x\psi^0) + (\phi_R, \psi_R) \in X$, $\Lambda \in \mathbb{R}$ being free in all this procedure. The set of equation obtained is

$$c^1 = \mathscr{P}((R_1, R_2) - \mathscr{L}(\tilde{\phi}, \tilde{\psi})) \tag{1.30}$$

$$\mathscr{L}(\phi_R, \psi_R) = \mathscr{Q}((R_1, R_2) - \mathscr{L}(\tilde{\phi}, \tilde{\psi})) \tag{1.31}$$

For $M > M_0$, the auxiliary functions depend on c^1, ϕ and ψ so this system is non-linear and coupled, but at $M = M_0$, we have $R_1 = R_2 \equiv 0$, and the auxiliary functions depend only on ϕ_0, ψ_0, so in this case the system, as before, can be solved step by step. Moreover, since here c does not appear in the boundary condition, we do not need the duality argument of the previous section : (1.30) is trivially solvable.

Finally, as before the differential of this system of equations with respect to $c^1, (\phi_R, \psi_R) \in \mathbb{R} \times X_1$ at $M = M_0$ and the corresponding solutions yields an isomorphism since $\mathscr{L}$ is invertible on X_1, and the implicit function theorem provides for M close to M_0 a solution of (S_ε) apart from the limiting conditions.

The right-limit condition is then obtained by an adaptation of the computations of Proposition 1.3.4. We wish to emphasise on the fact that even though w has exponential growth as $x \to +\infty$, ϕ_1, ψ_1 are indeed bounded, as highlighted in Lemma 1.4.1.

Proposition 1.4.3. *Let*

$$c = c^0 + (M - M_0)c^1, \ \phi = \phi_0 + (M - M_0)\phi_1, \ \psi = \psi_0 + (M - M_0)\psi_1$$

If $M - M_0$ is small enough, we have

$$\lim_{x \to +\infty} \mu\phi(x), \psi(x, y) = 1$$

uniformly in y.

Proof. By treating the upper boundary as usual in the system case, the arguments of Proposition 1.3.4 hold. The only thing to check is the existence of another supersolution with exponential decrease in this case. We look for a solution of the type $(e^{-\gamma x}, e^{-\gamma x}h(y))$. The equations on $\gamma > 0, h > 0$ are

$$-h'' + \left(-\frac{1}{2d}f'(1) - \gamma\left(\gamma + \frac{c}{d}\right)\right) h = 0 \tag{1.32}$$
$$dh'(0) = \mu - h(0) \tag{1.33}$$
$$-h'(-L) = 0 \tag{1.34}$$
$$-D\gamma^2 - c\gamma = h(0) - \mu \tag{1.35}$$

Since $f'(1) < 0$, this can be solved by

$$h(y) = C\cosh\left(\beta(\gamma)\left(y + L\right)\right)$$

where

$$\beta(\gamma) = \sqrt{-f'(1)/(2d) - \gamma(\gamma + c/d)}$$

Moreover, equation (1.33) gives that

$$C = \frac{\mu}{d\beta(\gamma)\sinh(\beta(\gamma)L) + \cosh(\beta(\gamma)L)}$$

which, plugged in equation (1.35) yields the equation on γ :

$$D\gamma^2 + c\gamma = \frac{\mu d\beta(\gamma)}{1 + \tanh(\beta(\gamma)L)}$$

which has a solution $0 < \gamma < \gamma_{lim}$ (where γ_{lim} is the positive zero of $\beta(\gamma)$) for the same reasons as in Proposition 1.3.4.

□

1.4.3 Proof of Lemma 1.4.1

Throughout all this section, in order to simplify the notations, we have taken $M_0 = 1$ without loss of generality. In this section we show that $\mathscr{L}$ is Fredholm of index 0 on X, and that $N(\mathscr{L}) = N(\mathscr{L}^2)$ is generated by $(\phi_0', \partial_x\psi_0)$. The proof of the second property does not change : $(\phi_0', \partial_x\psi_0)$ is indeed a solution of

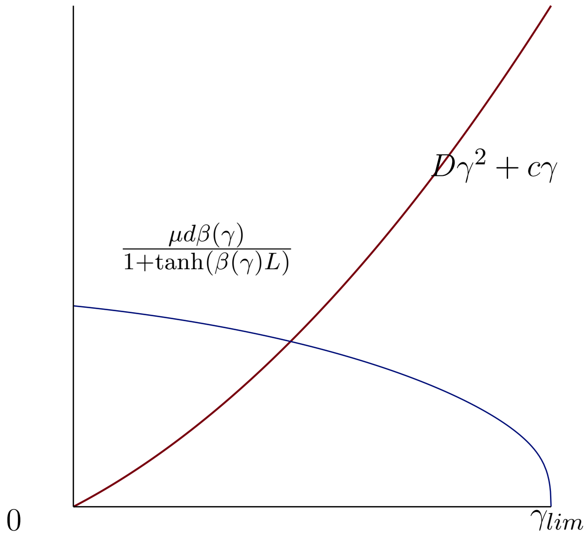


Figure 1.3: Equation (1.35) on γ

the problem, and by treating the boundary condition as usual in the system case, the proof of lemma 1.3.2 in section 1.3.3 still holds. The proof of the Fredholm property on the other hand changes a bit, since we did not take the usual weight but the exponential weight on the whole real line. This is because of the exchange condition : suppose we had taken the usual weight, and did all the machinery $\tilde{\mathscr{L}} = \tilde{\mathscr{T}} + \tilde{\mathscr{S}}$. Then we would not be able to show that $\tilde{\mathscr{T}}$ is invertible. Indeed, suppose we want to solve $\tilde{\mathscr{T}}(\phi, \psi) = (g, h) \in \mathcal{C}^\alpha(\mathbb{R}) \times \mathcal{C}^\alpha(\Omega_L)$. In order to obtain that $\tilde{\mathscr{T}}$ is injective (and that its inverse is bounded if it exists), we want to control (ϕ, ψ) by the data (g, h), by starting with the L^∞ norm. So suppose ψ reaches a maximum somewhere. Then if it is on $y = 0$ and on $x > 1$, we have a problem. Indeed, the Hopf lemma only gives $\psi < \mu\phi$ and then looking the equation for ϕ gives nothing : that is why we want to pull a bit the 0-order term of the equation on ϕ, and that is why we have chosen $w(x) = e^{rx}$ everywhere, so that $\tilde{\mathscr{T}}(\phi, \psi) = (g, h)$ is no more than

$$
\begin{array}{c}
-D\phi'' + (c^0 - 2Dr)\phi' + (\mu + \alpha_r)\phi - \psi = g \\
\hline
d\partial_y \psi = \mu\phi - \psi \\
\\
(-d\Delta + (c^0 - 2dr)\partial_x + \gamma(x))\psi = h \\
\\
\partial_y \psi = 0 \\
\hline
\end{array}
\tag{1.36}
$$

with $\alpha_r = -Dr^2 + c^0 r > 0$. In this setting, a maximum point of ψ reached on the road is no more a problem, we always have that $\psi < \frac{\mu}{\alpha_r}|g|_\infty$ in this case, and actually, in every case

$$|\psi|_\infty \leq \frac{1}{\min\gamma}|h|_\infty + \frac{\mu}{\alpha_r}|g|_\infty$$

$$|\phi|_\infty \leq \frac{|g|_\infty + |\psi|_\infty}{\mu + \alpha_r} \leq \frac{1}{\min\gamma(\mu+\alpha_r)}|h|_\infty + \frac{1}{\alpha_r}|g|_\infty$$

For the surjectivity, unlike before, the literature does not give any existence theorem for such a linear problem, so we have to do it by ourselves : just observe that the estimate above gives that $\tilde{\mathscr{T}}$ has closed range. Indeed, if $\tilde{\mathscr{T}}(\phi_n, \psi_n) = (g_n, h_n)$ and (g_n, h_n) converges in $\mathcal{C}^\alpha$ to a (g, h), then by the above estimate and the Cauchy criteria, (ϕ_n, ψ_n) converges uniformly to a bounded continuous (g, h). But also $\tilde{\mathscr{T}}(\phi_n, \psi_n)$ is bounded in $\mathcal{C}^\alpha$, so by regularity as in Proposition 1.4.1, (ϕ_n, ψ_n) converges up to extraction and diagonal process in $\mathcal{C}^{2,\beta}$. Uniqueness of the limit implies that (ϕ, ψ) is indeed $\mathcal{C}^{2,\beta}$ and the convergence holds in the $\mathcal{C}^{2,\beta}$ sense. Finally, passing to the limit we get $\tilde{\mathscr{T}}(\phi, \psi) = (g, h)$ and $(\phi, \psi) \in \mathcal{C}^{2,\alpha}$ so that (g, h) lies in $R(\tilde{\mathscr{T}})$. Finally, observe that the above estimate also holds for the formal adjoint of $\tilde{\mathscr{T}}$. Then, since the operators have smooth coefficients, the duality can be obtained thanks to the formal adjoint, and so $R(\tilde{\mathscr{T}}) = \overline{R(\tilde{\mathscr{T}})} = N(\tilde{\mathscr{T}}^*)^\perp = \mathcal{C}^\alpha$.

The only thing left to see is that solving something for tilded operators really yields something back in the untilded world : what we mean is that since w has exponential growth as $x \to +\infty$, we might have a problem. Indeed, we wanted to solve $\mathscr{T}u = (g, h)$ in the weighted spaces, so we saw this equation as $\frac{1}{w}\mathscr{T}(w \times (\phi, \psi)) = \frac{1}{w}(g, h) \in \mathcal{C}^\alpha(\mathbb{R}) \times \mathcal{C}^\alpha(\Omega_L)$ and obtained a solution $(\phi, \psi) \in \mathcal{C}^{2,\alpha}$. In the former cases, since $w \in \mathcal{C}^{\infty,\alpha}$ we did not have any problem to claim that also $wv \in \mathcal{C}^{2,\alpha}$ but here it is not the case any more, w is not even bounded, and we might not have $w\psi \in \mathcal{C}^{2,\alpha}$, we might even not have that it is bounded. Actually, $\mathcal{C}^{2,\alpha}$ and boundedness for $w \times (\phi, \psi)$ are equivalent because of Schauder estimates, so we just have to see that it is indeed bounded. We will do that by showing that ϕ, ψ have actually a Ce^{-rx} decay as $x \to +\infty$.

For this, let $K = \max(|g|_\infty, |h|_\infty)$ and observe that if $A \geq \max(K, K/-f'(1))$ then

$$(\overline{\phi}, \overline{\psi}) = \left(\frac{A}{\mu}e^{-rx}, Ae^{-rx}\right)$$

is a supersolution of (1.36) on $x > 1$, where (1.36) has constant coefficients and a positive 0-order term. Now just multiply this supersolution by a constant large enough so that it is above (ϕ, ψ) on $x = 1$ and apply the usual maximum principle and compactness argument to $(\overline{\phi} - \phi, \overline{\psi} - \psi)$: it can neither reach a negative minimum, nor have a negative infimum as $x \to +\infty$, which yields that $\phi, \psi \leq Ce^{-rx}$ for some constant $C > 0$. The same argument works for finding $C' < 0$ such that $\phi, \psi \geq -C'e^{-rx}$.

1.5 The case $\varepsilon \simeq 0$

We start with $\left(c_w, \psi_w, \phi_w = \frac{1}{\mu}\psi_w(\cdot, 0)\right)$. We want to continue this solution to a solution of (S_ε) for small $\varepsilon > 0$. If we set as usual $\phi = \phi_w + \varepsilon\phi_1, \psi = \psi_w + \varepsilon\psi_1, c = c_w + \varepsilon c_1$, using

$$\phi_1 = \frac{\psi_1 + d\partial_y\psi_w + \varepsilon d\partial_y\psi_w}{\mu}$$

from the exchange condition yields the equation

$$\begin{aligned} &W\psi_1 + \varepsilon\frac{c_1}{\mu}\partial_x\psi_1 + \left(-\frac{\varepsilon D}{\mu}\partial^2_{xx} + \varepsilon\frac{c_w + c_1\varepsilon}{\mu}\partial_x\right) d\partial_y\psi_1 \\ =& -\frac{c_1}{\mu}\partial_x\psi_w - \left(-\frac{D}{\mu}\partial^2_{xx} + \frac{c_w + c_1\varepsilon}{\mu}\partial_x\right) d\partial_y\psi_w \end{aligned} \tag{1.37}$$

where

$$W = d\partial_y - D/\mu\partial^2_{xx} + c_w/\mu\partial_x$$

as the upper boundary condition for the usual linearised problem in ψ_1 :

$$-d\Delta\psi_1 + c_w\partial_x\psi_1 - f'(\psi_w)\psi_1 = -c_1\partial_x\psi_w + R(\varepsilon, c_1, \psi_1) \tag{1.38}$$

In particular, by taking $\varepsilon = 0$ in (1.37), (1.38) we retrieve a linear Wentzell problem, i.e. (1.37), (1.38) is a singular perturbation of a Wentzell problem on which we already applied the implicit function theorem. Conversely, we can see (1.37) as an integro-differential regularisation of the Wentzell boundary condition, but the regularity theory of [30] does not apply easily to this situation.

As before, we want to transform (1.37) in a fixed Wentzell problem by using an auxiliary function. This time, since we do not have any existence or regularity theorem for such problems, we will have to compute everything by hand. Hopefully, since we work in a strip, we can use the partial (in x) Fourier transform which will be a very helpful tool. On the other hand, this time we will have to work with a constant coefficient operator instead of the linearised itself in order to be able to do the computations, but we will see that this is not a problem. From now on, let w denote the same weight function as in the Wentzell section. We now give two simple technical lemmas that we will use throughout the next computations.

Lemma 1.5.1. *If $k \in L^1$, $\hat{k} \in \mathcal{C}^\infty \cap L^2$ and $h \in L^\infty$, $\hat{h} \in \mathcal{S}'$ then the formula*

$$\mathcal{F}^{-1}(\hat{k}\hat{h}) = k * h$$

makes sense and holds (where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform).

Proof. Since $\hat{k}$ is a smooth function, the product distribution $\hat{k}\hat{h}$ makes sense and we can compute its inverse Fourier transform : result follows by using the classical properties of the Fourier transform on L^2 and the Fubini-Tonelli theorem.

□

Lemma 1.5.2. *Let $r > 0$. If $h \in L^\infty(\mathbb{R})$ and $e^{-rx}h(x) \in L^\infty(\mathbb{R})$ as well as $K \in L^1(\mathbb{R})$ and $e^{-rt}K(t) \in L^1(\mathbb{R})$ then*

$$K * h \in L^\infty(\mathbb{R}) \text{ and } e^{-rx}(K * h)(x) \in L^\infty(\mathbb{R})$$

If moreover $e^{-rx}h \in \mathcal{C}^\alpha(\mathbb{R})$, then $e^{-rx}(K * h)(x) \in \mathcal{C}^\alpha(\mathbb{R})$.

Proof.

$$\begin{aligned}|e^{-rx}(K * h)(x)| \leq \int_{\mathbb{R}} |K(t)e^{-rx}h(x-t)|dt &\leq \int_{\mathbb{R}} |e^{-rt}K(t)||e^{-r(x-t)}h(x-t)|dt \\ &\leq |e^{-rt}K(t)|_{L^1}|e^{-rx}h(x)|_{L^\infty}\end{aligned}$$

For the second part of just observe that

$$\begin{aligned}\frac{|e^{-ry}(K * h)(y) - e^{-rx}(K * h)(x)|}{|x-y|^\alpha} &\leq \int_{\mathbb{R}} K(t)e^{-rt}\frac{|e^{-r(y-t)}h(y-t) - e^{-r(x-t)}h(x-t)|}{|y-x|^\alpha}dt \\ &\leq |e^{-rt}K(t)|_{L^1}|e^{-rx}h(x)|_\alpha\end{aligned}$$

□

Now we assert the following :

Lemma 1.5.3. *By taking $r > 0$ small enough in the definition of w, we have*

$$\tilde{\psi} \in \mathcal{C}^1([0,1] \times \mathbb{R} \times \mathcal{C}^{3,\alpha}_w(\Omega_L); \mathcal{C}^{2,\alpha}_w(\Omega_L))$$

where $u = \tilde{\psi}(\varepsilon, c_1, v)$ solves

$$Wu + \varepsilon\tfrac{c_1}{\mu}\partial_x u + \left(-\tfrac{\varepsilon D}{\mu}\partial^2_{xx} + \varepsilon\tfrac{c_w + c_1\varepsilon}{\mu}\partial_x\right) d\partial_y u = h_0 - \varepsilon\tfrac{c_1}{\mu}\partial_x v - \left(-\tfrac{\varepsilon D}{\mu}\partial^2_{xx} + \varepsilon\tfrac{c_w + c_1\varepsilon}{\mu}\partial_x\right) d\partial_y v$$

$$-\Delta u + u = 0$$

$$\partial_y u = 0$$

and $h_0 := -\frac{c_1}{\mu}\partial_x\psi_w - \left(-\frac{D}{\mu}\partial^2_{xx} + \frac{c_w + c_1\varepsilon}{\mu}\partial_x\right) d\partial_y\psi_w \in \mathcal{C}^\alpha(\mathbb{R})$. Moreover, we have the estimate

$$|u|_{\mathcal{C}^{2,\alpha}(\Omega_L)} \leq C_1|h_0|_\infty + C_2\left|\frac{1}{\varepsilon}K_0\left(\frac{|x|}{d\varepsilon}\right) * (h_0 + \varepsilon h(v))\right|_\alpha + C_3|h_0 + \varepsilon h(v)|_\alpha$$

where K_0 denotes the 0-th modified Bessel function of the second kind (which is integrable[7] and whose Fourier transform is $\frac{\pi}{\sqrt{1+x^2}}$) so that $\frac{1}{\varepsilon}K_0\left(\frac{|x|}{d\varepsilon}\right)$ realises an approximation to the identity, and where $h(v)$ denotes $\partial_x v + \partial_{xy} v + \partial_{xxy} v$. Finally, we also have

$$\mathcal{L}\tilde{\psi}(\varepsilon, c_1, v) \in \mathcal{C}^{1,\alpha}_w(\Omega_L)$$

[7]it increases in a logarithmic fashion as $x \to 0$ and decreases as e^{-x}/x as $x \to \infty$, see [69] p.532.

Proof. The proof is based on the kernel analysis of this problem after applying a partial Fourier transform. First, let us see that $v \in \mathcal{C}^{3,\alpha}_w(\mathbb{R})$ implies that the right-hand side in the boundary condition for u is in $\mathcal{C}^{\alpha}_w(\mathbb{R})$. In the following, for the sake of notations we will only write it h. Applying formally the x-Fourier transform, we get a one parameter (in ξ) family of two-points boundary problems (in y) which are solved necessarily by

$$\hat{u}(\xi, y) = C(\xi)\cosh(\sqrt{\xi^2+1}(y+L))$$

and the upper boundary condition yields, if we set $\beta(\xi) = \sqrt{\xi^2+1}$

$$C(\xi) = \frac{\hat{h}(\xi)}{d\beta\left(\xi\right)\sinh\left(\beta\left(\xi\right)L\right)\left(1+\frac{\varepsilon D}{\mu}\xi^2+\varepsilon\frac{c_w+c_1\varepsilon}{\mu}i\xi\right)+\left(\frac{D}{\mu}\xi^2+\frac{c_w+c_1\varepsilon}{\mu}i\xi\right)\cosh\left(\beta\left(\xi\right)L\right)}$$

i.e. we get

$$\hat{u}(\xi, y) = C(\xi)\cosh\left(\beta\left(\xi\right)\left(y+L\right)\right)\hat{h}(\xi) =: \hat{k}_y(\xi)\hat{h}(\xi)$$

Now for each $-L \leq y < 0$, this kernel is in the Schwartz space $\mathscr{S}(\mathbb{R})$ and $u(x, y)$ for such y can be obtained by the usual convolution product between the Fourier inverse of $\hat{k}_y(\xi)$ and $h(x)$. Moreover, since for $-L \leq y < -\delta$ with $\delta > 0$ the kernels are a $\mathcal{C}^\infty$ family that is uniformly bounded in the Schwartz space $\mathscr{S}(\xi)$, we have by dominated convergence that u is a $\mathcal{C}^\infty$ function in Ω_L, in particular it is locally $\mathcal{C}^{2,\alpha}$. We now want to investigate the regularity of u on the line $y = 0$ in order to use Schauder estimates to conclude to a uniform $\mathcal{C}^{2,\alpha}$ regularity.

On $y = 0$, things get a little more complicated since the kernel involved is

$$\hat{k}_0(\xi) = \frac{1}{d\beta\left(\xi\right)\tanh\left(\beta\left(\xi\right)L\right)\left(1+\frac{\varepsilon D}{\mu}\xi^2+\varepsilon\frac{c_w+c_1\varepsilon}{\mu}i\xi\right)+\left(\frac{D}{\mu}\xi^2+\frac{c_w+c_1\varepsilon}{\mu}i\xi\right)}$$

which decays only like $\dfrac{1}{1+\frac{D}{\mu}\xi^2+\frac{\varepsilon dD}{\mu}|\xi|^3}$, and $(i\xi)^2\hat{k}_0(\xi)$ like $\dfrac{\xi^2}{1+\frac{D}{\mu}\xi^2+\frac{\varepsilon dD}{\mu}|\xi|^3}$. Keep in mind that we are interested in ε independent estimates, so we cannot use the little bonus decay it gives. Nonetheless, observe that $\hat{k}_0$ is $\mathcal{C}^1$ with respect to the parameters $(c_1 \in \mathbb{R}, \varepsilon \in [0,1])$ (this is something we will need in the end to apply the implicit function theorem) and decays at worst (when $\varepsilon = 0$) as $\frac{\mu}{D(1+\xi^2)}$. Heuristically, we see that $\varepsilon > 0$ is not a problem in the sense that it adds decay and does not prevent analyticity, so in a Fourier point of view, the worst case is when $\varepsilon = 0$, and in this case the kernels are nothing more than the kernels for the Wentzell problem in a strip, which is known to be well posed. We use a Paley-Wiener type theorem to prove this :

- $\hat{k}_0(\xi)$ is a $\mathcal{C}^1$ in $(c_1 \in \mathbb{R}, \varepsilon \in [0,1])$ family of integrable (because the worst decay is $\frac{1}{1+\frac{D}{\mu}\xi^2}$ for $\varepsilon = 0$) and real analytic functions (as the inverse of real analytic functions that have no zero). Moreover, independently from

ε and c_1, these real analytic functions admit an analytic continuation to a complex strip $|\Im\zeta| < a$ with $a > 0$ that have a η-uniformly bounded L^1 norm on the real lines $\mathbb{R} + i\eta$, $-a < \eta < a$, see lemma 1.5.4. By virtue of the Paley-Wiener type theorem of [73] (IX.14) and the dominated convergence theorem we know that $k_0(x)$ is a $\mathcal{C}^1$ in $c_1 \in \mathbb{R}, \varepsilon \in [0,1]$ family of bounded continuous real functions that satisfy all $|k_0(x)| \leq C_a e^{-a|x|}$. Now, we can say that $u(x,0) = k_0 * h$ is a bounded continuous function that is $\mathcal{C}^1$ with respect to the parameters ε, c^1 and v (since h is $\mathcal{C}^1$ in those parameters as product and sum of affine functions).

- For the sake of simplicity, we divide the analysis of $\xi^2\hat{k}_0(\xi)$ in two cases : $\varepsilon > 0$ or $\varepsilon = 0$ and we will see that the result is smooth in ε.
  Case $\varepsilon = 0$: in this case, the asymptotic behaviour of $\xi^2\hat{k}_0(\xi)$ as $|\xi| \to \infty$ yields $\xi^2\hat{k}_0(\xi) = \frac{\mu}{D} - \frac{\mu^2 d}{D^2\sqrt{1+\xi^2}} + r_1(\xi)$ where r_1 denotes an integrable function (it decays like $1/\xi^2$) that has an analytic continuation in some complex strip $|\Im| < a$, i.e. to which the same analysis as above applies. Thus, the Fourier transform of $\xi^2\hat{k}_0(\xi)$ is given by $\frac{\mu}{D}\delta - \frac{\mu^2 d}{D^2}\frac{1}{\pi}K_0(|x|) + \check{r}_1$, where δ denotes the Dirac distribution, K_0 the modified Bessel function of order 0, and where $\check{r}_1$ has the properties described in the section above. By lemma 2.3.2 we get

$$\partial^2_{xx}u(x,0) = \frac{\mu}{D}h_0 - \frac{\mu^2 d}{D^2}\frac{1}{\pi}K_0(|\cdot|) * h_0 + \check{r}_1 * h_0 \in \mathcal{C}^\alpha(\mathbb{R})$$

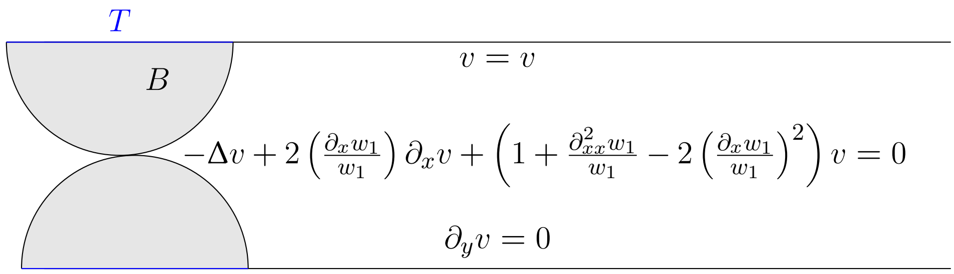
  since $h_0 \in \mathcal{C}^\alpha(\mathbb{R})$

  Case $\varepsilon > 0$. This changes the decay of the kernel from constant to $1/\xi$, so we will not get a Dirac term in the Fourier transform. Nonetheless, what is tricky is that we want $u = \tilde{\psi}$ to be a $\mathcal{C}^1$ function in ε to be able to use the implicit function theorem, i.e. we separated the computations for $\varepsilon > 0$ or $= 0$, but in the end the results should agree when $\varepsilon \to 0$. This will be based on the fact that the functions we will obtain will behave as an approximation to the identity as $\varepsilon \to 0$.

  Indeed, $\xi^2\hat{k}_0(\xi) = \frac{\mu}{D}\frac{1}{\sqrt{1+(d\varepsilon)^2\xi^2}} + r_2$. Notice that we chose to put ε in front of ξ inside the square root rather than just let it appear as $\frac{1}{\varepsilon}$: this is the right way to get smoothness in ε, since this gives the correct decay even if $\varepsilon = 0$. Now, observe that the inverse Fourier transform of the first term is $\frac{\mu}{D}\frac{1}{\pi}\frac{1}{d\varepsilon}K_0(\frac{|x|}{d\varepsilon})$: since $K_0(|\cdot|)$ is an integrable function on $\mathbb{R}^1$ whose integral equals to π, this clearly is $\frac{\mu}{D}$ times an approximation to the identity. We finish by saying that the term r_2 can be computed as $\frac{C(\varepsilon)}{\sqrt{1+\xi^2}} + r_3$ where $C(\varepsilon)$ is a smooth function that satisfies $C(0) = -\frac{\mu^2 d}{D}$ and r_3 is a smooth family with respect to (c_1, ε) of integrable functions to which the same analysis as r_1 applies, and that goes to r_1 as $\varepsilon \to 0$.

This analysis gives that $u \in \mathcal{C}^{2,\alpha}(y=0)$ and then by applying Schauder estimates for the Dirichlet problem, we get that $u \in \mathcal{C}^{2,\alpha}(\Omega_L)$. We describe now with more details the same technique applied on $w_1 u$.

We are now left to show that a weighted data yields a weighted solution, i.e. that $w_1 u \in \mathcal{C}^{2,\alpha}(\Omega_L)$. We observe that $v = w_1 u$ solves the following equation in Ω_L :



Thanks to the expression of w_1, the coefficients of this equation are smooth bounded functions and we can use local estimates up to the boundary for the Dirichlet or the Neumann problem (see Cor. 6.7 and Lemma 6.29 in [46]), so it suffices to show that $w_1 u$ is bounded and that $w_1 u(\cdot, 0) \in \mathcal{C}^{2,\alpha}(\mathbb{R})$, which thanks to the expression of w_1, is similar to $w_1 \partial^2_{xx} u \in \mathcal{C}^{\alpha}(\mathbb{R})$. We show that these are true provided $r < \min(\rho, 1)$ (see lemma 1.5.4 for the definition of ρ).

- $w_1 u$ is bounded thanks to lemma 1.5.2 : indeed $w_1 u(x,y) = w_1(x)(k_y * h)(x)$. As we already said, k_y is a family of bounded continuous functions uniformly bounded in L^1. Moreover, they have a uniform $Ce^{-\rho|x|}$ decay as $x \to \pm\infty$: for this see lemma 1.5.4 below and use [73], Theorem IX.14.
- $w_1 \partial^2_{xx} u(\cdot, 0)$ is bounded and has $\mathcal{C}^{\alpha}$ regularity since $K_0(|\cdot|)$ has $e^{-|x|}/|x|$ decay and the other kernels appearing in $\partial^2_{xx} u(\cdot, 0)$ satisfy lemma 1.5.2 too, thanks to their common analyticity ; see lemma 1.5.4.

□

Lemma 1.5.4. *Replacing ξ with the complex variable ζ in $\hat{k}_y(\xi)$ yields a meromorphic continuation of $\hat{k}_y$ in the strip $-1 < \Im z < 1$ that has no pole in a strip $-\rho < \Im z < \rho$ for $\rho > 0$ small enough.*

Proof. First, observe that apart from $\beta(\xi)$, the denominator of $\hat{k}_y$, which we will note $F(\xi)$ in this proof :

$$d\beta(\xi)\sinh(\beta(\xi) L)\left(1 + \frac{\varepsilon D}{\mu}\xi^2 + \varepsilon\frac{c_w + c_1\varepsilon}{\mu} i\xi\right) + \left(\frac{D}{\mu}\xi^2 + \frac{c_w + c_1\varepsilon}{\mu} i\xi\right)\cosh(\beta(\xi) L)$$

is composed of holomorphic functions over the whole complex plane. The only limiting function is $\beta(\xi)$ which is holomorphic in the strip $-1 < \Im z < 1$. As a result, $\hat{k}_y$ is meromorphic in this strip. Moreover, thanks to the $\frac{D}{\mu}\xi^2\cosh(\beta(\xi)L)$ term we can see that if ξ is large enough, $|F(\xi)|$ is large enough (independently from $\Im\zeta$ in the strip and from c_1, ε), so its zeroes have to be in a rectangle centred at the complex origin whose length depends on the parameters (but not on ε or c_1). Since the zeros of a non-zero holomorphic functions are isolated, we know that F has a finite number of zeros in such a rectangle. Moreover, a direct computation shows that it cannot have any zero on the real line. Thus, there exists $\rho > 0$ small enough such that on the strip $-\rho < \Im\xi < \rho$, F does not vanish.[8]

□

We now turn to the implicit function theorem procedure that concludes this section. Searching as usual for ψ_1 with form $\psi_1 = v + \tilde{\psi}(v)$, we are reduced to solving the following problem on v :

$$d\partial_y v - \tfrac{D}{\mu}\partial^2_{xx}v + \tfrac{c_w}{\mu}\partial_x v = 0$$

$$\mathcal{L}v + c_1\partial_x\psi^0 = R(\varepsilon, c^1, v) - \mathcal{L}\tilde{\psi}(s, c^1, v)$$

$$\partial_y v = 0$$

We now use the analysis of section 1.3.2 to claim that $\mathcal{L}$ endowed with this Wentzell boundary condition has the Fredholm property of index 0 between $\mathcal{C}^{3,\alpha}_w$ and $\mathcal{C}^{1,\alpha}_w$. Since R lies also in $\mathcal{C}^{1,\alpha}_w$, the procedure is then exactly the same as in 1.3.2 and for $\varepsilon > 0$ small enough leads to a solution $c_1, \psi_1 = v + \tilde{\psi}(\varepsilon, c_1, v)$ of (1.37), (1.38) that lies in $\mathcal{C}^{2,\alpha}_w(\Omega_L)$.

Then, setting $\phi_1(x) = \dfrac{\psi_1(x,0) + d\partial_y\psi_w(x,0) + \varepsilon d\partial_y\psi_1(x,0)}{\mu}$ we get that $\phi_1 \in \mathcal{C}^{2,\alpha}_w(\mathbb{R})$ and that $\phi = \frac{1}{\mu}\psi_w(x,0) + \varepsilon\phi_1, \psi = \psi_w + \varepsilon\psi_1$ solves (S_ε) except for the right limit condition. But the analysis of Proposition 1.3.4 with the maximum principle and Hopf lemma for the system gives that if $\varepsilon > 0$ is taken small enough, this limit holds.

Remark 1.5.1. Observe that solving this singular perturbation had a price of one derivative: we started by assuming $\psi_w \in \mathcal{C}^{3,\alpha}(\Omega_L)$ but we end up with a solution of (S_ε) that is only $\mathcal{C}^{2,\alpha}(\mathbb{R}) \times \mathcal{C}^{2,\alpha}(\Omega_L)$.

Remark 1.5.2. Finally, we wish to detail the changes that are to be made when f satisfies Assumption B :

- On the one hand, since $f'(0) < 0$, no weighted spaces are needed : the linearised operators will be Fredholm in the usual function spaces. This simplifies considerably the above method.
- On the other hand, the estimates by below on c_n will not hold anymore since we lose the positivity of $\displaystyle\int f(\psi)$. Nonetheless if one multiplies by $\partial_x\psi$ the

[8] Actually, contour integrals of F'/F show that F has only one or two zeros, depending on d, D, μ, c_w and L, and we can easily see that these are on the imaginary axis by solving for $\zeta = i\eta$ the equation $d\sqrt{1-\eta^2}\tanh(\sqrt{1-\eta^2}L)(1 - \varepsilon\frac{D}{\mu}\eta^2 - \varepsilon\frac{c_w + c_1\varepsilon}{\mu}\eta) = \frac{D}{\mu}\eta^2 + \frac{c_w + c_1\varepsilon}{\mu}\eta$.

equation satisfied by ψ in (S_ε) and integrate by parts, one gets

$$c\left(\int_{\Omega_L}(\partial_x\psi)^2+\int_{\mathbb{R}}\phi'^2\right)=L\int_0^1 f(s)\mathrm{d}s$$

Thus, since f is of positive total mass one ensures the positivity of c and one can take $c_{min}=0$. This will not be a problem thanks to $f'(0)<0:$ for instance in Prop. 1.3.2 the exponent r can now be defined as

$$r\ :=\sqrt{\frac{-f'(0)}{2d}}>0$$

and this will yield a suitable supersolution if one normalises the solutions by translation in such a way that $f(\psi)\leq\frac{f'(0)}{2}\psi$ on $x<0$.

Finally, the end of the proof of Prop. 1.3.3 adapts as follows :

$$c^\infty(L+s^\infty/\mu)=\int_{\Omega_L}f(\psi^\infty)$$

but this is impossible if $\beta=\theta$ since $c^\infty\geq 0$ and $\int f(\psi^\infty)<0$.

Acknowledgement

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Chapter 2

The large diffusion limit

“ If you try and take a cat apart to see how it works, the first thing you have on your hands is a non-working cat. ”

—Douglas Adams (1952-2001)

We study the velocity of travelling waves of a reaction-diffusion system coupling a standard reaction-diffusion equation in a strip with a one-dimensional diffusion equation on a line. We show that it grows like the square root of the diffusivity on the line. This generalises a result of Berestycki, Roquejoffre and Rossi in the context of Fisher-KPP propagation where the question could be reduced to algebraic computations. Thus, our work shows that this phenomenon is a robust one. The ratio between the asymptotic velocity and the square root of the diffusivity on the line is characterised as the unique admissible velocity for fronts of an hypoelliptic system, which is shown to admit a travelling wave profile.

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Contents

2.1 Introduction

This paper deals with the limit $D \to +\infty$ of the following system with unknowns $c > 0, u(x), v(x,y)$:

$$\begin{cases} -d\Delta v + c\partial_x v = f(v) \text{ for } (x,y) \in \Omega_L := \mathbb{R}\times]-L,0[\\ d\partial_y v(x,0) = \mu u(x) - v(x,0) \text{ for } x \in \mathbb{R} \\ -d\partial_y v(x,-L) = 0 \text{ for } x \in \mathbb{R} \\ -Du''(x) + cu'(x) = v(x,0) - \mu u(x) \text{ for } x \in \mathbb{R} \end{cases}$$

along with the uniform in y limiting conditions

$$\mu u, v \to 0 \text{ as } x \to -\infty$$

$$\mu u, v \to 1 \text{ as } x \to +\infty$$

These equations will be represented from now on as the following diagram

$$\begin{array}{ccc} 0 \leftarrow u & -Du'' + cu' = v - \mu u & u \to 1/\mu \\ \hline & d\partial_y v = \mu u - v & \\ & & \\ 0 \leftarrow v & -d\Delta v + c\partial_x v = f(v) & v \to 1 \\ & & \\ & -\partial_y v = 0 & \\ \hline \end{array} \tag{2.1}$$

In [22], Berestycki, Roquejoffre and Rossi introduced the following reaction-diffusion system :

$$\begin{array}{c} \partial_t u - D\partial_{xx} u = v - \mu u \\ \hline d\partial_y v = \mu u - v \\ \\ \partial_t v - d\Delta v = f(v) \\ \\ -\partial_y v = 0 \\ \hline \end{array} \tag{2.2}$$

but in the half plane $y < 0$ with $f(v)$ of the KPP-type, i.e $f > 0$ on $(0,1)$, $f(0) = f(1) = 0$, $f'(1) < 0$ and $f(v) \leq f'(0)v$. Such a system was proposed to give a mathematical description of the influence of transportation networks on biological invasions. If (c,u,v) is a solution of (3.1), then $(u(x+ct), v(x+ct))$ is a travelling wave solution of (3.2), connecting the states $(0,0)$ and $(1/\mu, 1)$. In [22], the following was shown :

Theorem. *([22])*

i) Spreading. There is an asymptotic speed of spreading $c_* = c_*(\mu, d, D) > 0$ such that the following is true. Let the initial datum (u_0, v_0) be compactly supported, non-negative and $\not\equiv (0,0)$. Then :

- *for all $c > c_*$*

$$\lim_{t\to+\infty}\sup_{|x|\geq ct}(u(x,t),v(x,y,t)) = (0,0)$$

uniformly in y.

- *for all $c < c_*$*

$$\lim_{t\to+\infty}\inf_{|x|\leq ct}(u(x,t),v(x,y,t)) = (1/\mu,1)$$

locally uniformly in y.

ii) The spreading velocity. If d and μ are fixed the following holds true.

- *If $D \leq 2d$, then $c_*(\mu,d,D) = c_{KPP} =: 2\sqrt{df'(0)}$*
- *If $D > 2d$ then $c_*(\mu,d,D) > c_{KPP}$ and $\lim_{D\to+\infty} c_*(\mu,d,D)/\sqrt{D}$ exists and is a positive real number.*

Thus a relevant question is whether the result of [22] is due to the particular structure of the nonlinearity or if it has a more universal character. This is a non trivial question since the KPP case benefits from the very specific property $f(v) \leq f'(0)v$: in such a case propagation is dictated by the linearised equation near 0, and the above question can be reduced to algebraic computations. Observe also that some enhancement phenomena really need this property : for instance, for the fractional reaction-diffusion equation

$$\partial_t u + (-\Delta)^s u = u(1-u)$$

in [27, 29], Cabré, Coulon and Roquejoffre proved that the propagation of an initially compactly supported datum is exponential in time. Nonetheless, this property becomes false and propagation stays linear in time with the reaction term studied here, as proved by Mellet, Roquejoffre and Sire in [64]. In this paper, we will show that the phenomenon highlighted in [22] persists under a biologically relevant class of nonlinearities that arise in the modelling of Allee effect. Namely f will be of the ignition type :

Assumption A. *$f : [0,1] \to \mathbb{R}$ is a smooth non-negative function, $f = 0$ on $[0,\theta]\cup\{1\}$ with $\theta > 0$, $f(0) = f(1) = 0$, and $f'(1) < 0$. For convenience we will still call f an extension of f on $\mathbb{R}$ by zero at the left of 0 and by its tangent at 1 (so it is negative) at the right of 1.*

With our choice of f, dynamics in the system (3.2) is governed by the travelling waves, which explains our point of view to answer the question through the study of equation (3.1). Replacing the half-plane of [22] by a strip is a technical simplification, legitimate since we are only interested in the propagation in the direction x. Observe that in the light of [23] and the numerical simulations in Chapter 3, translating our results in the half-plane setting seems to be a deep and non-trivial question that goes outside the scope of this paper and will be studied elsewhere.

Our starting point is the following result :

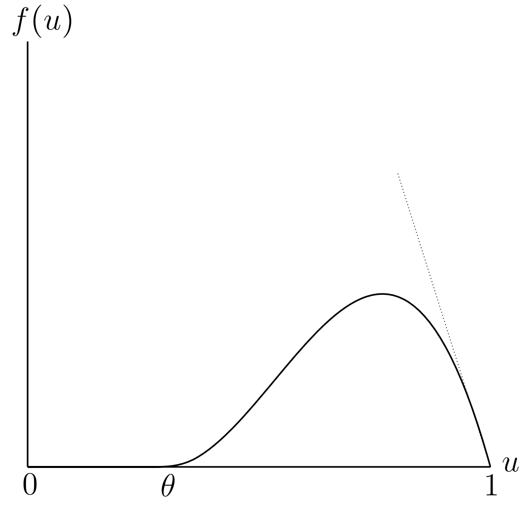


Figure 2.1: Example $f = \mathbf{1}_{u>\theta}(u-\theta)^2(1-u)$

Theorem 1. *([39])*

Let f satisfy Assumption (A). *Then there exists $c(D) > 0$ and u, v smooth solutions of* (3.1). *Moreover, $c(D)$ is unique, u and v are unique up to translations in the x direction, and $u', \partial_x v > 0$.*

The first result we will prove is the following :

Theorem 2. *There exists $c_\infty > 0$ such that*

$$c(D) \sim_{D\to+\infty} c_\infty \sqrt{D}$$

Remark 2.1.1. We would like to point out that in the homogeneous equation in $\mathbb{R}^n$:

$$-d\Delta v + c\partial_x v = f(v) \tag{2.3}$$

it is trivial by uniqueness (see the works of Kanel [52]) that $c(d) = c_0\sqrt{d}$ where c_0 is the velocity solution of (2.3) with $d = 1$. Indeed, to see this, just rescale (2.3) by $\tilde{u}(x) = u(x\sqrt{d})$ and $\tilde{c} = c/\sqrt{d}$. Thus, in Theorem 2 we retrieve the same asymptotic order for $c(D)$ as in the homogeneous case. The comparison between c_0 and c_∞ is an interesting question and we wish to answer it in another paper.

A by-product of the proof of Theorem 2 is the well-posedness for an a priori degenerate elliptic system, where the species of density v would only diffuse vertically, which can be seen as an hypoellipticity result :

Theorem 3. *c_∞ can be characterised as follows : there exists a unique $c_\infty > 0$ and $u \in \mathcal{C}^{2+\alpha}(\mathbb{R}), v \in \mathcal{C}^{1+\alpha/2,2+\alpha}(\Omega_L)$ with $u', \partial_x v > 0$, unique up to translations* *in x that solve*

$$
\begin{array}{ccc}
0 \leftarrow u & -u'' + c_\infty u' = v - \mu u & u \to 1/\mu \\
\hline
 & d\partial_y v = \mu u - v & \\
 & & \\
0 \leftarrow v & c_\infty \partial_x v - d\partial_{yy} v = f(v) & v \to 1 \\
 & & \\
 & \text{-}\partial_y v = 0 & \\
\hline
\end{array}
\tag{2.4}
$$

We will present two proofs of Theorem 3. One by studying the asymptotic behaviour of $c(D)$ thanks to estimates in the same spirit as the ones of Berestycki and Hamel in [9]. Another one of independent interest, by a direct method, showing that the system (2.4) is not degenerate despite the absence of horizontal diffusion in the strip. Both proofs consist in showing a convergence of some renormalised profiles to a limiting profile, solution of the limiting system (2.4).

From now on, we renormalise (c, u, v) in (3.1) by making

$$x \leftarrow \sqrt{D}x, c \leftarrow c/\sqrt{D}$$

ending up with the following equations

$$
\begin{array}{ccc}
0 \leftarrow u & -u'' + cu' = v - \mu u & u \to 1/\mu \\
\hline
 & d\partial_y v = \mu u - v & \\
 & & \\
0 \leftarrow v & -\frac{d}{D}\partial_{xx} v - d\partial_{yy} v + c\partial_x v = f(v) & v \to 1 \\
 & & \\
 & -\partial_y v = 0 & \\
\hline
\end{array}
\tag{2.5}
$$

for which we need to show

$$\lim_{D \to +\infty} c(D) = c_\infty > 0$$

in order to prove Theorem 2.

Before getting into the substance, we would like to mention that there is an important literature about speed-up or slow-down of propagation in reaction-diffusion equations in heterogeneous media and we wish to briefly present some of it.

Some other results

Closest to our work is the recent paper of Hamel and Zlatoš [49], concerned with the speed-up of a combustion front by a shear flow. Their model is :

$$\partial_t v + A\alpha(y)\partial_x v = \Delta v + f(v), \qquad t \in \mathbb{R}, (x, y) \in \mathbb{R} \times \mathbb{R}^{N-1} \tag{2.6}$$

where $A > 1$ is large, and where $\alpha(y)$ is smooth and $(1, \cdots, 1)$-periodic. They show that there exists $\gamma^*(\alpha, f) \geq \fint_{\mathbb{T}^{N-1}} \alpha(y)dy$ such that the velocity $c^*(A\alpha, f)$ of travelling fronts of (2.6) satisfies

$$\lim_{A\to+\infty} \frac{c^*(A\alpha, f)}{A} = \gamma^*(\alpha, f)$$

and under an Hörmander type condition on α[1] they characterise γ^* as the unique admissible velocity for the following degenerate system where $\gamma \in \mathbb{R}, U \in L^\infty$ and $\nabla_y U \in L^2 \cap L^\infty$:

$$\begin{cases} \Delta_y U + (\gamma - \alpha(y))\partial_x U + f(U) = 0 \text{ in } \mathscr{D}'(\mathbb{R} \times \mathbb{T}^{N-1}) \\ 0 \leq U \leq 1 \text{ a.e. in } \mathbb{R} \times \mathbb{T}^{N-1} \\ \lim_{x\to+\infty} U(x, y) \equiv 0 \text{ uniformly in } \mathbb{T}^{N-1} \\ \lim_{x\to-\infty} U(x, y) \equiv 1 \text{ uniformly in } \mathbb{T}^{N-1} \end{cases} \tag{2.7}$$

Let us also give a brief account of other results concerning enhancement of propagation of reaction-diffusion fronts, especially motivated by combustion modelling and in heterogeneous media. In the presence of heterogeneities, quantifying propagation is considerably more difficult than the argument of Remark 2.1.1. The pioneering work in this field goes back to the probabilistic arguments of Freidlin and Gärtner [45] in 1979. They studied KPP-type propagation in a periodic environment and showed that the speed of propagation is not isotropic any more : propagation in any direction is influenced by all the other directions in the environment, and they gave an explicit formula for the computation of the propagation speed.

Reaction-diffusion equations in heterogeneous media since then is an active field and the question of the speed of propagation has received much attention. Around 2000, Audoly, Berestycki and Pommeau [3], then Constantin, Kiselev and Ryzhik [33] started the study of speed-up or slow-down properties of propagation by an advecting velocity field. This study is continued in [54] and later by Berestycki, Hamel and Nadirashvili [14] and Berestycki, Hamel and Nadin [13] through the study of the relation between the principal eigenvalue and the amplitude of the velocity field.

Apart from speed-up by a flow field, the influence of heterogeneities in reaction-diffusion is studied in a series of paper [15, 16] published in 2005 and 2010, where Berestycki, Hamel and Nadirashvili, following [8] gave some new information about the influence of the geometry of the domain and the coefficients of the equation. The first paper deals with a periodic environment, the second with more general domains. In 2010 also, explicit formulas for the spreading speed in slowly oscillating environments were also given for the first time by Hamel, Fayard and Roques in [48].

The influence of geometry on the blocking of propagation was also studied in periodic environment by Guo and Hamel [47] and in cylinders with varying cross-section by Chapuisat and Grenier [32].

[1]Namely, α is smooth and there exists $r \in \mathbb{N}^*$ such that $\sum_{1\leq|\zeta|\leq r} |D^\zeta \alpha(y)| > 0$

The present paper highlights a totally different mechanism of speed-up by the heterogeneity, through a fast diffusion on a line.

Organisation of the paper

The strategy of proof is the following : first, we show that there exists constants $0 < m < M$ independent of D such that $m < c(D) < M$. Then, we show that the limit point of $c(D)$ as $D \to +\infty$ is unique and we characterise it, which proves Theorem 2 and 3. Another section is devoted to the proof of direct existence for system (2.4). More precisely, the organisation is as follows :

- In Section 2.2 we compute positive exponential solutions of the linearised near 0 of (2.5). Those are fundamental to study the tail of the solutions as $x \to -\infty$ for comparison purposes. We use them to show that $c(D) \leq M$.
- Section 2.3 is devoted to showing that $c(D) \geq m$ by proving some integral estimates.
- Section 2.4 proves Theorem 2 by showing the uniqueness of the limiting point $c_\infty > 0$ of $c(D)$. This uses integral identities and a mixed parabolic-elliptic sliding method.
- Finally, in Section 2.5 we construct travelling waves to the limiting system (2.4) by a direct method, proving Theorem 3. For this, we treat x as a time variable and combine standard parabolic and elliptic theory.

2.2 Positive exponential solutions, upper bound

We compute positive exponential solutions of (2.5) with $f = 0$. Those play an important role for comparison purposes as $x \to -\infty$ and in the construction of supersolutions. Looking for $\phi(x) = e^{\lambda x}, \psi(x, y) = e^{\lambda x} h(y)$ with $h > 0$ we get the equations

$$\begin{cases} -h'' + \lambda \left(\frac{c}{d} - \frac{1}{D}\lambda \right) h = 0 \text{ for } y \in (-L, 0) \\ h'(-L) = 0 \\ dh'(0) = \mu - h(0) \\ -D\lambda^2 + c\lambda = h(0) - \mu \end{cases} \tag{2.8}$$

Since we are interested in the asymptotic behaviour of $c(D)$, we can assume $D > d$ and get a solution given by

$$\begin{cases} \psi^D(x, y) = \dfrac{\mu e^{\lambda x} \cosh\left(\beta(\lambda)\left(y + L\right)\right)}{\cosh\left(\beta(\lambda)L\right) + d\beta(\lambda)\sinh\left(\beta(\lambda)L\right)} = e^{\lambda x} h^D(y) \\ \phi^D(x) = e^{\lambda x} \end{cases}$$

where $\beta(\lambda) = \sqrt{\lambda(\frac{c}{d} - \frac{\lambda}{D})}$ and with $c < \lambda < c\frac{D}{d}$ solving

$$-\lambda^2 + c\lambda = \frac{-\mu d\beta(\lambda)\tanh(\beta(\lambda)L)}{1 + d\beta(\lambda)\tanh(\beta(\lambda)L)} \tag{2.9}$$

as pictured in figure 2.2. Moreover, since the right-hand side of (2.9) is a decreasing

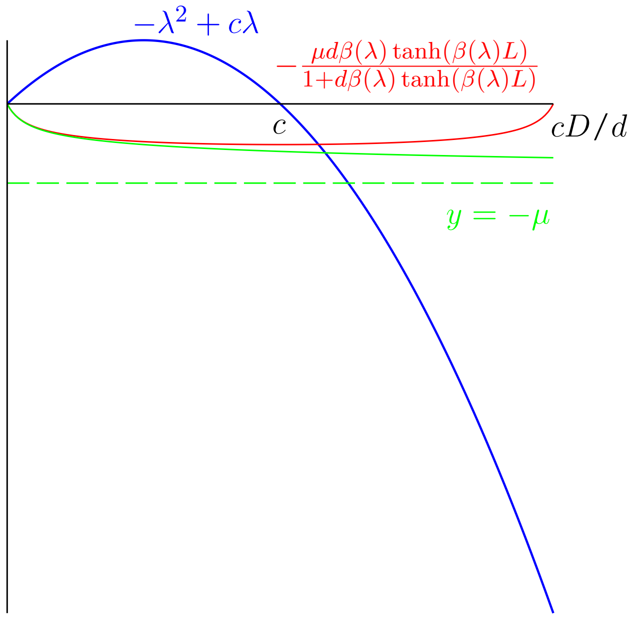


Figure 2.2: Eq. on λ in (2.8)

function of D, we know that λ is an increasing function of D, so $\lambda < \frac{c+\sqrt{c^2+4\mu}}{2}$ (see on Figure 2.2 the horizontal asymptote $-\mu$ of the graph of the right-hand side of the equation when $D = +\infty$). Thus we have the uniform bounds in D :

$$c < \lambda < \frac{c + \sqrt{c^2 + 4\mu}}{2}$$

Remark 2.2.1. Actually we have even better : since the right-hand side of (2.9) converges to

$$\frac{-\mu\sqrt{dc\lambda}\tanh(\sqrt{dc\lambda}L)}{1 + \sqrt{dc\lambda}\tanh(\sqrt{dc\lambda}L)}$$

as $D \to +\infty$, we know that λ increases to the solution of

$$-\lambda^2 + c\lambda = \frac{-\mu\sqrt{dc\lambda}\tanh(\sqrt{dc\lambda}L)}{1 + \sqrt{dc\lambda}\tanh(\sqrt{dc\lambda}L)}$$

as pictured on Figure 2.2.

We will also keep in mind that for every $D \in (d, +\infty]$, $c \mapsto \lambda(D, c)$ is an increasing function, indeed $-\lambda^2 + c\lambda$ is increasing and the right-hand side of the equation is decreasing (it can be written $\frac{-\mu g(c)}{1+g(c)}$ with $g'(c) < 0$).

From now on, we normalise $h(y)$ such that $\min_{y\in[-L,0]} h(y) = 1$ and study the tail of the fronts as $x \to -\infty$.

Proposition 2.2.1. *Let* (c, u, v) *denote the unique solution of* (2.5) *that satisfies*

$$\max_{x\le 0, y\in[-L,0]} (\mu u(x), v(x, y)) = \theta \tag{2.10}$$

and call $m = \min\Big(\min_{y\in[-L,0]} v(0, y), \mu u(0)\Big)$. *Then on* $x \le 0$,

$$\frac{m}{\max h} e^{\lambda x} h(y) \le \mu u, v \le \theta e^{\lambda x} h(y)$$

with the notations of Section 2.2.

Proof. Call $\mu\bar{u} = \theta e^{\lambda x}, \bar{v} = \theta e^{\lambda x} h(y)$. Then $\mu U := \mu(\bar{u} - u), V := \bar{v} - v$ satisfy :

$$0 \leftarrow U \qquad -U'' + cU' = V(x, 0) - \mu U \qquad U \ge 0$$

$$d\partial_y V = \mu U - V(x, 0)$$

$$0 \leftarrow V \qquad -\tfrac{d}{D}\partial^2_{xx} V - d\partial^2_{yy} V + c\partial_x V = 0 \qquad V \ge 0$$

$$\partial_y V = 0$$

Suppose there is a point where $V < 0$. Since V decays to 0 uniformly in y as $x \to -\infty$, V reaches a negative minimum somewhere. By the normalisation condition (2.10), the strong maximum principle and Hopf's lemma (see [11, 46]), it can only be on $x < 0, y = 0$ and at this point we have $\mu U < \min V$.

This is a contradiction : looking at the equation on U, its limit as $x \to -\infty$ and its non-negative value at $x = 0$, we can assert that it reaches a minimum at some $x_U < 0$ where the equation gives $\mu U(x_U) = V(x_U) + U''(x_U) \ge V(x_U) \ge \min V$. In the end, $V \ge 0$ and the maximum principle applied on U gives $U \ge 0$.

The exact same argument applied on $u - \underline{u}, v - \underline{v}$ give the other inequality. $\square$

Proposition 2.2.2. *There is a uniform bound in* D *on the velocity* $c(D)$ *of solutions of* (2.5) *:*

$$c(D) \le \sqrt{\frac{D}{D-d} Lipf} \sim_{D\to+\infty} \sqrt{Lipf}$$

Proof. Call $\mu\bar{u} = \bar{v} = e^{cx}$. A simple computation shows that if

$$c^2(1 - d/D) \ge \mathrm{Lip} f$$

then $(\bar{u}, \bar{v})$ is a supersolution of (2.5). We now use a sliding argument (see [19], [78]):

Since $\lambda > c$ in Prop. 2.2.1, we know that the graph of e^{cx} is asymptotically above the ones of μu and v. Knowing this and since $\mu u, v \leq 1$, we can translate the graph of e^{cx} to the left above the ones of μu and v. Now we slide it back to the right until one of the graphs touch, which happens since $\mu u, v \to 1$ as $x \to +\infty$ whereas $e^{cx} \to 0$ as $x \to -\infty$, uniformly in y. What we just said proves that

$$r_0 = \inf\{t \in \mathbb{R} \mid \bar{v}(t+x,y) - v(x,y) > 0 \textbf{ and } \bar{u}(t+x,y) - u(x) > 0\}$$

exists as an inf over a set that is non-void and bounded by below. Now call

$$U(x) := \bar{u}(r_0 + x) - u(x)$$
$$V(x,y) := \bar{v}(r_0 + x) - v(x,y)$$

By continuity, $U, V \geq 0$. But $\mu U, V$ satisfy

$$\begin{array}{ccc} 0 \leftarrow U & -U'' + cU' = V(x,0) - \mu U & U \to +\infty \\ \hline & d\partial_y V = \mu U - V(x,0) & \\ & & \\ 0 \leftarrow V & -\frac{d}{D}\partial_{xx}^2 V - d\partial_{yy}^2 V + c\partial_x V + k(x,y)V \geq 0 & V \to +\infty \\ & & \\ & \partial_y V = 0 & \\ \hline \end{array}$$

where $k(x,y) = -\dfrac{f(\bar{v}(r_0+x)) - f(v(x,y))}{\bar{v}(r_0+x) - v(x,y)} \in L^\infty$ since f is Lipschitz. Using the strong maximum principle to treat a minimum that is equal to 0 (so that no assumption on the sign of k is needed) and treating the boundary $y = 0$ as above, knowing that $V \not\equiv 0$ we end up with $V > 0$.

But then for any fixed compact $K_a = [-a,a] \times [-L,0]$, $\min_{K_a} V, \min_{[-a,a]} U > 0$ so that we can translate the graph of e^{cx} a little bit more to the right while still being above the ones of μu and v on K_a, i.e. $\bar{u}(r_0 - \varepsilon_a + x) - u(x) > 0$ on $[-a,a]$ and $\bar{v}(r_0 - \varepsilon_a + x) - v(x,y) > 0$ on K_a for $\varepsilon_a > 0$ small enough.

Now just chose a large enough so that on resp. $x < -a$ and $x > a$, $\mu\bar{u}(r_0 + \varepsilon_a + x), \bar{v}(r_0 + \varepsilon_a + x, y), \mu u, v$ are resp. close enough to 0 or large enough so that $k_a(x,y) = -\frac{f(\bar{v}(r_0-\varepsilon_a+x)) - f(v(x,y))}{\bar{v}(r_0-\varepsilon_a+x)-v(x,y)}$ has the sign of $-f'(0) = 0$ or $-f'(1) > 0$. Now the maximum principle applies just like above on $x < -a$ and $x > a$ and concludes that $\bar{u}(r_0 - \varepsilon_a + x) - u(x), \bar{v}(r_0 - \varepsilon_a + x) - v(x,y) > 0$ on the whole $\mathbb{R} \times \Omega_L$, which is a contradiction with the definition of r_0.

In the end, no such $\bar{u}, \bar{v}$ can exist, i.e. $c^2(1 - d/D) \leq \text{Lip} f$.

□

Remark 2.2.2. This proof shows how rigid the equations of fronts are when involving a reaction term with $f'(0), f'(1) \leq 0$: it is shown in [39] that there is no supersolution or subsolution (in a sense defined in [39]) except the solution itself and its translates. This fact was already noted in [18, 78] for Neumann boundary value problems.

2.3 Proof of the lower bound

In this section we show the following :

Proposition 2.3.1.

$$\inf_{D>d} c(D) = c_{min} > 0$$

We proceed by contradiction. Suppose that $\inf c(D) = 0$. Then there exists a sequence $D_n \to \infty$ (since c is a continuous function of D, see [39]) such that the associated solutions (c_n, ϕ_n, ψ_n) satisfy $c_n \to 0$. Moreover, integrating by parts the equation on v in (2.5) and using elliptic estimates to assert $u', \partial_x v \to 0$ as $x \to \pm\infty$ we get

$$c_n = \frac{1}{L + 1/\mu} \int_{\Omega_L} f(\psi_n)$$

so we know that $\int_{\Omega_L} f(\psi_n) \to 0$ which also gives

$$\int_{\Omega_L} f(\psi_n)\psi_n \leq \int_{\Omega_L} f(\psi_n) \to 0$$

Multiplying the equation by ψ_n and integrating by parts yields

$$\frac{d}{D_n}\int_{\Omega_L} \partial_x \psi_n^2 + d\int_{\Omega_L} \partial_y \psi_n^2 + \int_{\mathbb{R}} \phi_n' \partial_x \psi_n(\cdot, 0) + c_n \int_{\mathbb{R}} \phi_n' \psi_n(\cdot, 0) + \frac{c_n L}{2} = \int_{\Omega_L} f(\psi_n)\psi_n \tag{2.11}$$

All the terms in the left hand side of this expression are positive quantities, so each one of them must go to zero as $n \to \infty$. Now, we normalise ψ_n by

$$\psi_n(0, 0) = \theta_1 \in]\theta, 1[$$

and assert the following :

Lemma 2.3.1. *Fix $\delta > 0$ small. There exists $N > 0$ such that for all $n > N$ we have for all $-1 \leq x \leq 1$:*

$$\left(1 - \frac{\delta}{2}\right)\theta_1 < \psi_n(x, 0) < \left(1 + \frac{\delta}{2}\right)\theta_1$$

Before giving the proof, we mention an easy but technical lemma that will be used :

Lemma 2.3.2. *If $k \in L^1$, $\hat{k} \in \mathcal{C}^\infty \cap L^2$ and $h \in L^\infty$ then the formula*

$$\mathcal{F}^{-1}(\hat{k}\hat{h}) = k * h$$

makes sense and holds.

Proof. Since $\hat{k}$ is a smooth function, the product distribution $\hat{k}\hat{h}$ makes sense and we can compute its inverse Fourier transform : the result follows by using the classical properties of the Fourier transform on L^2 and the Fubini-Tonelli theorem. $\square$

We now turn to the proof of lemma 2.3.1.

Proof. We know that $\int_{\mathbb{R}} \phi_n' \partial_x \psi_n(\cdot, 0) \to 0$. Note that

$$-\phi_n'' + c_n \phi_n' + \mu \phi_n = \psi_n(\cdot, 0)$$

and so by lemma 2.3.2 and the fact that $\xi^2 - c_n i\xi + \mu$ has no real roots, we have $\phi_n = K_n * \psi_n(\cdot, 0)$ where $\hat{K_n}(\xi) = \frac{1}{\xi^2 - c_n i \xi + \mu}$, i.e.

$$K_n(x) = \sqrt{\frac{2\pi}{c_n^2 + 4\mu}} e^{-\frac{1}{2}\left(\sqrt{c_n^2+4\mu} - c_n\right)x} \left(e^{\sqrt{c_n^2+4\mu}x} H(-x) + H(x)\right)$$

This is a sequence of positive functions, uniformly bounded from below by a positive constant on any compact subset of $\mathbb{R}$.

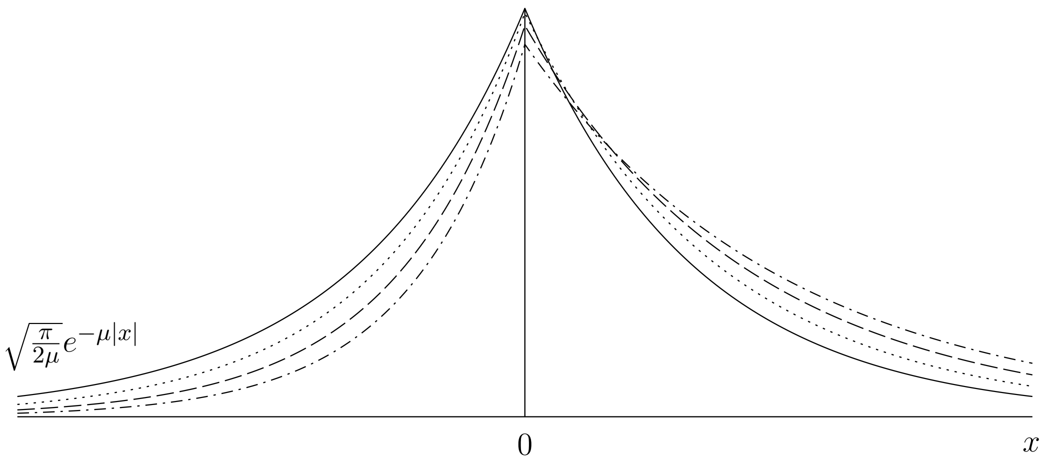


Figure 2.3: Graphs of K_n for $\mu = 1$, $c_n \in \{0, 0.3, 0.6, 0.9\}$

Now for $-1 \leq x \leq 1$ and since $\partial_x \psi_n \geq 0$ we have

$$|\psi_n(x, 0) - \theta_1| \leq \int_{-1}^{1} \partial_x \psi_n(\cdot, 0)$$

But since $K_n, \partial_x \psi_n \geq 0$ and since $K_n > \alpha_0 > 0$ on $[-2, 2]$:

$$\begin{aligned}
\int_{\mathbb{R}} (\partial_x \psi_n) \phi_n' \geq \int_{\mathbb{R}} (K_n * \partial_x \psi_n) \partial_x \psi_n &\geq \int_{-1}^{1} \left(\int_{\mathbb{R}} K_n(x-t) \partial_x \psi_n(t, 0) dt \right) \partial_x \psi_n(x, 0) dx \\
&\geq \int_{-1}^{1} \left(\int_{-1}^{1} K_n(x-t) \partial_x \psi_n(t, 0) dt \right) \partial_x \psi_n(x, 0) dx \\
&> \alpha_0 \left(\int_{-1}^{1} \partial_x \psi_n(\cdot, 0) \right)^2
\end{aligned}$$

Thus

$$\int_{-1}^{1} \partial_x \psi_n(\cdot,0) \leq \left(\frac{1}{\alpha_0}\int_{\mathbb{R}} (\partial_x \psi_n)\, \phi_n'\right)^{1/2} \to 0$$

and for n large enough this quantity is less than $\delta\theta_1/2$.

$\square$

Lemma 2.3.3. *Fix $0 < \delta < \frac{2\sqrt{2L}}{\theta_1}$. There exists $N' > 0$ such that for all $n > N'$, there exists a borelian J_n of $[-1,1]$ with measure ≥ 1 such that for all $x \in J_n$:*

$$\left(\int_{-L}^{0} \partial_y \psi_n(x,s)^2 ds\right)^{1/2} \leq \frac{\delta\theta_1}{2\sqrt{L}}$$

Proof. This result is based on a Markov-type inequality. Call

$$h(x) := \int_{-L}^{0} \partial_y \psi_n(x,s)^2 ds$$

We now use that

$$0 \leq \int_{-1}^{1} h(x)dx \leq \int_{\mathbb{R}} \left(\int_{-L}^{0} \partial_y \psi_n(x,s)^2 ds\right) dx \to 0$$

Thus for all $\varepsilon > 0$, there exists $N' > 0$ such that for all $n > N'$, $\int_{-1}^{1} h(x)dx \leq \varepsilon$. Using then that $h(x) \geq \sqrt{\varepsilon}\mathbf{1}_{h>\sqrt{\varepsilon}}$ we get that

$$|\{x \in [-1,1] \mid h(x) > \sqrt{\varepsilon}\}| < \frac{1}{2}\sqrt{\varepsilon}$$

where $|A|$ denotes the Lebesgue measure of A. We get the result by choosing

$$J_n = \{x \in [-1,1] \mid h(x) \leq \sqrt{\varepsilon}\}$$

and $\varepsilon = \delta^4\theta_1^4/16L^2$. The fact that $|J_n| \geq 1$ directly comes from the upper bound assumed on δ. $\square$

We can now finish the proof. Indeed, for $n > N, N'$ fixed and for $x \in J_n$ we have

$$\psi_n(x,0) - \int_{-L}^{0} |\partial_y(x,s)ds| \leq \psi_n(x,y) \leq \psi_n(x,0) + \int_{-L}^{0} |\partial_y(x,s)ds|$$

Using the Cauchy-Schwartz inequality and Lemmas 2.3.1 and 2.3.3, we get on $J_n \times [-L,0]$

$$(1-\delta)\theta_1 \leq \psi_n(x,y) \leq (1+\delta)\theta_1$$

And so

$$\left(1 + \frac{L}{\mu}\right) c_n = \int_{\Omega_L} f(\psi_n) \geq \int_{J_n\times[-L,0]} f(\psi_n) \geq L \inf_{J_n\times[-L,0]} f(\psi_n)$$

Choosing now θ_1 and δ such that $f([(1-\delta)\theta_1, (1+\delta)\theta_1]) > C > 0$ we get a contradiction with the assumption $c_n \to 0$.

2.4 The equivalent $c(D) \sim c_\infty\sqrt{D}$

We know that every sequence $c(D_n)$ associated to a sequence $D_n \to +\infty$ is trapped between two positive constants. Now we just have to show the uniqueness of the limit point. We divide the proof in three steps.

- Compactness : we prove that any (ϕ_n, ψ_n) associated to $D_n \to \infty$ and $c_n \to c > 0$ is bounded in H^3_{loc}. This uses integral identities.
- Treating the x variable as a time variable, we extract from such a family a (c, ϕ, ψ) that solves (2.5) with $D = +\infty$.
- We show uniqueness of c for such a problem using a parabolic version of the arguments in Proposition 2.2.2.

But first, let us give an easy but technical lemma that will be used in the next computations.

Lemma 2.4.1. *Gagliardo-Nirenberg and Ladyzhenskaya type inequalities in Ω_L*

- *For all $\alpha \in]\frac{1}{3}, 1[$, there exists $C_{GN} > 0$ s.t. for all $u \in H^1(\Omega_L)$*

$$|u|_{L^3(\Omega_L)} \le C_{GN} |u|^{1-\alpha}_{L^2(\Omega_L)} |u|^{\alpha}_{H^1(\Omega_L)}$$

- *For all $\beta \in]\frac{1}{2}, 1[$, there exists $C_L > 0$ s.t. for all $u \in H^1(\Omega_L)$*

$$|u|_{L^4(\Omega_L)} \le C_L |u|^{1-\beta}_{L^2(\Omega_L)} |u|^{\beta}_{H^1(\Omega_L)}$$

Proof. The inequalities above with $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$ are resp. Gagliardo-Nirenberg and Ladyzhenskaya inequalities. We can prove that these are still valid in Ω_L by re-doing the computations of Nirenberg and using trace inequalities, but the inequalities above will suffice for us and are easier to prove.

For this, we just use the Hölder interpolation inequality : if $p, q, r \ge 1$ and $\alpha \in [0, 1]$ such that $\frac{1}{q} = \frac{1-\alpha}{r} + \frac{\alpha}{s}$, then $|u|_{L^q} \le |u|^{1-\alpha}_{L^r} |u|^{\alpha}_{L^s}$. We apply this with $q = 3$ resp. 4, $r = 2$, $s \to +\infty$, and control the $|u|^{\alpha}_{L^s}$ part with the Sobolev embedding $|u|^{\alpha}_{L^s} \le C|u|^{\alpha}_{H^1}$ for $s \ge 2$. □

We are now able to treat Step 1 :

Lemma 2.4.2. *Let $D_n \to \infty$, (c_n, ϕ_n, ψ_n) the sequence of solutions associated to D_n and suppose $c_n \to c > 0$. Denote $\Omega_{L,M} := [-M, M] \times [-L, 0]$. Then for every $M \in \mathbb{N}$, there exists $C_M > 0$ s.t. $|\psi_n|_{H^3(\Omega_{L,M})} \le C_M$.*

Proof. a) The H^1 bound.

First, since $0 \le \psi_n \le 1$, we know that (ψ_n) is bounded in $L^2(\Omega_{L,M})$. Then we start from (2.11) : since (c_n) is bounded and because the right-hand side of (2.11) is bounded, we have that $(\partial_y \psi_n)$ is bounded in $L^2(\Omega_L)$.

Then multiply the equation inside Ω_L in (2.5) by $\partial_x\psi_n$ and in a similar fashion as (2.11), integrate by parts on $\Omega_{L,M}$. Boundary terms along the y axis decay thanks to elliptic estimates. Using $\partial_x\psi_n > 0$ and $|\phi_n|_{W^{2,\infty}(\mathbb{R})} \leq C$ (use Fourier transform of the variation of constants) we get the sum of following terms

- $-\dfrac{d}{D_n}\displaystyle\int_{\Omega_L} \partial^2_{xx}\psi_n\partial_x\psi_n = 0$
- $\displaystyle\int_{\Omega_L} -d\partial^2_{yy}\psi_n\partial_x\psi_n = \cancel{\int_{\Omega_L}\partial_x\left(\frac{1}{2}d\partial_y\psi_n^2\right)} + \int_{\mathbb{R}}(-\phi_n'' + c_n\phi_n')\partial_x\psi_n$ so
$$\left|\int_{\Omega_L} -d\partial^2_{yy}\psi_n\partial_x\psi_n\right| \leq C$$
- $\displaystyle\left|\int_{\Omega_L} f(\psi_n)\partial_x\psi_n\right| \leq \sup f \times L$
- Finally, $c_n\displaystyle\int_{\Omega_L}\partial_x\psi_n^2$, which we want to bound.

so that in the end

$$|\partial_x\psi_n|^2_{L^2(\Omega_L)} \leq \frac{L\sup f + C}{c_{min}} =: C_1$$

b) The H^2 bound

We apply the same process, on the equation satisfied by $(z_n, w_n) := (\phi_n', \partial_x\psi_n)$. The linear structure is the same, but this time we do not have positivity of the first x derivative any more. Multiplying by $w_n := \partial_x\psi_n > 0$ the equation satisfied by w_n and integrating gives rise to the following terms :

- $\displaystyle\int_{\Omega_L}\left(-\frac{d}{D_n}\partial^2_{xx}w_n\right)w_n = \frac{d}{D_n}\int_{\Omega_L}\partial_x w_n^2 \geq 0$
- $c_n\displaystyle\int_{\Omega_L}(\partial_x w_n)\,w_n = 0$ thanks to elliptic estimates.
- $\displaystyle\left|\int_{\Omega_L} f'(\psi_n)w_n^2\right| \leq |f'|_\infty C_1$ by the result above.
- $\displaystyle\int_{\Omega_L}\left(-d\partial^2_{yy}w_n\right)w_n = \int_{\Omega_L} d\partial_y w_n^2 + \int_{\mathbb{R}}(w_n - \mu z_n)w_n$ where
$$\int_{\mathbb{R}}(w_n - \mu z_n)w_n \geq \int_{\mathbb{R}} w_n^2 - \mu C \geq -\mu C$$

so that in the end

$$|\partial_y w_n|^2_{L^2(\Omega_L)} \leq \frac{|f'|_\infty C_1 + \mu C}{d} =: C_2$$

We now multiply by $\partial_x w_n$ the equation satisfied by w_n. Integration yields the sum of the following terms :

- $\int_{\Omega_L} \left(-\frac{d}{D_n} \partial^2_{xx} w_n \right) \partial_x w_n = 0$
- $c_n \int_{\Omega_L} \partial_x w_n^2$ which we want to bound.
- $\int_{\Omega_L} (-d\partial^2_{yy} w_n) \partial_x w_n = \int_{\mathbb{R}} (w_n - \mu z_n) \partial_x w_n = \mu \int_{\mathbb{R}} z_n' w_n$ so that

$$\left| \int_{\Omega_L} (-d\partial^2_{yy} w_n) \partial_x w_n \right| \leq \mu C$$

- $\int_{\Omega_L} f'(\psi_n) w_n \partial_x w_n = -\frac{1}{2} \int_{\Omega_L} f''(\psi_n) w_n^3$ so that

$$\begin{aligned} \left| \int_{\Omega_L} f'(\psi_n) w_n \partial_x w_n \right| \leq \frac{|f''|_\infty}{2} |w_n|^3_{L^3} &\leq \frac{|f''|_\infty}{2} C^3_{GN} \sqrt{C_1}^{2-\varepsilon} |w_n|^{1+\varepsilon}_{H^1(\Omega_L)} \\ &=: C_3 |w_n|^{1+\varepsilon}_{H^1} \end{aligned}$$

  for a small $\varepsilon > 0$ of our choice thanks to the Gagliardo-Nirenberg type inequality of Lemma 2.4.1.

Finally, we end up with the inequality

$$|w_n|^2_{H^1(\Omega_L)} \leq C_1 + C_2 + \mu C + C_3 |w_n|^{1+\varepsilon}_{H^1(\Omega_L)}$$

which yields the bound

$$|w_n|^2_{H^1} \leq C_4$$

Then we use the original equation to assert that $\partial^2_{yy} \psi_n$ is bounded in $L^2(\Omega_L)$, so that in the end the lemma is true with H^2_{loc}.

c) The H^3 bound

For the H^3 estimate, we iterate one last time with the equation satisfied by $(\tau_n, \rho_n) := (\phi'', \partial^2_{xx} \psi_n)$. Multiplying the equation by ρ_n and integrating gives as before a non-negative and a zero term, and the boundary integral as well as the right-hand side have to be studied more carefully :

- $\int_{\Omega_L} \left(-d\partial^2_{yy} \rho_n \right) \rho_n = \int_{\Omega_L} d\partial_y \rho_n^2 + \int_{\mathbb{R}} \rho_n^2 - \mu \int_{\mathbb{R}} \tau_n \rho_n$. But

$$\left| \int_{\mathbb{R}} \rho_n \tau_n \right| = \left| \int_{\mathbb{R}} \tau_n' \partial_x \psi_n \right| \leq |\tau_n'|_{L^2} |\partial_x \psi_n(\cdot, 0)|_{L^2} \leq C_5$$

  by elliptic estimates, since the ϕ_n' satisfy an uniformly elliptic equation on $\mathbb{R}$ with uniformly bounded coefficients and with data $\partial_x \psi_n(\cdot, 0)$ bounded in $H^{1/2}$, so $(\phi_n)'$ is bounded in $H^{2+1/2}$, which gives (τ_n') bounded in $H^{1/2}$ so in L^2 (this can be seen easily through the Fourier transform). The second factor is bounded by the above result and the continuity of the trace operator on Ω_L.

- The right-hand side is $f''(\psi_n)\partial_x\psi_n^2\rho_n + f'(\psi_n)\rho_n^2$. For the first term, we use that

$$\begin{aligned}\left|\int_{\Omega_L}(\partial_x\psi_n)^2\rho_n\right| &\leq \left(\int_{\Omega_L}\partial_x\psi_n^4\right)^{1/2}\left(\int_{\Omega_L}\rho_n^2\right)^{1/2}\\ &\leq C_L^2|\partial_x\psi_n|_{L^2}^{1-\varepsilon}|\partial_x\psi_n|_{H^1}^{1+\varepsilon}|\rho_n|_{L^2}\\ &\leq C_L^2\sqrt{C_1}^{1-\varepsilon}\sqrt{C_4}^{2+\varepsilon} =: C_6\end{aligned}$$

for some small $\varepsilon > 0$ of our choice thanks to the Ladyzhenskaya type inequality of Lemma 2.4.1.

For the second term we just have

$$\left|\int_{\Omega_L} f'(\psi_n)\rho_n^2\right| \leq |f'|_\infty C_4$$

so this procedure gives

$$|\partial_y\rho_n|^2_{L^2(\Omega_L)} \leq \frac{C_5 + C_6 + |f'|_\infty C_4}{d} =: C_7$$

Now multiplying by $\partial_x\rho_n$ and integrating gives a zero term, the $\int\partial_x\rho_n^2$ term we want to bound, and a boundary integral as well as a right hand side that are the following :

- $\int_{\mathbb{R}}(\rho_n - \mu\tau_n)\partial_x\rho_n = -\mu\int_{\mathbb{R}}\tau_n\partial_x\rho_n = -\mu\int_{\mathbb{R}}\phi_n''\partial_{xxx}\psi_n$. We bound this term thanks to a "fractional integration by parts" : indeed, we need to transfer more than one derivative on ϕ, but we do not control ϕ in H^4, only in $H^{3+1/2}$. For this, we use Plancherel's identity :

$$\begin{aligned}&\left|\int_{\mathbb{R}}\phi_n''\partial_{xxx}\psi_n\right| = \left|\int_{\mathbb{R}} -\xi^2\hat{\phi}_n i\xi^3\hat{\psi}_n\right|\\ &\leq \int_{\mathbb{R}}|\xi|^{2+1/2}|i\xi\hat{\phi}_n||\xi|^{1/2}|i\xi\hat{\psi}_n|\\ &\leq \left(\int_{\mathbb{R}}|\xi|^{2(2+1/2)}|i\xi\hat{\phi}_n|^2\right)^{1/2}\left(\int_{\mathbb{R}}|\xi|^{2(1/2)}|i\xi\hat{\psi}_n|^2\right)^{1/2}\\ &\leq \left(\int_{\mathbb{R}}\left(1+|\xi|^2\right)^{2+1/2}|i\xi\hat{\phi}_n|^2\right)^{1/2}\left(\int_{\mathbb{R}}\left(1+|\xi|^2\right)^{1/2}|i\xi\hat{\psi}_n|^2\right)^{1/2}\\ &\leq |\phi_n'|_{H^{2+1/2}}|\partial_x\psi_n(\cdot,0)|_{H^{1/2}}\\ &\leq CC_{tr}\sqrt{C_4}\end{aligned}$$

where C_{tr} is a bound for the trace operator on Ω_L.

- The right-hand side is $\int f''(\psi_n)\partial_x\psi_n^2\partial_x\rho_n + f'(\psi_n)\rho_n\partial_x\rho_n$. The first term is controlled thanks to Cauchy-Schwarz and Ladyzhenskaya's inequality again :

$$\begin{aligned}\left|\int_{\Omega_L} f''(\psi_n)\partial_x\psi_n^2\partial_x\rho_n\right| &\le |f''|_\infty \int_{\Omega_L} |2\rho_n\partial_x\psi_n\partial^2_{xx}\psi_n| \\ &\le 2|f''|_\infty|\rho_n|_{L^2}|\partial_x\psi_n\partial^2_{xx}\psi_n|_{L^2} \\ &\le 2|f''|_\infty|\rho_n|_{L^2}|\partial_x\psi_n|_{L^4}|\partial^2_{xx}\psi_n|_{L^4} \\ &\le 2|f''|_\infty C_L^2\sqrt{C_4}^{5/2-\varepsilon}|\rho_n|^{1/2+\varepsilon}_{H^1(\Omega_L)} \\ &=: C_8|\rho_n|^{1/2+\varepsilon}_{H^1(\Omega_L)}\end{aligned}$$

by applying Ladyzhenskaya's inequality twice.
The last term gives $\int_{\Omega_L} f'(\psi_n)\rho_n\partial_x\rho_n = -\int_{\Omega_L}\frac{1}{2}\rho_n^2 f''(\psi_n)\partial_x\psi_n$ so by Cauchy-Schwarz :

$$\begin{aligned}\left|\int_{\Omega_L} f'(\psi_n)\rho_n\partial_x\rho_n\right| &\le \frac{|f''|_\infty}{2}\left(\int_{\Omega_L}|\rho_n|^4\right)^{1/2}|\partial_x\psi_n|_{L^2(\Omega_L)} \\ &\le \frac{|f''|_\infty}{2}|\rho_n|^2_{L^4}\sqrt{C_1} \\ &\le \frac{|f''|_\infty}{2}C_L^2\sqrt{C_1}|\rho_n|^{1-\varepsilon}_{L^2}|\rho_n|^{1+\varepsilon}_{H^1} \\ &\le \frac{|f''|_\infty}{2}C_L^2\sqrt{C_1}\sqrt{C_4}^{1-\varepsilon}|\rho_n|^{1+\varepsilon}_{H^1} \\ &:= C_9|\rho_n|^{1+\varepsilon}_{H^1}\end{aligned}$$

by Ladyzhenskaya's inequality again.
As before, in the end we get

$$|\rho_n|^2_{H^1} \le C_4 + C_7 + CC_{tr}\sqrt{C_4} + C_8|\rho_n|^{1/2+\varepsilon}_{H^1(\Omega_L)} + C_9|\rho_n|^{1+\varepsilon}_{H^1}$$

which yields the boundedness of $|\rho_n|_{H^1}$.
Finally, the terms $\partial_{xyy}\psi_n$ and $\partial_{yyy}\psi_n$ are bounded in $L^2(\Omega_L)$ thanks to the equation : if we differentiate the original equation on ψ_n in y and then in x, the result is immediate.

□

We could go on again to H^4 by looking at the third derivatives, but the right-hand side would involve too much computations and interpolation inequalities. Instead, we stop here and use the following lemma.

Lemma 2.4.3. *With the assumptions of Lemma 2.4.2, there exists* $(\phi,\psi) \in \mathcal{C}^2(\mathbb{R}) \times \mathcal{C}^{1x,2y}(\Omega_L)$ *such that* (c,ϕ,ψ) *satisfies* (2.4) *and* $\phi', \partial_x\psi \ge 0$.

Proof. Thanks to Lemma 2.4.2, (ψ_n) is bounded in H^3_{loc} so by Rellich's theorem we can extract from it a sequence that converges strongly in H^2_{loc} to $\psi \in H^3_{loc}$. Moreover, thanks to the Sobolev embedding $H^3(\Omega_{L,M}) \hookrightarrow \mathcal{C}^{1,\gamma}$ for every $0 < \gamma <$

1, by Ascoli's theorem and the process of diagonal extraction we can assume that ψ_n converges in $\mathcal{C}^{1,\beta}$ to $\psi \in \mathcal{C}^{1,\beta}$ for some $0 < \beta < 1$ fixed. By elliptic estimates, (ϕ_n) is bounded in $\mathcal{C}^{3,\gamma}(\mathbb{R})$ for every $0 < \gamma < 1$, so again, we can still extract and assume that ϕ_n converges to a $\phi \in \mathcal{C}^{3,\beta}$ in the $\mathcal{C}^{3,\beta}$ norm.

Since f satisfies $f(0) = 0$ and is Lipschitz continuous, we can assert that $f(\psi_n)$ converges to $f(\psi)$ in H^2_{loc}. Then we can pass to the L^2 limit $n \to \infty$ in equation (2.5) satisfied by (c_n, ϕ_n, ψ_n) and see that (2.4) is satisfied a.e. Moreover, ϕ' and $\partial_x\psi$ are non-negative as locally Hölder limits of positive functions. Finally, if we fix $x_0 \in \mathbb{R}$ we assert that $\psi(x_0, \cdot) \in L^2(-L, 0)$ and $\psi \in \mathcal{C}^1([x_0, +\infty[, L^2(]-L, 0[) \cap \mathcal{C}^0(]x_0, +\infty[, H^2(]-L, 0[)$. Indeed, this comes from $\psi \in H^3_{loc}$ and Jensen's inequality. For instance for $x > x_0$ and h small :

$$\begin{aligned}
\left|\partial^2_{yy}\psi(x+h,\cdot) - \partial^2_{yy}\psi(x,\cdot)\right|_{L^2(-L,0)} &= \left(\int_{-L}^0 \left(\partial^2_{yy}\psi(x+h,y) - \partial^2_{yy}\psi(x,y)\right)^2 dy\right)^{1/2} \\
&= \left(\int_{-L}^0 \left(\int_x^{x+h} \partial_{xyy}\psi(s,y)ds\right)^2 dy\right)^{1/2} \\
&\le \left(\int_{-L}^0 \left(\int_x^{x+h} \partial_{xyy}\psi(s,y)^2 ds\right) dy\right)^{1/2} \\
&\le |\psi|_{H^3(]x,x+h[\times[-L,0])} \to 0
\end{aligned}$$

as $h \to 0$ as the integral of an integrable function over a set whose measure tends to zero. It is known (see [26], Section 10) that such a solution is unique and has $\mathcal{C}^{1_x, 3+\gamma_y}$ regularity on every $[x_0 + \varepsilon, +\infty[\times[-L, 0]$. Since we can do this for every $x_0 \in \mathbb{R}$, the regularity announced in the lemma is proved. The uniform limit to the left is obtained thanks to Prop. 2.2.1 : on $x \le 0$

$$\mu\phi_n, \psi_n \le \theta e^{\lambda_n x} h_n(y) \le \theta e^{mx} h_-$$

where $h_- > 0$ is a uniform lower bound on h_n whose existence is proved in the next lemma.

The right limits are obtained in a similar fashion as in [17] by integration by parts and by using standard parabolic estimates instead of elliptic ones. See Lemma 2.5.1 and Prop. 2.5.10 in the next section for similar and complete computations. $\square$

We conclude this section with the following lemmas, that prove the uniqueness of the limit point c.

Lemma 2.4.4. *Suppose c and $\bar{c} > c$ are two limit points of $c(D)$ and denote (c_n, ϕ_n, ψ_n) and $(\bar{c}_n, \underline{\phi}_n, \underline{\psi}_n)$ some associated sequences of solutions that converge to (c, ϕ, ψ) and $(\bar{c}, \underline{\phi}, \underline{\psi})$ as in the previous theorem. Then there exists $X \in \mathbb{R}$ and $N \in \mathbb{N}$ s.t. for all $x \le X$ and $n \ge N$, $\underline{\psi}_n(x,y) < \psi_n(x,y)$.*

Proof. This relies on comparison with exponential solutions computed in section 2.2 and on the uniform convergence of ψ_n resp. $\underline{\psi}_n$ to ψ resp. $\underline{\psi}$. Indeed, if as in section 2.2 we denote $\bar{\lambda}_n$ and $\underline{h}_n$ the exponent and the y part of the exponential solutions, we claim that :

$$\exists \underline{h}_-, \underline{h}_+ > 0 \mid \underline{h}_- < \underline{h}_n(y) < \underline{h}_+$$

Indeed, since $\bar{c}_n \to \bar{c}$, there exists $\bar{c}_+ > 0$ s.t. $c_n < \bar{c}_+$. Then

$$\beta_n(\bar{\lambda}_n) < \sqrt{\frac{\bar{c}_+\left(\bar{c}_+ + \sqrt{\bar{c}_+^2 + 4\mu}\right)}{2d}} =: \underline{\beta}_+$$

so that

$$\underline{h}_n(y) \leq \mu \cosh(\underline{\beta}_+ L) =: \underline{h}_+$$

and

$$\underline{h}_n(y) \geq \frac{\mu}{\cosh(\underline{\beta}_+ L) + d\underline{\beta}_+ \sinh(\underline{\beta}_+ L)} =: \underline{h}_-$$

The same holds for $h_n(y)$ with constants $h_+, h_- > 0$.

Now normalise $\underline{h}_n$ s.t. $\min_{[-L,0]} \underline{h}_n = 1$. Then we have $\underline{h}_n \leq \frac{\underline{h}_+}{\underline{h}_-}$, and Prop. 2.2.1 yields on $x \leq 0$:

$$\mu\underline{\phi}_n, \underline{\psi}_n \leq \theta e^{\bar{\lambda}_n x} \frac{\underline{h}_+}{\underline{h}_-}$$

On the other hand, there exists $N \in \mathbb{N}$ s.t. for $n \geq N$

$$m_n = \min\left(\min_{y\in[-L,0]} \psi(0,y), \mu\phi(0)\right) \geq \frac{1}{2} \min\left(\min_{y\in[-L,0]} \psi(0,y), \mu\phi(0)\right) =: m$$

so that using Prop. 2.2.1 again on $x \leq 0$, for $n \geq N$,

$$\mu\phi_n, \psi_n \geq m e^{\lambda_n x} \frac{h_-}{h_+}$$

Finally, by monotonicity of λ (see Remark 2.2.1) $\lambda_n \to \lambda(+\infty, c)$ and $\bar{\lambda}_n \to \lambda(+\infty, \bar{c}) > \lambda(+\infty, c)$, so for $n \geq N$ large enough, $\overline{\lambda}_n - \lambda_n > d := \frac{1}{2}(\lambda(+\infty, \bar{c}) - \lambda(+\infty, c)) > 0$. Now for

$$x < \frac{1}{d} \ln\left(\frac{m h_- \underline{h}_-}{\theta h_+ \underline{h}_+}\right) =: X$$

we have the inequality announced.

□

Lemma 2.4.5. *If (c, ϕ, ψ) and $(\bar{c}, \underline{\phi}, \underline{\psi})$ solve the equations (2.4) in the conclusion of Lemma 2.4.3, then $c = \bar{c}$.*

Proof. Since these solutions have classical regularity, we can apply the strong parabolic maximum principle and the parabolic Hopf's lemma (see for instance [11, 43]) in a similar fashion as the elliptic case of Proposition 2.2.2.

First, observe that

$$c\partial_x\underline{\psi} - d\partial^2_{yy}\underline{\psi} = f(\underline{\psi}) + (c-\bar{c})\partial_x\underline{\psi} \leq f(\underline{\psi})$$

Now, slide $\mu\phi, \psi$ to the left above $\mu\underline{\phi}, \underline{\psi}$ this way : just do it on a slice $x = a$ with $a > 0$ large enough so that on $x > a$ we know the sign of $\frac{f(\psi)-f(\underline{\psi})}{\psi-\underline{\psi}}$ and can use the parabolic maximum principle with initial "time" $x = a$ (dotted line below) :

$$\begin{array}{ccc}
0 \leftarrow \phi - \underline{\phi} & -(\phi-\underline{\phi})'' + c(\phi-\underline{\phi})' = (\psi-\underline{\psi}) - \mu(\phi-\underline{\phi}) & \phi - \underline{\phi} \to 0 \\
\hline
 & d\partial_y(\psi-\underline{\psi}) + (\psi-\underline{\psi}) = \mu(\phi-\underline{\phi}) & \\
0 \leftarrow \psi - \underline{\psi} & c\partial_x(\psi-\underline{\psi}) - d\partial^2_{yy}(\psi-\underline{\psi}) - \frac{f(\psi)-f(\underline{\psi})}{\psi-\underline{\psi}}(\psi-\underline{\psi}) \geq 0 & \psi - \underline{\psi} \to 0 \\
 & \partial_y(\psi-\underline{\psi}) = 0 & \\
\hline
\mu\phi, \psi > \mu\underline{\phi}, \underline{\psi} > 1-\varepsilon & &
\end{array}$$

Treating the upper boundary as before, we obtain that $\mu\phi, \psi > \mu\underline{\phi}, \underline{\psi}$ on $x \geq a$. Using Proposition 2.4.4, the order is also true for x negative enough, so that there is only a compact rectangle left where the order is needed : for this, just slide $\mu\phi, \psi$ enough to the left.

Now, as before, slide back to the right until the order is not true any more, finishing with the minimum possible translate $\mu\phi(r_0 + x) \geq \mu\underline{\phi}(x), \psi(r_0 + x, y) \geq \underline{\psi}(x, y)$. The strong parabolic maximum principle (without sign assumption) gives that the order is still strict (use a starting x smaller than the x where an eventual contact point happens) since $\underline{\psi} \not\equiv \psi$. Thus, on any compact K_a as large as we want, we can slide $\mu\phi, \psi$ ε_a more to the right again, the order still being true on K_a. Now just chose a large enough so that $-a < X + r_0 - \varepsilon_a$, and so that on $x > a$ we know the sign of $\frac{f(\psi)-f(\underline{\psi})}{\psi-\underline{\psi}}$: Prop. 2.4.4 and the parabolic strong maximum principle give that $\mu\phi(r_0 - \varepsilon_a + x) > \mu\underline{\phi}(x), \psi(r_0 - \varepsilon_a + x, y) > \underline{\psi}(x, y)$ which is a contradiction.

□

Remark 2.4.1.

- We could avoid the use of exponential solutions in the proof of Lemma 2.4.4 : indeed, by considering some fixed translates ϕ^r_n, ψ^r_n of ϕ, ψ we can have, for n large enough thanks to the locally uniform convergence and if r is large enough

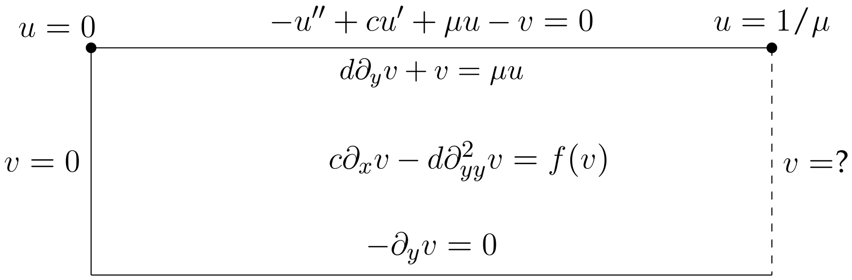
  $$\psi^r_n(-a, y) \geq (1-\delta)\psi^r(-a, y) \geq (1+\delta)\underline{\psi}(-a, y) \geq \underline{\psi}_n(-a, y)$$

- This idea of using the parabolic maximum principle to treat a degenerate elliptic equation motivates the following section where we answer the question : can the solution of (2.4) be recovered by a direct method, without seeing it as the limit of the more regular solutions of (2.5) ?

2.5 Direct study of the limiting problem

We investigate the following elliptic-parabolic non-linear system in

$$[0, M] \times ([0, M] \times [-L, 0]) = [0, M] \times \Omega_{L,M}$$



(2.12)

with $c, u(x), v(x, y)$ as unknowns. We call a supersolution of (2.12) a solution of (2.12) where the $=$ signs are replaced by $\geq$. The plan of this section is the following :

- First, we study the linear background of (2.12) in order to use Perron's method.
- Then we prove the well posedness of (2.12) and study monotonicity and uniqueness properties of the solution.
- In a third subsection, we study the influence of c.
- Finally, under a suitable normalisation condition on c_M obtained thanks to the previous step, we study the limit $M \to +\infty$ of (2.12) and recover the solution of (2.4).

2.5.1 Linear background

In this subsection, we recreate the standard tools behind Perron's method. Even though these are quite standard, we give the proofs in our precise case because of the specificity of mixing parabolic and elliptic theory. Let $k > 0$ be a constant. We look at the following linear system of inequations

$$\begin{cases} c\partial_x v - d\partial^2_{yy} v + kv = h \text{ in } \Omega_{L,M} \\ d\partial_y v(\cdot, 0) + v(\cdot, 0) \geq \mu u \text{ on } (0, M) \\ -\partial_y v \leq 0 \text{ on } y = -L \end{cases}$$

$$\begin{cases} -\partial^2_{xx} u + c\partial_x u + (\mu + k)u - v(\cdot, 0) = g \text{ in } (0, M) \end{cases}$$

along with the parabolic and elliptic limiting conditions

$$v \geq 0 \text{ on } x = 0$$

$$u(0) \geq 0, u(M) \geq 0$$

represented from now on as the following diagram

$$u \geq 0 \qquad -u'' + cu' + (\mu + k)u - v = g \qquad u \geq 0$$

$$d\partial_y v + v \geq \mu u$$

$$v \geq 0 \qquad c\partial_x v - d\partial^2_{yy} v + kv = h \qquad v = ?$$

$$-\partial_y v \geq 0 \tag{2.13}$$

Proposition 2.5.1. *(Maximum principle.) If (u, v) are $\mathcal{C}^2$ functions up to the boundary of resp. $(0, M)$ and $\Omega_{L,M}$ that solve inequation (2.13) with $g, h \geq 0$ then*

$$u, v \geq 0$$

Moreover, $u, v > 0$ in resp. $(0, M)$ and $\Omega_{L,M}$ or $u \equiv 0, v \equiv 0$.

Proof. We simply mix parabolic and elliptic strong maximum principles. Suppose that $\min v < 0$. By the parabolic strong maximum principle and Hopf's lemma, $\min v$ is necessarily reached on $y = 0$ and at this point, $\mu u < \min v$. This is a contradiction with

$$-\partial^2_{xx} u + c\partial_x u + (\mu + k)u \geq v$$

with its endpoints conditions that ensure

$$u \geq \frac{\min v}{\mu + k}$$

Thus we have $v \geq 0$ and the elliptic maximum principle gives also $u \geq 0$. Finally, if $v(x_0, y_0) = 0$ with $x_0 > 0$ then by the strong parabolic maximum principle, $v \equiv 0$ on $x < x_0$, thus $u \equiv 0$ on $x < x_0$ which by the elliptic strong maximum principle gives $u \equiv 0$, so that $v = g = u \equiv 0$ and $\partial_y v \geq 0$ on $y = 0$, and by parabolic Hopf's lemma, $v \equiv 0$. □

Corollary 2.5.1. *(Comparison principle.) Let $k > Lip(f)$. Then we have the following comparison principle : if $g_1 \leq g_2$ and $h_1 \leq h_2$, then if (u_1, v_1) and (u_2, v_2) are solutions $\mathcal{C}^2$ up to the boundary of*

$$u_i = 0 \qquad -u_i'' + cu_i' + (\mu + k)u_i - v_i = kg_i \qquad u_i = 1/\mu$$

$$d\partial_y v_i + v_i = \mu u_i$$

$$v_i = 0 \qquad c\partial_x v_i - d\partial^2_{yy} v_i + kv_i = f(h_i) + kh_i \qquad v_i = ?$$

$$-\partial_y v_i = 0 \tag{2.14}$$

then

$$u_1 \leq u_2, v_1 \leq v_2$$

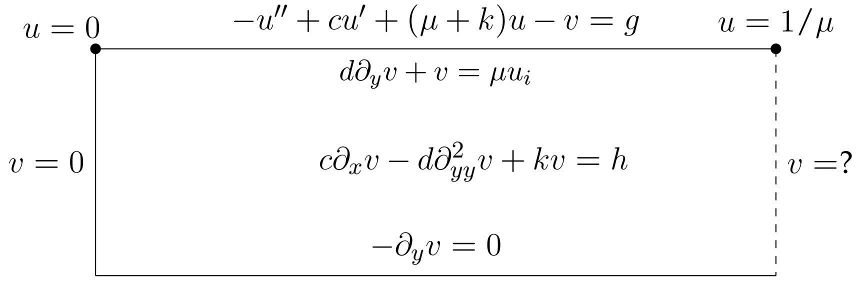
Proof. Just observe that $(u_2 - u_1, v_2 - v_1)$ solve (2.13) with $g = k(g_2 - g_1) \geq 0$ and $h = f(h_2) - f(h_1) + k(h_2 - h_1) \geq (-Lip(f) + k)(h_2 - h_1) \geq 0$. $\square$

Corollary 2.5.2. *(Supersolution principle.) Let $(\bar{u}, \bar{v})$ be a supersolution of (2.12). If (u, v) is a solution of (2.14) with data $(\bar{u}, \bar{v})$ then*

$$u \leq \bar{u}, v \leq \bar{v}$$

Proof. Observe that $(\bar{u} - u, \bar{v} - v)$ solves an inequation (2.13) with $g, h \geq 0$. $\square$

Proposition 2.5.2. *(Unique solvability of the linear system.) Let $c > 0$, $(g, h) \in \mathcal{C}^{\alpha}([0, M]) \times \mathcal{C}^{\alpha/2,\alpha}(\Omega_{L,M})$ and $k > Lipf$. Then there exists a unique solution $(u, v) \in \mathcal{C}^{2,\alpha}([0, M]) \times \mathcal{C}^{1+\alpha/2,2+\alpha}(\Omega_{L,M})$ of*



(2.15)

Proof. The classical parabolic theory allows us to set

$$S \ : \mathcal{C}^{1+\alpha/2} \to \mathcal{C}^{1+\alpha/2}, \quad U \mapsto v(\cdot, 0)$$

where v solves the last four equations in (2.15) with u replaced by U. S is affine and thanks to parabolic Hopf's lemma, uniformly continuous for the L^{∞} norm :

$$|SU_1 - SU_2|_{\infty} \leq \mu |U_1 - U_2|_{\infty}$$

Since $\mathcal{C}^{1+\alpha/2}([0, M])$ is dense in $BUC([0, M])$, we can extend S to a uniformly continuous affine function $\tilde{S}$ on $X = BUC([0, M])$.

On the other hand thanks to classical ODE theory we can set

$$T \ : L^{\infty} \to W^{2,\infty}, \quad V \mapsto u$$

where u is solution of the first equation in (2.13) with $v(\cdot, 0)$ replaced by V. Observe also thanks to elliptic regularity that T sends $\mathcal{C}^{\alpha}$ to $\mathcal{C}^{2,\alpha}$.

By the strong elliptic maximum principle, observe that $T \circ \tilde{S} \ : X \to X$ is a contraction mapping :

$$|T\tilde{S}U_1 - T\tilde{S}U_2|_{\infty} \leq \frac{\mu}{\mu + k} |U_1 - U_2|_{\infty}$$

By use of Banach fixed point theorem, it has a unique fixed point $u \in X$. Observe now that $u = T(\tilde{S}(u))$ and since $\tilde{S}(u) \in L^{\infty}$, $u = T(\tilde{S}(u)) \in W^{2,\infty} \subset \mathcal{C}^{1+\alpha/2}$ and in the end $\tilde{S}(u) = S(u) \in \mathcal{C}^{1+\alpha/2}$ so that $u = T(S(u)) \in \mathcal{C}^{2+\alpha}$. Finally, parabolic regularity gives $v \in \mathcal{C}^{1+\alpha/2,2+\alpha}$ and (u, v) solves (2.13) in the classical sense.

$\square$

2.5.2 The non-linear system

Combining all the results from the previous section we get :

Theorem 2.5.1. *There exists a smooth solution* $0 \leq \mu u, v \leq 1$ *of* (2.12).

Proof. Use $(0,0)$ and $(1/\mu, 1)$ as sub and supersolutions and start an iteration scheme from $(1/\mu, 1)$. We get a decreasing sequence bounded from below by $(0,0)$. It converges point wise but the L^∞ bound on u_n, v_n gives a $\mathcal{C}^{1+\alpha/2}$ bound on u_n which then gives a $\mathcal{C}^{1+\alpha/2,2+\alpha}$ bound on v_n, which then gives a $\mathcal{C}^{2+\alpha}$ bound on u_n. By Ascoli's theorem we can extract from (u_n, v_n) a subsequence that converges to $(u,v) \in \mathcal{C}^{2+\beta} \times \mathcal{C}^{1+\beta/2,2+\beta}$. The point wise limit then gives the uniqueness of this limit point and thus that (u_n, v_n) converges to it. Finally, (u,v) has to be a solution of the equation.

Observe also that the only possible loss of regularity comes from the non-linearity f. Actually, if f is of class $\mathcal{C}^\infty$, by elliptic and parabolic regularity described above, u and v are $\mathcal{C}^\infty$ too. More precisely, $f \in W^{k+1,\infty}$ implies $(u,v) \in C^{2+k,\alpha}([0,M]) \times \mathcal{C}^{k+1+\alpha/2,k+2+\alpha}(\Omega_{L,M})$. □

We are now interested in sending $M \to +\infty$ to recover the travelling wave observed in the last section. For this, we need to normalise the solution in $\Omega_{L,M}$ in such a way that we do not end up with the equilibrium $(0,0)$ or $(1/\mu, 1)$. We trade this with the freedom to chose c : this motivates the investigation of the influence of c on (u,v) as well as a priori properties of (u,v). To this end, we use a sliding method in finite cylinders. Because we apply the parabolic maximum principle on v, we will not have to deal with the corners of the rectangle.

Proposition 2.5.3. *If* $c > 0$ *and* $0 \leq \mu u, v \leq 1$ *is a classical solutions of* (2.12) *then*

$$u', \partial_x v > 0$$

Proof. First observe that $m_0 := \min_{[-L,0]} v(M,y) > 0$ for the same reasons as in Prop. 2.5.1. Observe also that

$$\lim_{\varepsilon \to 0} \max_{[0,\varepsilon]\times[-L,0]} v = 0$$

thanks to the uniform continuity of v. Denote

$$v_r(x,y) = v(x+r, y)$$

and

$$\Omega^r_{L,M} = [-r, M-r] \times [-L, 0]$$

The previous observation asserts that $v_r - v > 0$ on $\Omega^r_{L,M} \cap \Omega_{L,M}$ if r is close enough to M. Call (r_0, M) a maximal interval such that for all r in this interval, $v_r - v > 0$ on $\Omega^r_{L,M} \cap \Omega_{L,M}$. We know that such an interval exists by the previous observation. Let us show that $r_0 = 0$ by contradiction. Suppose $r_0 > 0$. By continuity, $v_{r_0} - v \geq 0$. But $(v_{r_0} - v)(0,y) > 0$ and $V := v_{r_0} - v, U := u_{r_0} - u$ satisfy on $[0, M-r_0], [0, M-r_0] \times [-L, 0]$:

$$U > 0 \qquad -U'' + cU' + \mu U = V \qquad U(M - r_0) > 0$$

$$d\partial_y V + V = \mu U$$

$$V > 0 \qquad c\partial_x V - d\partial_{yy}^2 V + \frac{f(v_{r_0}) - f(v)}{v_{r_0} - v} V \geq 0$$

$$-\partial_y V = 0$$

By the mixed elliptic-parabolic strong maximum principle and Hopf's lemma for comparison with 0 as in prop. 2.5.1 we know that $v_{r_0} - v > 0$ (we cannot have $v_{r_0} \equiv v$ because then $u(M - r_0) = 1/\mu$ and that is impossible thanks to strong elliptic maximum principle since $r_0 > 0$). Then we may translate a little bit more, since $v_{r_0-\varepsilon} - v$ is continuous in ε, so that $v_{r_0-\varepsilon} - v > 0$, which is a contradiction with the definition of r_0.

As a result, u and v are non-decreasing in x, that is $u', \partial_x u \geq 0$. Now differentiating the equation with respect to x and applying the same mixed maximum principle as above for comparison with 0 yields $u', \partial_x v > 0$. □

Proposition 2.5.4. *For fixed $c > 0$, there is a unique solution (u, v) of (2.12) such that $0 \leq \mu u, v \leq 1$.*

Proof. The proof is essentially the same as monotonicity. Suppose (u_1, v_1) and (u_2, v_2) are solutions. The same observations as above allow us to translate v_2 to the left above v_1. Now translate it back until this is not the case any more. We show by contradiction that this never happens : suppose $v_2^{r_0} - v \geq 0$ with $r_0 > 0$. Then $v_2^{r_0} - v > 0$ or $\equiv 0$, but the latter case is not possible since it would give $u(M - r_0) = \frac{1}{\mu}$. Thus $v_2^{r_0} - v > 0$ and we can still translate a bit more, that is a contradiction with the definition of r_0 so $r_0 = 0$ and $v_2 \geq v$. By symmetry, $v_2 \equiv v_1$, and then $u_2 \equiv u_1$. □

Proposition 2.5.5. *The function $c \mapsto (u, v)$ is decreasing in the sense that if $0 < c < \bar{c}$ and (c, u, v) and $(\bar{c}, \underline{u}, \underline{v})$ solve (2.12) then*

$$\underline{u} < u, \underline{v} < v$$

Proof. The proof is again the same, just observe that $(\underline{u}, \underline{v})$ is a subsolution of the equation with c, thanks to monotonicity since $(c - \bar{c})\underline{\partial_x} v < 0$. □

2.5.3 Limits with respect to c

For now, we assert the following properties, which will be enough to conclude this section. Nonetheless, note that the study of solutions of (2.12) with $c = 0$ shows interesting properties. Namely, the solutions are necessarily discontinuous, which implies that the regularisation comes also from the $c\partial_x$ term. This has to be seen in the light of hypoelliptic regularisation in kinetic equations (see [25, 50]).

Proposition 2.5.6. *Let (u_c, v_c) denote the solution in Prop. 2.5.4. Then for every $\alpha \in (0,1)$*

$$\lim_{c\to 0} u_c(\alpha M) \geq \alpha$$

Proof. By monotonicity we already know that (u_c, v_c) converges point wise as $c \to 0$ to some (u,v). Since $0 \leq v_c \leq 1$, by uniqueness of the limit and Ascoli's theorem we know that u is in $W^{2,\infty} \hookrightarrow \mathcal{C}^{1,1/2}$.

Multiplying the equation satisfied by v_c by a test function $\phi(y) \in \mathcal{C}_c^\infty(-L,0)$, integrating, and then multiplying by $\psi(x) \in \mathcal{C}_c^\infty(0,M)$ and integrating again, yields after passing to the limit $c \to 0$ thanks to dominated convergence : for all $x \in (0,M)$

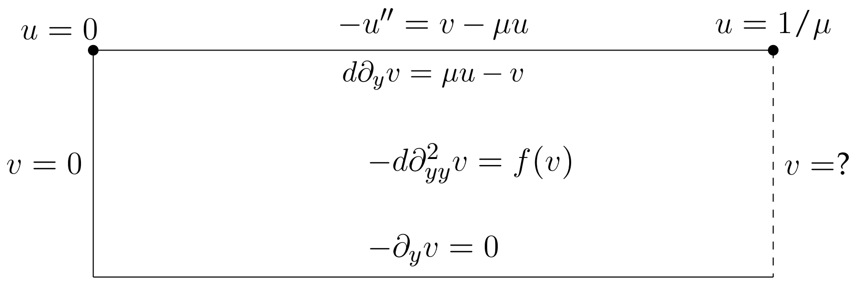
$$-d\int_{-L}^{0} v\phi''(y)\mathrm{d}y = \int_{-L}^{0} f(v(x,y))\phi(y)\mathrm{d}y$$

i.e. $v(x,\cdot)$ satisfies $-v(x,\cdot)'' = f(v) \in L^\infty$ in the sense of distributions, so that $v(x,\cdot) \in W^{2,\infty} \hookrightarrow \mathcal{C}^{1,1/2}$, so that actually $f(v) \in \mathcal{C}^1$ and these equations are satisfied in the classical sense. Moreover $v(x,\cdot)'$ is bounded and by concavity, $v_y(-L)$ exists and is seen to be necessarily 0 by using a test function ϕ whose support intersects $y = -L$ and integrating by parts.

At this point, $v(x,\cdot) \in \mathcal{C}^2(-L,0) \cap \mathcal{C}^1([-L,0])$ and satisfies $-d\partial_{yy}^2 v(x,y) = f(v(x,y))$ and $\partial_y v(-L) = 0$. Using now a test function ϕ whose support intersects $y = 0$ and integrating by parts we obtain $d\partial_y v(x,0) = \mu u(x) - v(x,0)$.

Moreover, $u(0) = v(0,y) = 0$, $u(M) = 1/\mu$, v is non-decreasing with respect to the x variable so differentiable a.e. and continuous up to a countable set (which in fact is of cardinal 1, 2 or 3).

Finally, passing to the limit in the equation for u yields that $u \in \mathcal{C}^1$ satisfies $-u'' = \mu u - v$ in the sense of distributions, so that we have the following picture :



(2.16)

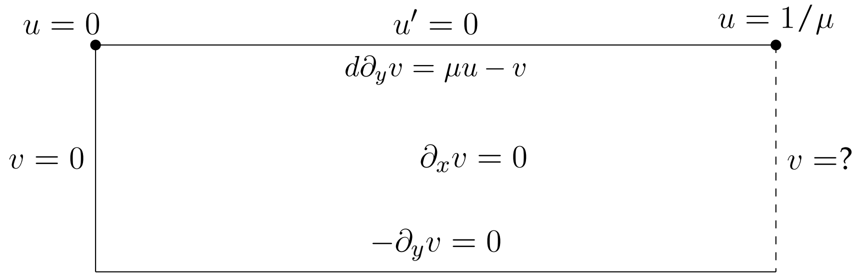
Notice that since $f \geq 0$, $v(x,\cdot)$ is concave, so that $d\partial_y v(x,0) \leq 0$, i.e. $v - \mu u \geq 0$ so that u is concave. As a consequence, it is over the chord between $(0,0)$ and $(M, 1/\mu)$ which is the desired conclusion.

□

Proposition 2.5.7.

$$\lim_{c\to +\infty} u_c = 0 \textit{ pointwise on } [0, M[$$

Proof. Applying the exact same method as above we obtain as $c \to +\infty$, (u, v) a classical solution of



(2.17)

Now just observe that $u_c'(0)$ is decreasing with respect to c and non-negative, so that u can be extended in a $\mathcal{C}^1$ way on $[0, M)$ by the Cauchy criterion : we end-up with $u \equiv 0$ on $[0, M)$. We observe that u is thus necessarily discontinuous at $x = M$, which is consistent with the limit $c \to \infty$ in the integration by parts of the equation on (u_c, v_c) :

$$c\left(\frac{1}{\mu} + \int_{-L}^{0} v_c(M, y)\mathrm{d}y\right) = \int_{\Omega_{L,M}} f(v_c) + u_c'(M) - u_c'(0)$$

since the left-hand side goes to $+\infty$ and everything except $u_c'(M)$ is bounded by above in the right-hand side.

□

2.5.4 Limit as $M \to \infty$

We now call $\theta' = \frac{1+\theta}{2}$, $\theta'' = \frac{2+\theta}{3}$ and chose $c = c_M$ such that $u_M(\theta'' M) = \theta'$. We are now interested in compactness on c_M to pass to the limit $M \to +\infty$ in the equations. From now on we change the coordinate x by $x - \theta' M$ so that u_M, v_M are resp. defined on

$$\begin{aligned}\Omega_M &:= [-\theta'' M, (1-\theta'')M] \\ \Omega_{L,M} &:= [-\theta'' M, (1-\theta'')M] \times [-L, 0]\end{aligned}$$

and

$$u_M(0) = \theta'$$

Proposition 2.5.8. *For M large enough, $c_M < \sqrt{Lipf}$*

Proof. Since we do not know how the level lines $\{v_M = \theta\}$ behave, we cannot apply the argument of Proposition 2.2.2. Nonetheless, this is counterbalanced by taking advantage of being in a rectangle. We look for

$$\bar{u}(x) = e^{rx}, \bar{v}(x, y) = \mu e^{rx}$$

to solve (2.12) with the $=$ signs replaced by $\geq$. A direct computation yields

$$-r^2 + c_M r \geq 0, c_M r \geq \mathrm{Lip} f$$

so the best choice is

$$r = c_M, c_M \geq \sqrt{\mathrm{Lip} f}$$

So now suppose

$$c_M \geq \sqrt{\mathrm{Lip} f}$$

and set

$$t_0 = \inf\{t \in \mathbb{R} \mid \bar{u}(t+\cdot) - u_M) > 0 \text{ on } \Omega_M \textbf{ and } \bar{v}(t+\cdot, y) - v_M > 0 \text{ on } \Omega_{L,M}\}$$

This infimum exists as it is taken over a set that is non-void and bounded by below (using the limits of e^{rx} and the bounds on u, v). By continuity

$$\bar{u}(t_0+\cdot) - u_M, \bar{v}(t_0+\cdot, y) - v_M \geq 0$$

and

$$\text{There exists } x_0 \in \Omega_M \text{ s.t. } \bar{u}(t_0+x_0) - u(x_0) = 0$$
$$\textbf{or}$$
$$\text{There exists } (x_0', y_0') \in \Omega_{L,M} \text{ s.t. } \bar{v}(t_0+x_0', y_0') - v(x_0', y_0') = 0$$

Since $u = v = 0$ at the left boundaries, $x_0, x_0' > -\theta'' M$. Thanks to the normalisation condition, the first case is impossible, since $\bar{u}(t_0+\cdot) - u_M$ satisfies the strong elliptic maximum principle with non-negative boundary values and data. Indeed, the only thing to check is that

$$\bar{\mu} u(t_0 + (1-\theta'')M) \geq \mu u_M((1-\theta'')M) = 1$$

This is obtained provided $M \geq \frac{2}{\sqrt{\mathrm{Lip} f}} \ln\left(\frac{1}{\mu}\right)$: the level lines θ' should touch before the level lines 1 since

$$\begin{aligned}
|\mu u_M^{-1}(\theta') - \mu \bar{u}^{-1}(\theta')| &= \frac{1}{c} \ln\left(\frac{\theta'}{\mu}\right) \\
&< \frac{1}{c} \ln\left(\frac{1}{\mu}\right) \\
&< M - \frac{1}{c} \ln\left(\frac{1}{\mu}\right) \\
&= |\mu u_M^{-1}(1) - \mu \bar{u}^{-1}(1)|
\end{aligned}$$

The second case is impossible also, by using the strong parabolic maximum principle and Hopf's lemma as usual. In every case, there is a contradiction so that in the end

$$c_M \leq \sqrt{\mathrm{Lip} f}$$

□

Proposition 2.5.9. *There exists $c_- > 0$ that does not depend on M s.t. $c_M > c_-$*

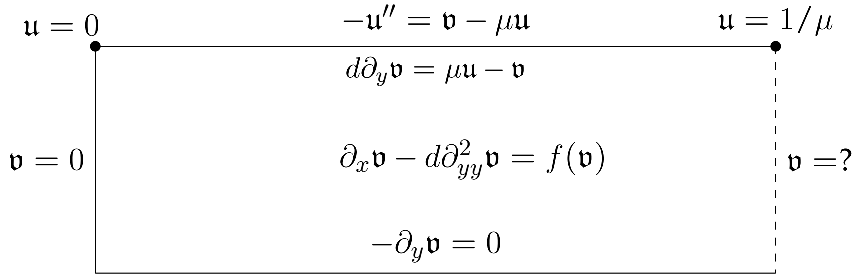
Proof. We argue by contradiction by supposing $\inf c(M) = 0$. Then there exists $M_n \to \infty$ (by continuity of $c(M)$) such that $c_{M_n} =: c_n \to 0$. Denote u_n, v_n the associated normalised sequence of solutions. Since u_n is uniformly in $\mathcal{C}^{1,\beta}$ we extract from it a subsequence that converges in $\mathcal{C}^{1,\alpha}$. We now assert the following :

For every $A > 0$, $(v_n(y)(x))_n$ is equicontinuous and bounded in $\mathcal{C}((-L,0), L^1(-A,A))$

The boundedness comes directly from the fact that $v_n(x,y) \in [0,1]$ is increasing, thus it is bounded uniformly in n and y in $BV(-A,A)$ which is compactly embedded in $L^1(-A,A)$. For the equicontinuity, we have

$$\int_{-A}^{A} |v_n(y,x) - v_n(y+\varepsilon,x)|dx \le \int_{-A}^{A} \left(\int_{y}^{y+\varepsilon} |\partial_y v_n(x,s)|\, ds \right) dx$$

so that a uniform bound on $\partial_y v_n$ will suffice. This bound is classical and comes from parabolic regularity after rescaling, but let us give it here for the sake of completeness. Consider $\mathfrak{u}_n(x) = u(c_n x)$ and $\mathfrak{v}_n(x,y) = v(c_n x, y)$ so that with the new variables, $x \in (-\frac{\theta' M_n}{c_n}, \frac{(1-\theta')M_n}{c_n})$ and $\mathfrak{u}, \mathfrak{v}$ satisfy



(2.18)

Now we reduce to a local estimate :

$$\begin{aligned}
|\mu\mathfrak{u}_n|_{\mathcal{C}^{1,\alpha}(-A,A)} &= |\mu\mathfrak{u}_n|_{L^\infty(-A,A)} + |\mu\mathfrak{u}_n'|_{L^\infty(-A,A)} + \sup_{x\neq y} \frac{|\mu\mathfrak{u}_n'(x) - \mu\mathfrak{u}_n'(y)|}{|x-y|^\alpha} \\
&\le |\mu\mathfrak{u}_n|_{L^\infty(-A,A)} + 3|\mu\mathfrak{u}_n'|_{L^\infty(-A,A)} + \sup_{|x-y|<1} \frac{|\mu\mathfrak{u}_n'(x) - \mu\mathfrak{u}_n'(y)|}{|x-y|^\alpha} \\
&\le |\mu\mathfrak{u}_n|_{L^\infty(-A,A)} + 4 \sup_{x_0\in(-A,A)} |\mu\mathfrak{u}_n|_{\mathcal{C}^{1,\alpha}(B_1(x_0))}
\end{aligned}$$

And finally, $|\mu\mathfrak{u}_n|_{L^\infty(-A,A)} \le 1$ as well as

$$\begin{aligned}
|\mu\mathfrak{u}_n|_{\mathcal{C}^{1,\alpha}(B_1(x_0))} &\le C_0 |\mu\mathfrak{u}_n|_{W^{2,p}(B_1(x_0))} \\
&\le C_0 C_1 \left(|\mu\mathfrak{u}_n|_{L^p(B_1(x_0))} + |\mathfrak{v}_n|_{L^p(B_1(x_0))} \right) \\
&\le 2C_0C_1
\end{aligned}$$

thanks to the Sobolev inequality, the standard $W^{2,p}$ estimates with $p = 1/(1-\alpha)$, and $0 \leq \mu u \leq 1$, so that in the end

$$|\mu \mathfrak{u}_n|_{\mathcal{C}^{1,\alpha}(-A,A)} \leq 1 + 8C_0C_1$$

Finally, we plug this in the classical Schauder parabolic estimate up to the boundary to get

$$|\mathfrak{v}_n|_{\mathcal{C}^{1+\alpha/2,2+\alpha}(-A,A)} \leq C_3 \left(|\mathfrak{v}_n|_{L^\infty(-A,A)} + |u_n|_{\mathcal{C}^{1+\alpha/2}(-A,A)} \right) \leq C_3(2 + 8C_0C_1)$$

even independently from A. So that in the end

$$|\partial_y v_n|_{L^\infty((-A,A)\times(-L,0))} = |\partial_y \mathfrak{v}_n|_{L^\infty((-A/c_n,A/c_n)\times(-L,0))} \leq C_3(2 + 8C_0C_1)$$

The fact is now proved, and thanks to Ascoli's theorem and a diagonal extraction, we can extract from u_n, v_n some u, v that converges in $\mathcal{C}((-L,0), L^1(-n,n))$ for every $n \in \mathbb{N}$. Just as in the previous computations by integrating by parts, we get that u, v ends up to be a classical solution of (since $M_n/c_n \to \infty$)

$$\begin{array}{c} -u'' = v(x,0) - \mu u \\ \hline d\partial_y v = \mu u - v(x,0) \\ \\ -d\partial_{yy}^2 v = f(v) \\ \\ \partial_y v = 0 \\ \hline \end{array} \qquad (2.19)$$

along with $\mu u(0) = (1+\theta)/2$. But this is impossible : indeed, u is bounded and thanks to $f \geq 0$, v is concave on each y-slice, which gives that u is also concave, on the whole $\mathbb{R}$ so it is constant. Thanks to the normalisation condition, $\mu u \equiv (1+\theta)/2$, so that $v(x,0) \equiv \mu u \equiv (1+\theta)/2$, so that $\partial_y v(x,0) \equiv 0$, but then by concavity and the Neumann condition, $v \equiv (1+\theta)/2$ which is a contradiction with $f((1+\theta)/2) > 0$.

□

We can now pass to the limit $M \to \infty$ in the equations and prove Theorem 3.

Proof. Taking $M_n \to +\infty$, thanks to the bounds on c_{M_n} we can extract from it a subsequence converging to some $c > 0$. We can also use the elliptic-parabolic regularity discussed in the beginning to extract from (u_{M_n}, v_{M_n}) some subsequence that converges in $\mathcal{C}^{1+\alpha/2,2+\alpha}_{loc}$ to some (u,v) that solve the equations in (2.4). Bounds and monotonicity are inherited from the $\mathcal{C}^1$ limit. The last thing to check are the uniform limits as $x \to \mp\infty$, which are obtained thanks to the following lemmas. □

Lemma 2.5.1.

$$\iint_{[0,+\infty]\times[-L,0]} f(v) < +\infty, \qquad \iint_{[0,+\infty]\times[-L,0]} |\partial_y v|^2 < +\infty$$

and the same is true with $[-\infty, 0]$.

Proof. Integrate on $[0,M]\times[-L,0]$ the equation for v in (2.4) to get

$$\iint_{[0,M]\times[-L,0]} f(v) = c\left(\int_{-L}^{0} v(M,y) - \int_{-L}^{0} v(0,y)\right) + u'(M) - u'(0) + c\left(u(M) - u(0)\right)$$

Everything in the left-hand side is bounded, apart from $u'(M)$. Thus, for the integral to diverge as $M \to +\infty$, $u'(M) \to +\infty$, which is impossible since u is bounded.

For the second integral, multiply the equation by v and integrate by parts to get

$$\iint_{[0,M]\times[-L,0]} d|\partial_y v|^2 = \iint_{[0,M]\times[-L,0]} f(v)v - c\int_{-L}^{0} \frac{1}{2}(v(M,y)^2 - v(0,y)^2) + \int_0^M (u'' - cu')v$$

The first two integrals in the right-hand side are bounded. For the last one, we see that

$$c\int_0^M u'v \le c(u(M) - u(0)) \le \frac{c(1-\theta')}{\mu}$$

so that for the integral to diverge as $M \to +\infty$, $\int_0^M u''v \to +\infty$. But

$$\int_0^M u''v = u'(M)v(M) - u'(0)v(0) - \int_0^M u'v' \le u'(M)v(M) \le u'(M)$$

so that this is again impossible. The case of $[-\infty, 0]$ is similar.

$\square$

Proposition 2.5.10. *$\mu u(x), v(x,y) \to 1$ uniformly in y as $x \to +\infty$ and to some constant $v_- \le \theta$ as $x \to -\infty$.*

Proof. By bounds and monotonicity, $v(x,y)$ converges point wise to some $v_+(y)$ as $x \to +\infty$. Let us define the functions $v_j(x,y) = v(x+j,y)$ in $[0,1]\times[-L,0]$ for every integer j. Standard parabolic estimates and Ascoli's theorem tell us that up to extraction, $v_j \to \delta$ in the $\mathcal{C}^1$ sense for a $\mathcal{C}^1$ function δ. By uniqueness of the simple limit, $\beta = v_+ \in \mathcal{C}^1$. So v_j lies in a compact set of $\mathcal{C}^1([0,1]\times[-L,0])$ and has a unique limit point $\beta \in \mathcal{C}^1$: then it converges to it in the $\mathcal{C}^1$ topology. The y-uniform limits follow.

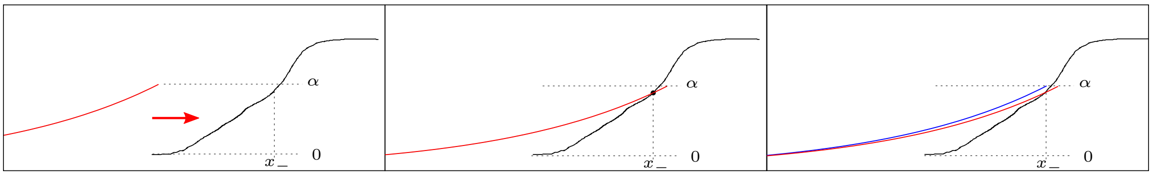
Now using the finiteness of the second integral above, we have that $v_+(y)$ is constant, moreover, $f(v_+) = 0$ thanks to the finiteness of the first integral. By the exchange condition, μu converges to v_+ as $x \to +\infty$ so necessarily $v_+ \ge \theta'$: the only possibility is $v_+ = 1$.

The exact same arguments apply to $-\infty$, but all of $[0,\theta]$ are admissible constants. Let us finish with the following. $\square$

Proposition 2.5.11.

$$v_- = 0$$

Proof. Here we use that v comes from v_{M_n}. As in the proof of the upper bound on c_M, we do not know what happens to the level lines $\{v_{M_n} = \theta\}$, which prevents us to use the usual comparison with positive exponential solutions as $v \leq \theta$: indeed, the sets $\{v_{M_n} = \theta\}$ could be sent to $-\infty$. We use a sliding method with a less sharp supersolution by looking at a level line $\{v_{M_n} = \alpha\}$ with $\alpha > \theta$ close to θ to prove that this is not the case. We give below a picture of the argument before writing it completely.



First, observe that thanks to the y-uniform convergence of $\mu u, v$ to $v_- \leq \theta$ resp. 1 as $x \to \mp\infty$:

$$\exists\, x_- \leq x_+ \in \mathbb{R} \text{ s.t. } \{\mu u_{M_n} = \alpha\} \subset [x_-, x_+] \text{ and } \{v_{M_n} = \alpha\} \subset [x_-, x_+] \times [-L, 0]$$

Indeed, there exists x_-, x_+ s.t. for all $x \leq x_-, y \in [-L, 0]$, $\mu u, v(x, y) \leq \frac{\theta+\alpha}{2}$ and for all $x \geq x_+, y \in [-L, 0]$, $\mu u, v(x, y) \geq \frac{\alpha+1}{2}$. Thanks to the uniform local convergence of u_{M_n}, v_{M_n} to u, v we can say that there exists $N \in \mathbb{N}$ s.t. for all $n \geq N$, on $[x_-, x_+] \times [-L, 0]$ we have $\frac{2\theta+\alpha}{3} \leq \mu u_{M_n}, v_{M_n} \leq \frac{\alpha+2}{3}$, and we conclude thanks to the monotonicity of u_{M_n}, v_{M_n}.

We now work on $x \leq x_-$ so that $\mu u_{M_n}, v_{M_n} \leq \alpha$ and use the fact that $\mathrm{Lip} f_{|[0,\alpha]}$ is as small as we want by taking α close enough to θ. The same computations as in Prop. 2.5.8 as well as $c > c_-$ and the monotonicity of $e^{c_- x}$ give that if α is chosen so that

$$\mathrm{Lip} f_{|[0,\alpha]} \leq c_-^2$$

then $(e^{c_- x}, \mu e^{c_- x})$ is a supersolution of (2.12) as long as $\mu e^{c_- x} \leq \alpha$: so look at the graph of $\mu e^{c_- x}$ and cut it after it reaches α. Now translate this half-graph to the left until it is disconnected with the graph of u_{M_n}, v_{M_n} and bring it back until it touches μu_{M_n} or v_{M_n} before x_-, which necessarily happens since $\mu u(x_-), v(x_-, y) \leq \alpha$. The arguments given in Prop. 2.5.8 assert here that the contact necessarily happens at x_- with μu_{M_n}, i.e. the graphs of $\mu u_{M_n}, v_{M_n}$ are below some translation of the cut graph of $\mu e^{c_- x}$ that touches it at x_-, where $\mu u_{M_n} \leq \alpha$ so that they are below the graph of $\alpha e^{c_-(x - x_-)}$, i.e.

$$\mu u_{M_n}, v_{M_n} \leq e^{c_-(x - x_-)} \text{ on } x \leq x_-$$

By making $n \to \infty$ we get that $\mu u, v$ decays as $x \to -\infty$ at least as $e^{c_- x}$, which is consistent with the computations of the exponential solutions in Section 2.2. $\square$

As a conclusion, I would like to mention that this study motivates the question of convergence towards travelling waves. This work suggests that the travelling wave of (**??**) is globally stable among initial data that are over θ on a set large enough (whose measure would scale as $\sqrt{D}$). I also conjecture that this convergence happens uniformly in D. This will be the purpose of a future work.

Acknowledgement

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Chapter 3

Transition between a low speed and the travelling waves speed

“ Do you feel like a remnant
Of something that’s past?
Do you find things are moving
Just a little too fast?
Do you hope to find new ways
Of quenching your thirst?
Do you hope to find new ways of doing
Better than your worst?
Hey slow, Jane, let me prove
Slow, slow, Jane, we’re on the move. ”

—Nick Drake (1948-1974), Hazey Jane I

Contents

3.1 Introduction

The question

This chapter investigates the dynamics of the travelling fronts exhibited in the previous chapters. We shed light on what kind of initial data are attracted by the travelling waves and on the precise mechanism of attraction, which ends up to be more sophisticated than we thought at first glance. We recall the system

$$\frac{\partial_t u - D\partial_{xx}^2 u = v - \mu u}{d\partial_y v = \mu u - v}$$

$$\partial_t v - d\Delta v = f(v)$$

$$\frac{\partial_y v = 0}{} \tag{3.1}$$

and its renormalisation ($x \leftarrow x\sqrt{D}$) for the study of the limit $D \to +\infty$

$$\frac{\partial_t u - \partial_{xx}^2 u = v - \mu u}{d\partial_y v = \mu u - v}$$

$$\partial_t v - \frac{d}{D}\partial_{xx}^2 v - d\partial_{yy}^2 v = f(v)$$

$$\frac{\partial_y v = 0}{} \tag{3.2}$$

Some results will be stated for equation (3.1) and some for (3.2) and the proofs will juggle between the two. Also, we will try to be as clear as possible.

It is well known since the pioneering works of Kolmogorov, Petrovskii and Piskunov [56] that "front like" initial data are attracted by associated travelling waves. This result was later refined and generalised in Aronson-Weinberger [1] and again in Fife-McLeod [42] to bistable non-linearities. Unlike the KPP case where even a Dirac measure as initial datum will converge to a pair of travelling waves, in the case of a non-linearity with a threshold, one has to assume largeness conditions on the initial data. In our case for instance it is clear that for initial data everywhere below θ the equation will be linear and the solution will decay to zero. Moreover, ignition type non-linearities present an additional technical difficulty compared to the case of [42] because of their degeneracy ($f'(0) = 0$) in the regime $v \leq \theta$.

Since the results of the afore mentioned authors, these phenomena have received much attention, see for instance Roquejoffre [75, 76] where the author studies the case of heterogeneous cylinders, or Zlatoš and Du-Matano [40, 80] where the authors provide sharp conditions on the initial data for extinction or invasion to occur. See the subsection "bibliographical study and discussion" below for a more precise description of all the above works. We also mention Mellet-Nolen-Ryzhik-Roquejoffre [65] where the ideas of [1, 76] were generalised and simplified to the context of generalised transition fronts. Our first theorem stated below is an adaptation of their work to our heterogeneous context.

First results

Theorem 3.1.1. *Let* (u_0, v_0) *be a front-like initial datum for equation* (3.2)*, that is* $(u_0, v_0) \in \mathcal{P}_{\alpha_0}$ *defined in the next section. There exists an exponent* $\omega > 0$ *that depends on the initial data only through* α_0 *and for all* $\varepsilon > 0$ *small enough there exists two shifts* $\xi_1^{\pm}$ *such that*

$$\begin{aligned} \phi(x + c\xi_1^- + ct) - C\varepsilon e^{-\omega t} &\le \mu u(t,x) \ \le \mu\phi(x + c\xi_1^+ + ct) + C\varepsilon e^{-\omega t} \\ \psi(x + c\xi_1^- + ct) - C\varepsilon e^{-\omega t} &\le v(t,x,y) \le \psi(x + c\xi_1^+ + ct) + C\varepsilon e^{-\omega t} \end{aligned}$$

where C *is a constant that depends only on* d *and* L*. Moreover,* ω *does not depend on* $D > d$*.*

Remark 3.1.1.

- The previous theorem does not give the convergence towards travelling waves strictly speaking. It only says that the solution stays strapped (up to an exponentially small error in time) between two translates of the wave. In [65], this is the starting point of an iterative argument showing a geometric decrease of the distance separating the two shifts with respect to a fixed time step. One can repeat the argument of [65] by adapting it to the system and replacing the Harnack inequality by the strong maximum principle. However this would not be uniform in $D > d$. A different mechanism is at work here and we prefer to focus on this different issue.

- As we will see later, the initial data can be trapped between two translates $c\xi_0^{\pm}$ of a wave also. We wish to mention that the distance between the above shifts $\xi_1^{\pm}$ and the shifts $\xi_0^{\pm}$ can be controlled in terms of ε. For a more precise statement of the Theorem in terms of shifting sub and supersolutions, see Theorem 3.2.1 below.

The associated theorem for compactly supported initial data asserts the following (see Theorem 3.3.1 for a precise statement) :

Theorem 3.1.2. *Let* (u_0, v_0) *be non-negative smooth compactly supported data for equation* (3.1)*. There exists* $\delta > 0$ *and* $M = O(\sqrt{D})$ *such that if* $\mu u_0, v_0 > 1 - \delta$ *for* $x \in (-M, M)$ *then* $\mu u, v$ *stays trapped (up to an exponentially decaying error) between two shifts of a pair of travelling waves evolving in both directions.*

This theorem is a natural consequence of the stability of front-like initial data using an argument initiated by [42] and was pretty much expected. Nonetheless, numerical simulations indicated a more subtle mechanism of attraction when the initial data is smaller. We provide them below for the case $\mu u_0, v_0 = \mathbf{1}_{(-a,a)}$ with $a > 0$ small compared to $\sqrt{D}$.

Numerical simulations

These simulations were produced using FreeFem++ : we used P2 finite elements using a mesh consisting in 400 points horizontally and 50 vertically. The time-scheme used is explicit Euler, which seems quite sufficient in terms of accuracy and speed for our context. We imposed Neumann boundary conditions on the vertical ends of a box of size $A \times L$ with $A \gg L$, which gives a good approximation of our setting as long as u and v are very close to 0 near the edges. Finally, we represented u as a function over the box so that it is visible. The following parameters were used :

a	3
d	0.1
D	100
μ	1.4
θ	0.3
$f(v)$	$10 \times \mathbf{1}_{v>\theta}(v-\theta)^2(1-v)$
A	500
L	50
Δt	0.1

We comment the figures below by the scenario which, we think, is the most plausible :

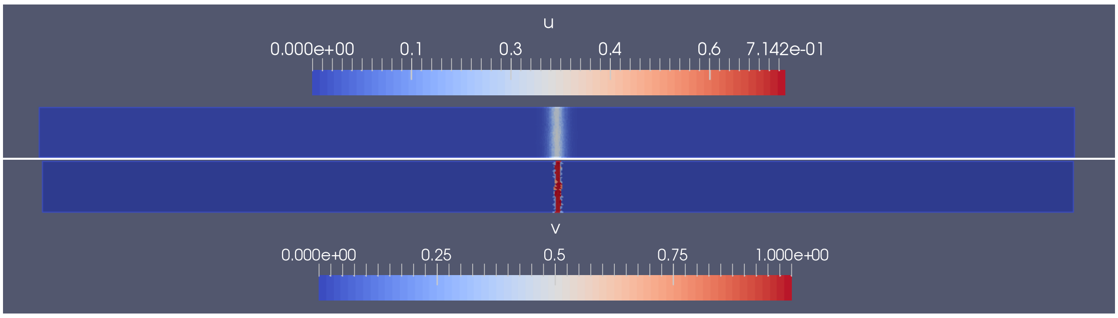

Figure 3.1: $t = 0$

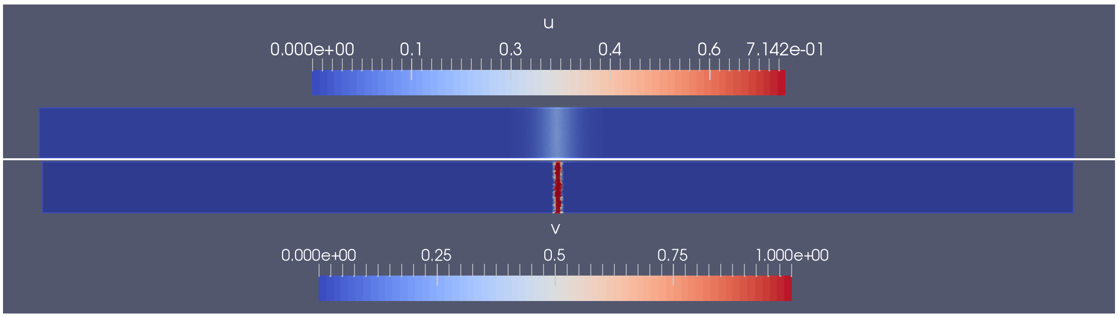

Figure 3.2: $t = 10\Delta t$: due to the large diffusivity D, u is quickly spread on all $\mathbb{R}$ and decays rapidly

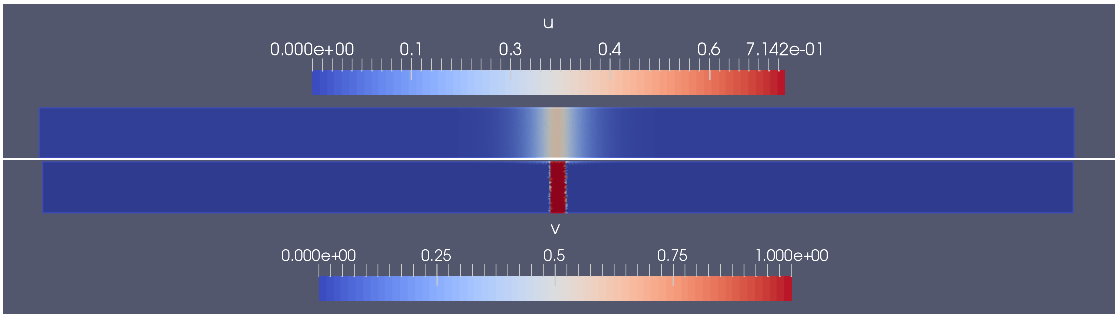

Figure 3.3: $t = 75\Delta t$: in the meanwhile, v grows slowly and transmits mass to u

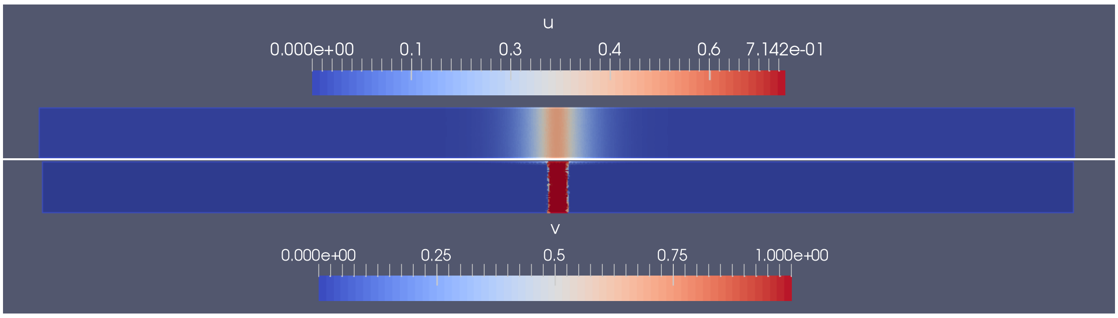

Figure 3.4: $t = 100\Delta t$

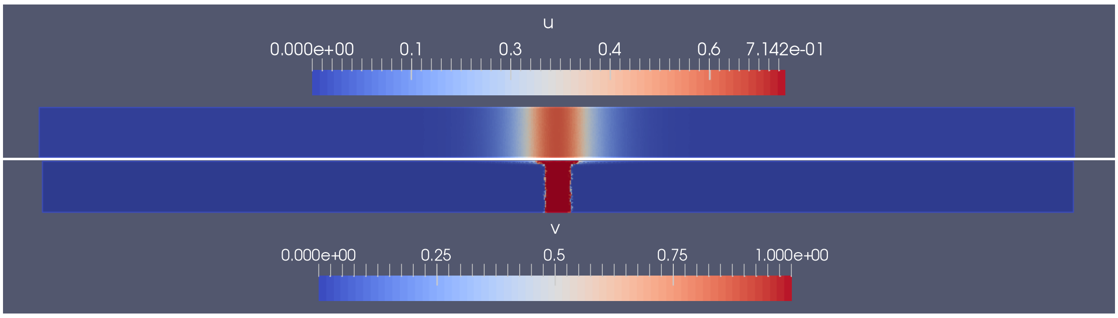

Figure 3.5: $t = 150\Delta t$: at some point, u has recovered enough mass and starts to lead the propagation

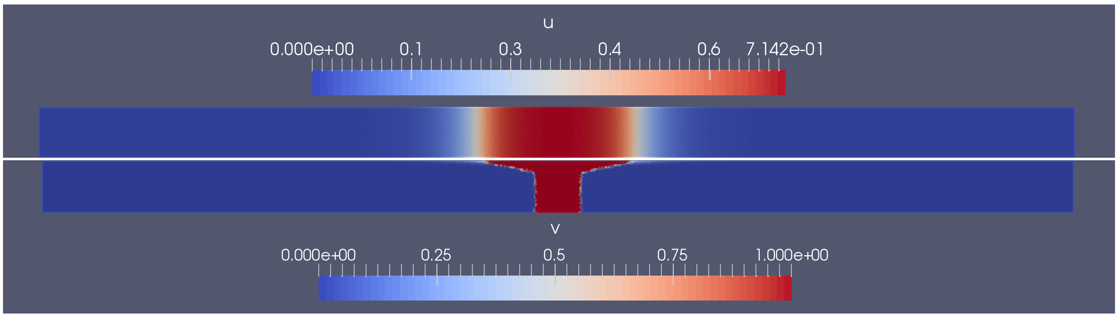

Figure 3.6: $t = 300\Delta t$

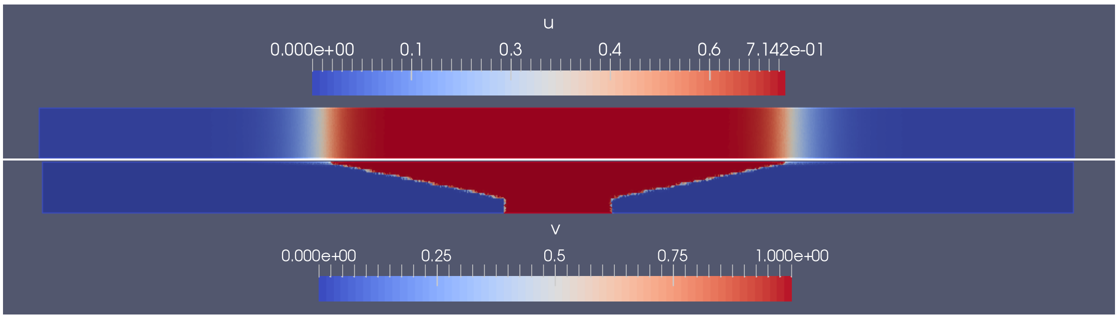

Figure 3.7: $t = 750\Delta t$: acceleration of the propagation is then transmitted downwards from the road to the bottom of the field, reaching the regime described in Theorem 3.3.1.

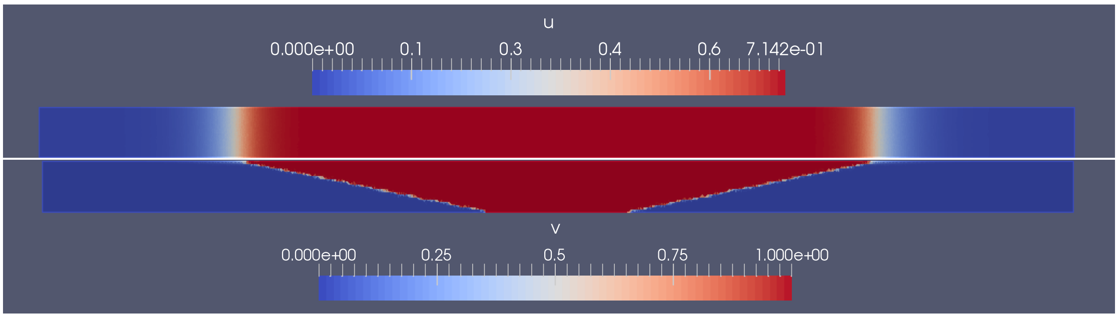

Figure 3.8: $t = 1000\Delta t$

Data with small support and additional effects

In this section we describe the result behind the above numerical simulations.

Theorem 3.1.3. *Let L be large enough (independently of D). There exists $M', \delta' > 0$ independent of $D > d$ such that if the initial datum of* (3.1) *satisfies*

$$v_0 > 1 - \delta' \text{ for } x \in (-M', M')$$

then after a finite time $t_D = D^{1/2} \ln D + O(1)$ one has μu and v satisfying the assumptions of Theorem 3.3.1, i.e. $\mu u, v \geq 1 - \delta$ for $x \in (-M\sqrt{D}, M\sqrt{D})$. As a consequence, starting from the time $t = t_D$, propagation occurs as described in Theorem 3.1.2.

Remark 3.1.2.

- Observe that we assume a size condition on L : indeed, it is closely related to the existence of a large steady state to a Robin boundary value problem (the equation satsified by v in (3.1) when one sets $u = 0$). It happens that this problem admits a large enough solution only for L large enough, as we will see below.

- We also believe that $t_D \geq kD^\alpha$ for some $\alpha \in]0, 1/2]$ so that there is an incompressible time tending to $+\infty$ as $D \to +\infty$ while the propagation is slow (that is, independent of D).

Finally, we investigate the symmetrical situation of an initial data supported only on the road. Unlike the case of initial data supported on the field, if $\mu u_0 \leq 1$ has a support of size $\leq C\sqrt{D}$ (in the model (3.1)) there will be extinction. On the other hand, we provide conditions on μ for invasion to happen in the case $\mu u_0 \equiv 1$. We sum up these results in the following theorem, stated for (3.2) this time :

Theorem 3.1.4. *Let $v_0 \equiv 0$ and $\mu u_0 = \mathbf{1}_{(-a,a)}$ be initial data for* (3.2) *and u, v the associated solutions. We have the following :*

- *There exists $a_0 > 0$ independent of D such that if $a < a_0$, μu and v decay to* 0 *uniformly as $t \to +\infty$.*
- *If $a = +\infty$ there are thresholds $\mu^\pm$ independent of D such that for $\mu < \mu^-$ invasion occurs and for $\mu > \mu^+$, μu and v converge uniformly to $1/(\mu(L + 1/\mu)) \leq \theta$.*
- *More generally, provided $\mu < \mu^-$, there exists $a_1 > 0$ independent of D such that if $a > a_1$, invasion occurs.*

Remark 3.1.3. It is quite natural that μ too large leads to extinction : indeed, we normalised u so that $u \leq 1/\mu$ and moreover μ acts as a death rate in the equation on u. In the meanwhile, v sees the same initial boundary Robin condition $\equiv 1$ independantly from μ.

This raises the question of the normalisation of u, which never arose before because μ was fixed until now. Even though $1/\mu$ is a natural threshold for u in light of the comparison principle, it is another interesting viewpoint to study Theorem 3.1.4 with $u_0 \equiv 1$ (and not μu_0).

Bibliographical study and discussion

Before getting into the substance, let us put our results in their context and mention some classical results about the stability of reaction-diffusion fronts. The first works concerning the behaviour of compactly supported initial data in reaction-diffusion equation of combustion (or bistable) type can be found in Kanel′ [53]. For the one dimensional equation

$$\partial_t v - \partial^2_{xx} v = f(v)$$

the author shows the existence of two thresholds $L_0, L_1 > 0$ such that if $v_0 = \mathbf{1}_{(-l,l)}$ with $l < L_0$, v ends up below θ in finite time (and as a consequence, decays to 0 uniformly) : we call this situation *quenching*. On the other hand, if $l > L_1$ it is shown that $v \to_{t\to+\infty} 1$ uniformly on compact sets. Zlatoš [80] showed that in this context $L_0 = L_1$, and more generally Du and Matano [40] showed the existence of such sharp thresholds for more general one-parameter families of solutions of such equations.

The result of Kanel′ is refined by Aronson-Weinberger [1] in higher dimensions where the authors give the existence of a speed c_* at which the level sets of $v(t, x)$

move. This result was in turn refined by Fife-McLeod [42], where the authors show a precise mechanism of convergence towards travelling waves.

Since then, a considerable amount of research has been devoted to proving such results in heterogeneous contexts, shedding light on some interesting properties. Roquejoffre [75, 76] generalised the result in cylinders and in the presence of a velocity field $(\alpha(y), 0, \cdots, 0)$. An important issue in this context is to understand whether propagation or quenching will occur in terms of the size of the support of the initial datum. One important instance is the influence of the amplitude of the flow, which has been studied in various papers starting from [3]. In the KPP case and for shear flows, Berestycki [6] showed a linear speed-up

$$c_*(A) \underset{A\to+\infty}{\sim} kA$$

of the travelling wave speed, as the amplitude of the flow satisfies $A \to +\infty$. The result was also obtained and generalised by Constantin-Kiselev-Oberman-Ryzhik [33] by introducing the notion of *bulk burning rate*. For ignition type nonlinearities, the same result holds as proved by Hamel and Zlatoš [49] (see the introduction of Chapter 2 for a precise description of their result which is closely related to the contents of the chapter). On the other hand, Constantin-Kiselev-Ryzhik [34] and Kiselev-Zlatoš [55] show that the price to pay for this speed-up is also a linear scaling in A for the thresholds introduced before :

$$L_0 \underset{A\to+\infty}{\sim} k_0 A, \quad L_1 \underset{A\to+\infty}{\sim} k_1 A$$

provided that the flow is not constant on too large intervals. In other words, one trades a linear speed up of propagation for a linear growth in the critical size of initial data that leads to quenching.

In the case of cellular flows, the same phenomenon happens but with a scaling in $A^{1/4}$ (up to a logarithmic factor) : the speed-up property was proved by Novikov and Ryzhik [68] for the KPP case and more recently by Zlatoš [81] for combustion type nonlinearities. On the other hand, Fannjiang-Kiselev-Ryzhik [41] proved (for flows with small enough cells) that if $L^4 \ln(L) < kA$ – where L represents the size of the square supporting the initial datum – quenching happens. See also the numerical simulations of [79].

Another interesting mechanism is studied in Constantin-Roquejoffre-Ryzhik-Vladimora [35] where the authors investigate a system coupling a reaction-diffusion equation and a Burgers equation. They show different quenching results with respect to a gravity parameter, one of them being that quenching happens independently on l when the gravity is large enough.

From this point of view, Theorem 3.1.3 may come up as a surprise since it shows a speed-up of the propagation ($c = c_\infty\sqrt{D}$) for free : D does not appear in the threshold size of the initial data v_0. The trade-off is the presence of what we call "two-speed" dynamics : propagation first happens at a small speed that does not depend on D, but accelerates towards the full speed $c(D)$. On the other hand, if one tries to initiate the invasion only thanks to $\mu u_0 = \mathbf{1}_{(-l,l)}$, Theorem 3.1.4 shows that quenching happens if $l < a_0 D^{1/2}$ (from the point of view of the initial model (3.1)).

Organisation of the paper

The organisation of the chapter is as follows :

- Section 3.2 is devoted to proving Theorem 3.1.1. Its subsections introduce the material needed : an initial trapping of the initial data between two translates of the wave and the construction of wave-like sub and supersolutions.
- Section 3.3 provides the details for the proof of Theorem 3.1.2 by using a classical argument.
- In section 3.4 we prove Theorem 3.1.3 and describe more precisely the mechanism we introduced above.
- Finally the last section investigates the case of initial data supported on the road only.

3.2 Front-like initial data

3.2.1 Trapping the initial data

We study initial data that are a perturbation of the front. Set $\Omega_L = \mathbb{R} \times (-L, 0)$ and

$$\mathcal{P}_\alpha = \{(\rho_1, \rho_2) \in UC_0(\mathbb{R}) \times UC_0(\Omega_L) \ \mid \exists C > 0 \text{ s.t. } \rho_i(x) \le Ce^{\alpha x}\} \tag{3.3}$$

Then we assume

$$0 \le \mu u_0, v_0 \le 1 \tag{3.4}$$

$$(u_0, v_0) = (\phi(x + \xi), \psi(x + \xi)) + (\rho_1, \rho_2) \tag{3.5}$$

for some (ρ_1, ρ_2) in $\mathcal{P}_{\alpha_0}$ for some $\alpha_0 > 0$ and $\xi \in \mathbb{R}$. In this subsection, we prove that such initial data can be trapped between two translates of the travelling front, which is conceptually simple but necessary.

Due to the degeneracy of $f(v)$ as $v \le \theta$, we will have to use the following function in our stability results. Let $L_0 > 3$ and

$$0 < \alpha < \min(\alpha_0, c)$$

and define $\Gamma(x)$ to be a smooth non-decreasing function as pictured in Figure 3.9 such that

$$\Gamma(x) = \begin{cases} 1 \text{ if } x > L_0 \\ e^{\alpha(x+L_0)} \text{ if } x < -L_0 - 1 \end{cases} \tag{3.6}$$

We also recall the exponential convergence towards 0 or 1 as $x \to \pm\infty$ proved in the previous chapters : there exists $\lambda, \tilde{\lambda} > 0$ (bounded from below uniformly in $D > d$) and one can enlarge $L_0 > 0$ such that

$$\begin{aligned} &\forall x < -L_0/2 \qquad \mu\phi, \psi \qquad \le \frac{\theta}{2} e^{\lambda(x+L_0/2)} \qquad \le \frac{\theta}{2} \\ &\forall x > L_0/2 \qquad 1 - \mu\phi, 1 - \psi \le \frac{1-\theta_1}{2} e^{-\tilde{\lambda}(x-L_0/2)} \le \frac{1-\theta_1}{2} \end{aligned} \tag{3.7}$$

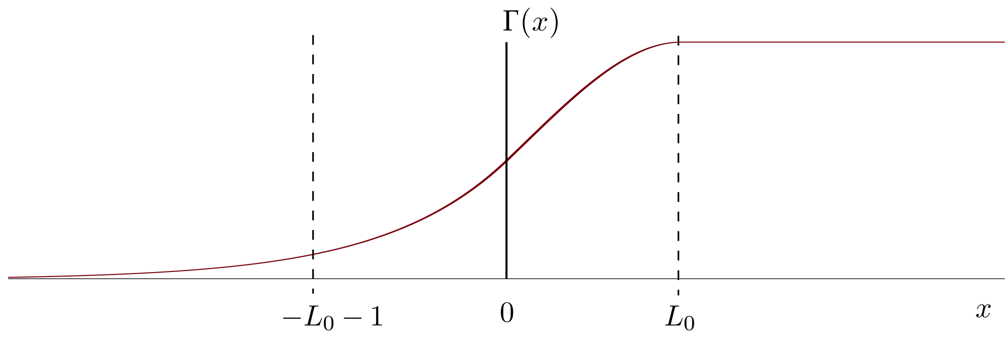

Figure 3.9: Shape of $\Gamma(x)$

where $\theta < \theta_1 < 1$ is chosen so that $-f'(s) \geq -f'(1)/2 =: \beta$ when $s > \theta_1$ as pictured in Figure 3.10. That way, ahead of the front the system becomes linear and behind the front one controls the monotonicity of f.

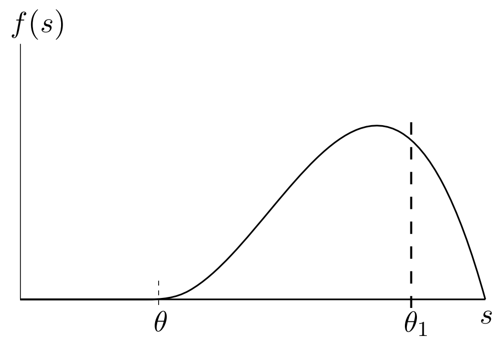

Figure 3.10: Example of f and definition of θ_1

We now have the setting to assert the following :

Proposition 3.2.1. *Assume* (3.4),(3.5)*. Then for any* $\varepsilon > 0$*, there exists* $\xi_0^- < 0$ *and* $\xi_0^+ > 0$ *large enough such that*

$$\mu\phi(x+\xi_0^-) - \varepsilon\Gamma(x+\xi_0^-) \leq \mu u_0(x) \leq \mu\phi(x+\xi_0^+) + \varepsilon\Gamma(x+\xi_0^+) \tag{3.8}$$

$$\psi(x+\xi_0^-, y) - \varepsilon\Gamma(x+\xi_0^-) \leq v_0(x,y) \leq \psi(x+\xi_0^+, y) + \varepsilon\Gamma(x+\xi_0^+) \tag{3.9}$$

Proof. We only prove (3.8). (3.9) is obtained simultaneously with the same arguments (y-uniform limits, y-uniform exponential decay) by taking $|\xi_0^\pm|$ large enough. We start with the right inequality. Let $\varepsilon > 0$. Thanks to the uniform limit of ϕ as $x \to +\infty$, there exists B_ε independent of ξ_0^+ such that for $x \geq -\xi_0^+ + L_0 + B_\varepsilon$,

$$\mu\phi(x+\xi_0^+) + \varepsilon\Gamma(x+\xi_0^+) \geq 1 - \varepsilon + \varepsilon = 1 \geq \mu u_0(x)$$

On the other hand, when $x \leq -\xi_0^+ - L_0 - 1$, $\mu u_0(x) \leq Ce^{\alpha_0 x}$ so here for the inequality to be true, one just needs

$$\varepsilon e^{\alpha(x+\xi_0^+ + L_0)} \geq Ce^{\alpha_0 x}$$

But since $x+\xi_0^+ + L_0 < 0$ and $0 < \alpha < \alpha_0$, one just needs

$$\varepsilon e^{\alpha_0(x+\xi_0^+ + L_0)} \geq C e^{\alpha_0 x}$$

which is ensured as soon as $\xi_0^+ > \ln(C/\varepsilon) - L_0$.

Now only the compact region $x \in (-\xi_0^+ - L_0 - 1, -\xi_0^+ + L_0 + B_\varepsilon)$ remains. Observe that on this interval, $\mu u_0(x)$ goes uniformly to 0 as $\xi_0^+ \to \infty$, whereas the right-hand side in (3.8) has a fixed positive infimum, so that the desired order becomes true by enlarging ξ_0^+ enough.

For the existence of ξ_0^- : observe that on $x \geq -\xi_0^- + L_0 + 1$

$$\mu\phi(x+\xi_0^-) - \varepsilon\Gamma(x+\xi_0^-) \leq 1 - \varepsilon \leq \mu u_0(x)$$

provided ξ_0^- is negative enough, thanks to the uniform limit of u_0 as $x \to +\infty$. Now for the rest of the proof, we need on $x \leq -\xi_0^- + L_0 + 1$

$$\varepsilon e^{\alpha(x+\xi_0^- + L_0)} \geq \mu\phi(x+\xi_0^-) - (\mu\phi(x) + \rho(x))$$

Because the exponential decay λ of ϕ and ψ satisfies $\lambda > c \geq \alpha$ (see [38]) this is true on $x \leq -\xi_0^- - L_0 - B_\varepsilon$ with $B_\varepsilon > 0$ large enough independent of ξ_0^- so that here

$$\varepsilon e^{\alpha(x+\xi_0^- + L_0)} \geq \theta e^{\lambda(x+\xi_0^-)} \geq \mu\phi(x+\xi_0^-)$$

Again, we cover the compact region left around the interface by enlarging $-\xi_0^-$ enough.

□

3.2.2 Wave-like sub and supersolution

We adapt the original result of Fife-McLeod [42] using the simplified notations and generalisation of Mellet-Nolen-Ryzhik-Roquejoffre [65]. The adaptation is non trivial from a computational point of view, so let us first explain the changes that we expect to happen. Our objective is to build a supersolution $\overline{u}, \overline{v}$ to (3.2) that is close to the front (in the frame moving at speed c). In the homogeneous case and for generalized transitions fronts the authors of [65] proposed

$$\bar{v} = \psi(x + c\xi(t)) + q(t)\Gamma(x + c\xi(t))$$

using the idea of Roquejoffre [76], where ψ is the front, $q(t) = \varepsilon e^{-\omega t}$ and Γ is defined above : this is a necessary correction to take into account the initial perturbation and the degeneracy of f on $v \leq \theta$ (which is not present in the bistable case and in the original paper [42]). $\xi(t)$ is a time increasing shift that starts from ξ_0^+ and converges to some $\xi_\infty^+ > \xi_0^+$: this shift has to be seen as a necessary correction because the fronts are only stable modulo translations and one has to shift suitably the wave over time to preserve a supersolution.

In our case, one has to look for

$$\begin{cases} \mu\bar{u} = \mu\phi(x + c\xi(t)) + q_u(t)\Gamma(x + c\xi(t)) \\ \bar{v} = \psi(x + c\xi(t), y) + q_v(t, y)\Gamma(x + c\xi(t)) \end{cases} \tag{3.10}$$

and

$$\begin{cases} \mu\underline{u} = \mu\phi(x - c\xi(t)) - q_u(t)\Gamma(x - c\xi(t)) \\ \underline{v} = \psi(x - c\xi(t), y) - q_v(t, y)\Gamma(x - c\xi(t)) \end{cases} \tag{3.11}$$

with ξ starting this time from ξ_0^-.

Indeed, the heterogeneity induced by the line of fast diffusion breaks the previous homogeneous supersolution ($q_u = q_v = q$) since the first component of $\mathcal{N}[\overline{u}, \overline{v}] \geq 0$ would yield $c\dot{\xi}\phi_x + \dot{q} \geq 0$, which is too restrictive for q if one wants exponential decay. The same problem happens if one assumes that q_v depends only on t, since the boundary inequation would give $q_v \geq q_u$ but the previous inequation would give $\dot{q}_u + q_u - q_v \geq 0$. As a consequence, one has to assume a y dependence in q_v : this has to be seen in the light of the observed dynamics, indeed, the solution $v(t, x, y)$ will stretch rapidly for y near 0 and only later for smaller values of y, so the correction has to take this into account. We now adapt the computations of [65] with this in mind.

The stability of this kind of travelling waves owes much to the fact that the x-derivatives of the fronts are uniformly positive on any compact set, more precisely one has

$$\forall M > 0, \exists \delta_M > 0 \mid \partial_x\phi, \partial_x\psi > \delta_M \text{ when } x \in (-M, M) \tag{3.12}$$

Now we reduce α a bit more and set

$$\alpha = \min(\alpha_0, c/5) \tag{3.13}$$

just so that the quantities

$$\alpha c - d/D\alpha^2 \geq \alpha c - \alpha^2 > \alpha c/4 - \alpha^2 > 0 \tag{3.14}$$

cannot be zero. These quantities will play an important role in the following computations. Observe that this condition on α means that the decay correction obtained through Γ is limited : solutions starting from large perturbations (i.e. small α_0) will be stabilized thanks to a correction with an α_0 decay also, but solutions from very small perturbations (i.e. very large α_0) will still need a $c/5$ correction in the decay at $-\infty$ to be stabilized.

Since we still want an exponential decay of $q_v(t)$ we look for

$$\begin{cases} q_u(t) = \varepsilon C e^{-\omega t} \\ q_v(t, y) = \varepsilon h(y) e^{-\omega t} \end{cases} \tag{3.15}$$

with separate variables. The boundary conditions yield $h'(-L) = 0$ and $h'(0) + h(0) = C$ so that we have a large choice for h. Nonetheless, it will become clear in the following computations that a good candidate is

$$C = \cosh\left(\sqrt{\kappa/d}L\right) + \sinh\left(\sqrt{\kappa/d}L\right) \tag{3.16}$$

$$h(y) = \cosh\left(\sqrt{\kappa/d}\,(y + L)\right) \tag{3.17}$$

with

$$\kappa = \min(\beta/2, (\alpha c - d/D\alpha^2)/2) > 0 \tag{3.18}$$

$$\omega = \min(G(\sqrt{\kappa/dL}), \beta/2, \mathrm{Lip} f, \alpha c/4 - \alpha^2) > 0 \tag{3.19}$$

and $G(x) = \frac{\mu \tanh(x)}{1+\tanh(x)}$. All of these conditions seem barbaric for the moment, but their role will appear clearly in the computations.

Observe that since $G'(0) > 0$, the decay exponent ω is then linearly small as β or α_0 or μ is small, but it should be noticed that it does not depend on $D \geq d$, and that it depends on the initial data only through α_0. We can now state the following :

Theorem 3.2.1. *Assume* (3.4),(3.5) *and let* u, v *denote the associated solutions of* (3.2)*. Let* $\varepsilon_0 = \min(\theta/4, (1-\theta_1)/4, \gamma_0)$ *where*

$$\gamma_0 = \frac{1}{4B}, B = \left(\frac{3Lipf + |\Gamma|_{C^2}}{c\delta_{L_0+2}}\right) C \max(1, 1/\mu)$$

There exists a constant K_0 *that depends on the initial data only through* α_0 *and such that if* $\varepsilon \in (0, \varepsilon_0)$*, there exists* $\xi_1^{\pm}$ *with*

$$\xi_1^+ \leq \xi_0^+ + \varepsilon K_0, \xi_1^- \geq \xi_0^- - \varepsilon K_0 \tag{3.20}$$

and for all $t \geq 0$*,*

$$\phi(x + c\xi_1^-) - q_u(t)\Gamma(x + c\xi_1^-) \leq \mu u(t, x - ct) \ \leq \mu\phi(x + c\xi_1^+) + q_u(t)\Gamma(x + c\xi_1^+) \tag{3.21}$$

$$\psi(x + c\xi_1^-) - q_v(t, y)\Gamma(x + c\xi_1^-) \leq v(t, x - ct, y) \leq \psi(x + c\xi_1^+) + q_v(t, y)\Gamma(x + c\xi_1^+) \tag{3.22}$$

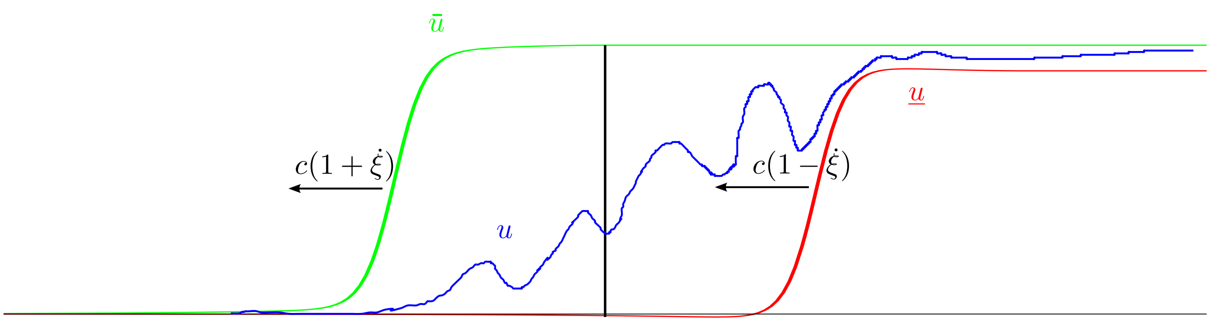

Figure 3.11: Trapping of the front-like data drawn at initial time

Proof. Inequations (3.21),(3.22) are set in the moving frame with variables $(t, x + ct)$. As a consequence, in the computations one has to replace ∂_t by $\partial_t + c\partial_x$. We now want to show that $\bar{u}, \bar{v}$ as defined in (3.10) yields indeed a supersolution :

$$\mathcal{N}\begin{pmatrix}\overline{u}\\ \overline{v}\end{pmatrix} \geq \begin{pmatrix}0\\ 0\end{pmatrix}$$

where

$$\mathcal{N}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_t - u_{xx} + cu_x + \mu u - v(\cdot, 0) \\ v_t - \frac{d}{D} v_{xx} - dv_{yy} + cv_x - f(v) \end{pmatrix}$$

and that $\underline{u}, \underline{v}$ as defined in (3.11) yields a subsolution. Then (3.21),(3.22) will follow by an application of the comparison principle, Prop. 3.2.1, and the monotonicity of ξ. Indeed, this will show that in the original frame u, v stays trapped between the fronts shifted initially by $\xi_0^{\pm}$ and moving at speed resp. $c(1 \pm \dot{\xi})$ (or speed $\pm\dot{\xi}$ in the moving frame) as pictured on Figure 3.11. A similar figure holds for v if one imagines it with a bounded continuous deformation along the y-axis due to the $h(y)$ factor in q_v. This deformation becomes of course exponentially small over time due to the $e^{-\omega t}$ factor. Observe also that $\dot{\xi} \leq 1/4$ and is exponentially decaying over time, so u, v will propagate at least and at most with speed $c + o(1)$.

We divide this computation in three zones concerning $x + \xi(t)$. In the following, ϕ and ψ will always mean $\phi(x + c\xi(t))$ and $\psi(x + c\xi(t), y)$, q_u will always mean $q_u(t)$, q_v will either mean $q_v(t, 0)$ or $q_v(t, y)$ and Γ will always mean $\Gamma(x + c\xi(t))$, all of these functions being defined as above in (3.6) and (3.15)-(3.17).

Behind the front : $x + \xi(t) > L_0 + 1$

Here $\Gamma \equiv 1$ and $\psi, \overline{v} \geq (1 + \theta_1)/2$ so that

$$\begin{aligned}
\mathcal{N}\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}_1 &= c\dot{\xi}\phi_x + \dot{q}_u/\mu - \phi_{xx} + c\phi_x + \mu\phi + q_u - \psi - q_v(t, 0) \\
&= c\dot{\xi}\phi_x + \dot{q}_u/\mu + q_u - q_v \\
&\geq \dot{q}_u/\mu + q_u - q_v \\
&= \varepsilon e^{-\omega t}\left(-Cw/\mu + C - \cosh\left(\sqrt{\kappa/d}L\right)\right) \\
&= \varepsilon e^{-\omega t}\left(-\left(\cosh\left(\sqrt{\kappa/d}L\right) + \sinh\left(\sqrt{\kappa/d}L\right)\right)w/\mu + \sinh\left(\sqrt{\kappa/d}L\right)\right) \geq 0
\end{aligned}$$

The first inequality holds because we look for $\dot{\xi} \geq 0$ and the last because $\omega \leq G(\sqrt{\kappa}L)$.

$$\begin{aligned}
\mathcal{N}\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}_2 &= c\dot{\xi}\psi_x + \dot{q}_v - d/D\psi_{xx} + c\psi_x - d\psi_{yy} - d\partial^2_{yy}q_v - f(\psi) + f(\psi) - f(\overline{v}) \\
&= c\dot{\xi}\psi_x + \dot{q}_v + f(\psi) - f(\overline{v}) - d\partial^2_{yy}q_v \\
&\geq \dot{q}_v - d\partial^2_{yy}q_v + \beta q_v \\
&= \varepsilon e^{-\omega t}h(y)(-\omega - \kappa + \beta) \\
&\geq \varepsilon e^{-\omega t}h(y)(-\omega + \beta/2) \geq 0
\end{aligned}$$

The last inequality holds because $w \leq \beta/2$ and the next to last because $\kappa \leq \beta/2$.

Ahead of the front : $x + \xi(t) < -L_0 - 1$

Heres $\Gamma(x + \xi(t)) = e^{\alpha(x + \xi(t) + L_0)}$, $\psi \leq \theta/2$ and $\bar{v} \leq \psi + \varepsilon \leq 3\theta/4 \leq \theta$ so there are no reaction terms.

$$\begin{aligned}
\mathcal{N}\begin{pmatrix}\overline{u}\\ \overline{v}\end{pmatrix}_1 &= c\dot{\xi}\phi_x + \left(\frac{\dot{q}_u}{\mu} + \frac{q_u}{\mu}\alpha c\dot{\xi} - \frac{q_u}{\mu}\alpha^2 + c\frac{q_u}{\mu}\alpha + q_u - q_v(\cdot,0)\right) e^{\alpha(x+\xi+L_0)}\\
&\geq \frac{1}{\mu}\left(\dot{q}_u + q_u\alpha c\dot{\xi} - q_u\alpha^2 + c\alpha q_u\right) e^{\alpha(x+\xi+L_0)}\\
&\geq \frac{1}{\mu}\left(-\omega + \alpha c - \alpha^2\right) e^{\alpha(x+\xi+L_0)} q_u \geq 0
\end{aligned}$$

The last inequality holds because $\omega \leq (\alpha c - \alpha^2)/2$, and the first because $q_u(t) \geq q_v(t,0)$.

$$\begin{aligned}
\mathcal{N}\begin{pmatrix}\overline{u}\\ \overline{v}\end{pmatrix}_2 &= c\dot{\xi}\psi_x + e^{\alpha(x+\xi+L_0)}\left(\dot{q}_v + q_v\left(\alpha(c\dot{\xi}+c) - d/D\alpha^2\right) - d\partial^2_{yy}q_v\right)\\
&\geq e^{\alpha(x+\xi+L_0)}q_v\left(-\omega + \alpha c\dot{\xi} + \alpha c - d/D\alpha^2 - \kappa\right)\\
&\geq e^{\alpha(x+\xi+L_0)}q_v\left(-\omega + (\alpha c - d/D\alpha^2)/2\right) \geq 0
\end{aligned}$$

The last inequality holds because of the condition on ω, and the next to last because of the condition on κ and because $\dot{\xi} \geq 0$.

The middle region : $|x+\xi(t)| < L_0 + 2$

$$\begin{aligned}
\mathcal{N}\begin{pmatrix}\overline{u}\\ \overline{v}\end{pmatrix}_1 &= c\dot{\xi}\phi_x + \frac{\dot{q}_u}{\mu}\Gamma + c\dot{\xi}\frac{q_u}{\mu}\Gamma_x - \frac{q_u}{\mu}\Gamma_{xx} + c\frac{q_u}{\mu}\Gamma_x + (q_u - q_v)\Gamma\\
&\geq c\dot{\xi}\phi_x + \frac{\dot{q}_u}{\mu}\Gamma - \frac{q_u}{\mu}\Gamma_{xx}\\
&\geq c\dot{\xi}\delta_{L_0+2} - (\omega + |\Gamma|_{C^2})\frac{q_u}{\mu} \geq 0
\end{aligned}$$

Provided

$$\dot{\xi} \geq \frac{\omega + |\Gamma|_{C^2}}{c\delta_{L_0+2}} q_u \tag{3.23}$$

$$\begin{aligned}
\mathcal{N}\begin{pmatrix}\overline{u}\\ \overline{v}\end{pmatrix}_2 &\geq c\dot{\xi}\psi_x - q_v\mathrm{Lip}f + \dot{q}_v\Gamma + c\dot{\xi}q_v\Gamma_x - d/Dq_v\Gamma_{xx} + cq_v\Gamma_x - d\partial^2_{yy}q_v\Gamma\\
&\geq c\dot{\xi}\delta_{L_0+2} - q_v\mathrm{Lip}f - \omega q_v - d/D|\Gamma|_{C^2}q_v - \kappa q_v \geq 0
\end{aligned}$$

Provided

$$\dot{\xi} \geq \frac{\mathrm{Lip}f + \omega + d/D|\Gamma|_{C^2} + \kappa}{c\delta_{L_0+2}} q_v \tag{3.24}$$

We obtain conditions (3.23), (3.24) by remarking that $\kappa \leq \beta < \mathrm{Lip}(f)$, $\omega < \mathrm{Lip}(f)$ and $d/D < 1$ and then we take

$$\dot{\xi}(t) = B\varepsilon e^{-\omega t}$$

so that

$$\xi(t) = \xi_0^+ + \frac{B\varepsilon(1-e^{-\omega t})}{\omega} \tag{3.25}$$

and

$$K_0 = B/\omega \tag{3.26}$$

answer our queries.

One should observe that the condition $B\varepsilon \leq 1/4$ has not been used yet as well as $\omega \leq c\alpha/4 - \alpha^2$ rather than just $1/2(c\alpha - \alpha^2)$. Observe that the computations concerning the subsolution (3.11) with this time

$$\xi(t) = -\xi_0^- + \frac{B\varepsilon(1-e^{-\omega t})}{\omega} \tag{3.27}$$

are exactly symmetric, except for a $c\alpha(1-\dot{\xi})$ term (instead of $c\alpha(1+\dot{\xi})$) that appears ahead of the front and in the middle region, which is treated thanks to the above still unused assumptions :

$$\begin{aligned}
-\omega + c\alpha(1-\dot{\xi}) - d/D\alpha^2 - \kappa &\geq -\omega + \frac{3c\alpha}{4} - d/D\alpha^2 - \kappa \\
&\geq -\omega + \frac{3c\alpha}{4} - \alpha^2 - \kappa \\
&\geq -\omega + \frac{c\alpha}{4} - \frac{\alpha^2}{2} \\
&\geq -\omega + \frac{c\alpha}{4} - \alpha^2 \geq 0.
\end{aligned}$$

$$\begin{aligned}
-\omega + c\alpha(1-\dot{\xi}) - \alpha^2 &\geq -\omega + \frac{3c\alpha}{4} - \alpha^2 \\
&\geq -\omega + \frac{c\alpha}{4} - \alpha^2 \geq 0.
\end{aligned}$$

□

3.3 Compactly supported initial data

In this section, we go back in the fixed original frame. Seeing the problem in the light of [42] and [76] it is natural to test :

$$\begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix} = \begin{pmatrix} \phi(x+ct+c\xi(t)) + \phi(-x+ct+c\xi(t)) - 1/\mu \\ \psi(x+ct+c\xi(t)) + \psi(-x+ct+c\xi(t)) - 1 \end{pmatrix}$$

as a subsolution to (3.2), i.e. a pair of waves evolving in opposite directions. Of course, in light of the previous section, for this to be a subsolution one needs a well chosen correction in time and in space (in the degeneracy regime of f). Let us define the symmetrized fronts

$$\begin{aligned}
\tilde{\phi}(\cdot) &= \phi(-\cdot) \\
\tilde{\psi}(\cdot) &= \psi(-\cdot)
\end{aligned} \tag{3.28}$$

In the sequel we will always use the following notations :

$$\begin{aligned}\phi &= \phi(x + ct + \xi_0 - c\xi(t))\\ \tilde{\phi} &= \tilde{\phi}(x - ct - \xi_0 + c\xi(t))\end{aligned} \tag{3.29}$$

and the same will hold for $\psi, \tilde{\psi}, \Gamma, \tilde{\Gamma}$. Here ξ_0 will be a large initial shift and $\xi(t)$ a time-increasing shift with $\xi(0) = 0$ and

$$c\xi(+\infty) \leq 1 \tag{3.30}$$

which will be realised as a smallness condition on ε_0. In this section

$$\alpha = \min(\lambda, \tilde{\lambda}, c/5) \tag{3.31}$$

where λ and $\tilde{\lambda}$ are already defined in (3.7) so that α yields the same inequations as above and moreover $\alpha < \lambda, \tilde{\lambda}$. Γ is defined as above, only with a little more margin. Precisely let us set this time :

$$\Gamma(x) = \begin{cases} 1 \text{ if } x > L_0 - 1 \\ e^{\alpha(x+L_0)} \text{ if } x < -L_0 + 1 \end{cases} \tag{3.32}$$

We will set the following :

$$\begin{cases} \underline{u} = \max\left(0, \phi + \tilde{\phi} - 1/\mu - q_u(t)/\mu \min(\Gamma, \tilde{\Gamma})\right) \\ \underline{v} = \max\left(0, \psi + \tilde{\psi} - 1 - q_v(t,y) \min(\Gamma, \tilde{\Gamma})\right) \end{cases} \tag{3.33}$$

The proof will consist in adapting the previous computations. We shall see that $(\underline{u}, \underline{v})$ yields a subsolution provided only a size condition on the initial shift ξ_0 (independently of $D > d$). This condition is important, because then for the initial data to lie above $(\underline{u}(0), \underline{v}(0))$ it has to be large enough on a large enough interval. Moreover, we wish to insist on the fact that to retrieve the original model (3.1) one has to change the variable $x \leftarrow x/\sqrt{D}$. As a consequence, when stated for (3.1), our result assumes that u_0, v_0 are large enough on an interval with length of order $\sqrt{D}$. Theorem 3.1.2 will be proved as soon as we have proved the

Theorem 3.3.1. *1. There exists $\varepsilon_0 > 0$ small enough and two constants $B, \xi_0 > 0$ large enough such that for all $0 < \varepsilon < \varepsilon_0$, there exists a small $\delta > 0$ and $M > 0$ such that if $0 \leq \mu u_0, v_0 \leq 1$ satisfy $\mu u_0, v_0 > 1 - \delta$ on $x \in (-M, M)$, then*

$$\underline{u} = \max(0, \phi + \tilde{\phi} - 1/\mu - q_u/\mu \min(\Gamma, \tilde{\Gamma}))$$

$$\underline{v} = \max(0, \psi + \tilde{\psi} - 1 - q_v \min(\Gamma, \tilde{\Gamma}))$$

where $q_u = \varepsilon C e^{-\omega t}$ and $q_v = \varepsilon h(y) e^{-\omega t}$ are defined as above and this time

$$\xi(t) = \frac{B\varepsilon(1 - e^{-\omega t})}{\omega} \tag{3.34}$$

defines a subsolution to (3.2) *with initial data* u_0, v_0 *for all times. By the comparison principle, we then have at all times*

$$\underline{u} \le u, \underline{v} \le v$$

As a consequence, (3.2) *propagates the initial data* u_0, v_0 *along the* x*-axis with speed at least as* $ct + o(1)$ *in both directions.*

2. *Using the notations of Section 2 we have the following : let* $\tilde{u}, \tilde{v}$ *denote the same functions as in* (3.10) *with* ϕ, ψ *and* Γ *replaced by* $\tilde{\phi}, \tilde{\psi}, \tilde{\Gamma}$*. As a consequence,* $\tilde{u}, \tilde{v}$ *will be a supersolution for decreasing front-like initial data. Up to enlarging the initial shifts, we assert that*

$$(\min(\bar{u}, \tilde{u}), \min(\bar{v}, \tilde{v}))$$

is a supersolution to (3.2) *with initial data* u_0, v_0 *for all times. Again, this implies that*

$$u \le \min(\bar{u}, \tilde{u}), v \le \min(\bar{v}, \tilde{v})$$

and so that the level lines of u, v *propagate at most as* $c(t + \xi(t)) = ct + o(1)$ *in both directions along the* x*-axis.*

Remark 3.3.1.
- The previous theorem says simply that the compactly supported initial data which are large enough on a large enough interval will stay trapped – up to an exponentially small correction in time – between two translates of a pair of fronts evolving in both directions, as pictured in Figure 3.12

- As noticed above, observe that one needs to replace $M \leftarrow M\sqrt{D}$ when Theorem 3.3.1 is stated for the original system (3.1).

- The size condition on u_0, v_0 is far from optimal and ensures only that $u_0 \ge \underline{u}(0), v_0 \ge \underline{v}(0)$ as pictured on Figure 3.12. It could be sharpened by replacing $1 - \delta$ with θ and by waiting long enough for the reaction to put u, v above $1 - \delta$.

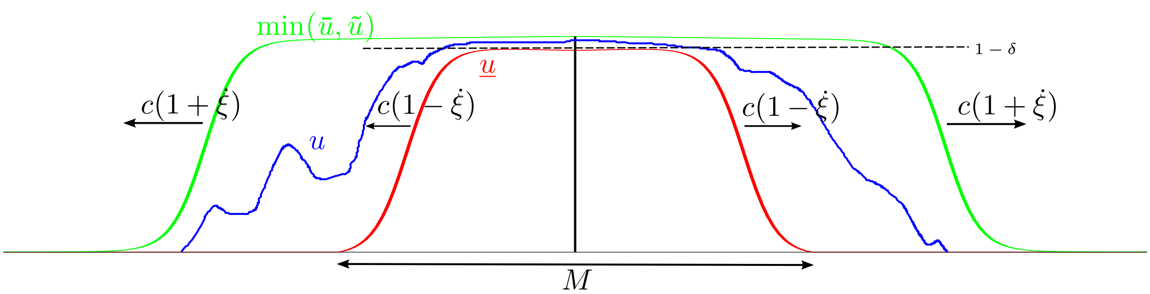

Figure 3.12: Trapping of the c.c. data drawn at initial time

Proof. The second part of Theorem 3.3.1 is easy because the minimum of two supersolutions is a supersolution and any front like initial data can be translated above any compactly supported initial data.

The first part is more intricate. Observe that $\underline{u}(0), \underline{v}(0)$ are zero except on a set of length $(-M, M)$ (with M proportional to ξ_0) and that on $(-M, M)$ they are less than some $1-\delta$: this directly gives the largeness condition asked so that $u_0 \geq \underline{u}(0), v_0 \geq \underline{v}(0)$. We now detail the computation of $\mathcal{N}(\underline{u}, \underline{v})$ in the following subsections by splitting the computations in three zones concerning $x + ct + \xi_0$.

3.3.1 $x + ct + c\xi_0 < -L_0$

In this zone, one has necessarily $x + ct + \xi_0 - \xi(t) < -L_0$ and also $x - ct - \xi_0 + \xi(t) < -L_0$ (by asking $2\xi_0 \geq 1$). As a consequence, in this zone we have $\mu\phi, \psi, \underline{v} \leq \theta/2$ and $\mu\tilde{\phi}, \tilde{\psi} \geq (1+\theta_1)/2$. Also $\min(\Gamma, \tilde{\Gamma}) \equiv \Gamma \equiv e^{\alpha(x+ct+\xi_0-\xi(t)+L_0)}$ will be denoted $e^{\alpha(\cdots)}$ from now on. Then

$$
\begin{aligned}
\mathcal{N}\begin{pmatrix}\underline{u}\\ \underline{v}\end{pmatrix}_1 &= -c\dot{\xi}\phi_x + c\dot{\xi}\tilde{\phi}_x - \frac{\dot{q}_u}{\mu}\tilde{\phi}_x e^{\alpha(\cdots)} - \frac{q_u}{\mu}c\alpha e^{\alpha(\cdots)} + \frac{q_u}{\mu}c\alpha\dot{\xi}e^{\alpha(\cdots)}\\
&+ \frac{q_u}{\mu}\alpha^2 e^{\alpha(\cdots)}\\
&- q_u e^{\alpha(\cdots)} + q_v e^{\alpha(\cdots)}\\
&\leq -\frac{q_u}{\mu}e^{\alpha(\cdots)}\left(-\omega + c\alpha(1-\dot{\xi}) - \alpha^2\right) + (q_v - q_u)e^{\alpha(\cdots)}
\end{aligned}
$$

Both terms are already negative thanks to the conditions stated in Section 2. Then a computation similar to the preceding section – thus not detailed here – leads to

$$
\mathcal{N}\begin{pmatrix}\underline{u}\\ \underline{v}\end{pmatrix}_2 \leq -q_v e^{\alpha(\cdots)}\left(-\omega + c\alpha(1-\dot{\xi}) - d/D\alpha^2 - \kappa\right) + f(\tilde{\psi})
$$

This quantity can be made negative provided $\omega \leq 2\alpha c$ (which is already the case) : indeed, using the exponential decay of $f(\tilde{\psi})$ in this zone, the above expression can be factorized as

$$
-q_v e^{\alpha(\cdots)} \times (\cdot)
$$

with $(\cdot)$ having the sign of $-\omega + c\alpha(1-\dot{\xi}) - d/D\alpha^2 - \kappa \geq 0$ provided only that ξ_0 is large enough (but depending on the initial data only through α_0).

3.3.2 $x + ct + \xi_0 \in (-L_0, L_0)$

First, we ensure $c\xi \leq 1$ by asking that $c\dfrac{B\varepsilon}{\omega} \leq 1$ so by taking

$$
\varepsilon_0 \leq \frac{\omega}{cB} \tag{3.35}
$$

As a consequence, $x + ct + \xi_0 - c\xi(t) \in (-L_0 - 1, L_0)$ and $x - ct - \xi_0 + c\xi(t) < -L_0$. Since $\omega < 2\alpha c$, the computations of section 3.2.2 still hold by enlarging the constant B enough.

3.3.3 $x + ct + \xi_0 > L_0$

Here three subcases can appear concerning $x - ct - \xi_0$. By exchanging ϕ, ψ and $\tilde{\phi}, \tilde{\psi}$ and since $\alpha < \lambda$, the cases $x - ct - \xi_0 \in (-L_0, L_0)$ and $x - ct - \xi_0 > L_0$ are already covered by the computations above. Only the case $x - ct - \xi_0 < -L_0$ remains. In this zone, $x - ct - \xi_0 + c\xi(t) < -L_0 + 1$, so here $\min(\Gamma, \tilde{\Gamma}) \equiv 1$ and both ψ and $\tilde{\psi}$ are close to 1.

Observe that the computations of section 2 still hold by splitting this zone in two subzones : $x < 0$ and $x > 0$. In the first one, one will bound $f(\psi) + f(\tilde{\psi}) - f(\underline{v})$ by $\text{Lip} f (1 - \psi) - \beta(1 - \tilde{\psi} + q_v)$ and in the second one by $\text{Lip} f (1 - \tilde{\psi}) - \beta(1 - \psi + q_v)$. Then, since $\omega < \min(\lambda, \tilde{\lambda}) c$ there holds

$$\mathcal{N} \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}_2 \leq -q_v \times (\cdot)$$

with $(\cdot)$ being positive provided ξ_0 is large enough. This proves Theorem 3.3.1. $\square$

3.4 Initial data with small compact support

We now go back to the original equation (3.1) and state the following.

Theorem 3.4.1. *Let L be large enough (independently of D). There exists $M', \delta' > 0$ independent of $D > d$ such that if the initial data of (3.1) satisfies*

$$v_0 > 1 - \delta' \textit{ for } x \in (-M', M')$$

then after a finite time $t_D = D^{1/2} \ln D + O(1)$ one has μu and v satisfying the assumptions of Theorem 3.3.1, i.e. $\mu u, v \geq 1 - \delta$ for $x \in (-M\sqrt{D}, M\sqrt{D})$. As a consequence, starting from the time $t = t_D$, propagation occurs as described in Theorem 3.3.1.

We will divide the proof in several steps :

- First, since $D > d$ is ought to be large, u should be very small for small times. Thus we first investigate the equation for v in (3.1) where u is replaced by 0, and we not only expect to use its solution as a subsolution but we really expect that it will reflect the dynamics of the full solution for some time. By the maximum principle and Hopf's lemma and since $u > 0$, the solution $(0, \underline{v})$ to this problem will serve as a subsolution for (3.1) :

$$\begin{cases} d\partial_y \underline{v} + \underline{v} = 0 \\ \partial_t \underline{v} - d\Delta \underline{v} = f(\underline{v}) \\ \partial_y \underline{v} = 0 \end{cases} \tag{3.36}$$

- Then we investigate the largest steady solution $p(y)$ of (3.36) (observe that 1 is not a steady state). This is where the assumption on L plays a crucial role : it is needed for this steady state to be large enough. This will be the purpose of Lemma 3.4.1 below.

- Then the existence of a travelling wave for (3.36) connecting 0 and $p(y)$ will serve to build a subsolution for (3.36) propagating just as in Theorem 3.3.1 but here at speed $c_p = O(1)$. This will give a lower bound on the boundary data $v(x,-L) \geq \underline{v}(x,-L) \geq \cdots$. This will be the purpose of Lemma 3.4.2.

- Using this lower bound, we then go back to (3.2) : we show that even without the reaction term, this lower bound suffices to have $\mu u, v \geq 1-\delta$ on $(-M, M)$ within a finite time t_D. As a consequence, this is the case also for the nonlinear problem. This will be proved in a final step.

We start with the following lemma : it is standard but we prove it for completeness.

Lemma 3.4.1. *There exists $L_0 > 0$ such that if $L > L_0$, there exists a solution $p(y)$ to*

$$\begin{cases} -dp'' = f(p) & (3.37a) \\ p'(-L) = 0 & (3.37b) \\ dp'(0) + p(0) = 0 & (3.37c) \end{cases}$$

with $p > 0$ concave decreasing and $p(-L) = 1 - \delta'' > 1 - \delta$. Moreover $\delta'' \to 0$ as $L \to +\infty$.

Proof. Since $f \geq 0$, every solution of (3.37a)-(3.37c) is concave decreasing. Multiplying (3.37a) by $p'(y)$, integrating and using (3.37b) yields

$$-d\frac{p'(y)^2}{2} = F(p(y)) - F(p(-L)) \tag{3.38}$$

where

$$F(v) = \int_0^v f(s)\mathrm{d}s \tag{3.39}$$

Now (3.37c) becomes

$$dp'(0) = \sqrt{2d}\sqrt{F(p(-L)) - F(p(y))}$$

Since F is increasing on $(\theta, 1)$ from 0 to $\displaystyle\int_0^1 f$ we define F^{-1} to be its inverse function in this range and we now impose $p(0)$ to satisfy

$$0 < \frac{p(0)^2}{2d} + F(p(0)) < \int_0^1 f$$

which in turn imposes a maximum value for $p(0)$:

$$p(0) < p_0^{\max} \tag{3.40}$$

And under condition (3.40) one can solve (3.37c) by taking :

$$p(-L) = F^{-1}\left(\frac{p(0)^2}{2d} + F\left(p(0)\right)\right) \tag{3.41}$$

Going back to (3.38) one now has

$$\sqrt{\frac{d}{2}}\frac{p'(y)}{\sqrt{F(p(-L)) - F(p(y))}} = 1$$

so that integration on $(-L, 0)$ and the change of variables $v = \psi(y)$ yields

$$\sqrt{\frac{d}{2}}\int_{p(0)}^{p(-L)} \frac{\mathrm{d}v}{\sqrt{F(p(-L)) - F(v)}} = L$$

And finally the changes of variables $v = F^{-1}(u)$ and an affine one yield

$$N(p_0) \ := \frac{p(0)}{2}\int_0^1 \frac{\mathrm{d}w}{f\left(F^{-1}\left(\frac{p(0)^2}{2d}w + F(p(0))\right)\right)\sqrt{1-w}} = L \tag{3.42}$$

Since $F'(\theta) = F'(1) = 0$ observe that one has $\lim_{p\to\theta} N(p) = \lim_{p\to p_0^{\max}} N(p) = +\infty$. Moreover, N is positive, starts to be decreasing and then becomes increasing. As a consequence it has a positive minimum L_0 which plays the role of a threshold value : if $L < L_0$, there is no solution of (3.42). If $L = L_0$ there is only one, and if $L > L_0$ there are two. See Figure 3.13 :

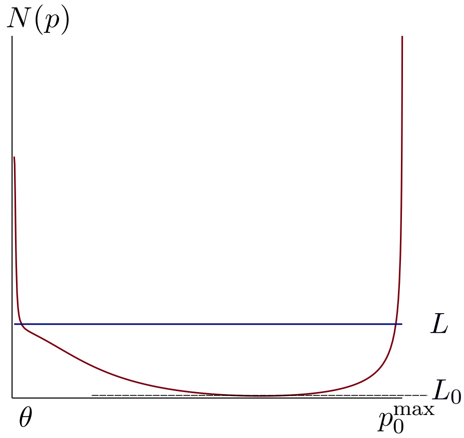

Figure 3.13: Behaviour of N

We now choose $p(0)$ to be the largest solution of the two and we call it p_0^+. One has clearly $p_0^+ \to p_0^{\max}$ as $L \to +\infty$ so by (3.41), $p(-L) \to 1$ as $L \to +\infty$. We now just enlarge L enough for $p(-L) > 1 - \delta$ and we conclude by solving the Cauchy problem

$$\begin{cases} -p'' = f(p) \\ p(-L) = F^{-1}\left(\frac{{p_0^+}^2}{2d} + F\left(p_0^+\right)\right) \\ p'(-L) = 0 \end{cases}$$

Figure 3.14 represents the qualitative behaviour of p :

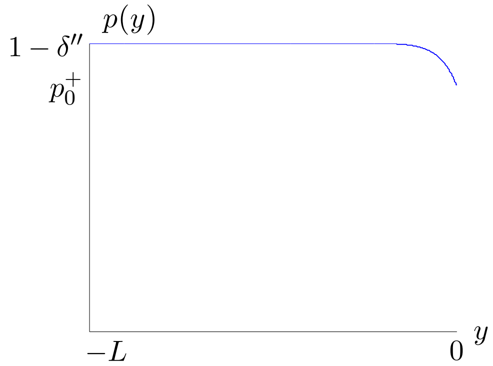

Figure 3.14: Behaviour of p

□

Lemma 3.4.2. *Let v be a solution of* (3.36)*. There exists $\delta', M' > 0$ independent of D such that if $v_0 > 1-\delta'$ for $x \in (-M', M')$, there holds*

$$v(t,x,-L) \geq (1-\delta'')\varphi_t(x) - Ce^{-bt}$$

where $C > 0$ and $b > 0$ are constants that do not depend on D and $\varphi_t(x)$ defines a family of smooth functions bounded in $\mathcal{C}^3$ such that $\varphi_t(x) = 1$ for $|x| \leq \frac{c_p}{2}t$ and $\varphi_t(x) = 0$ for $|x| \geq c_p t$ for some speed $c_p > 0$ independent of D.

Proof. First, that there exists a travelling wave solution with speed $c_p > 0$ independent of D of (3.36) connecting 0 and $p(y)$ has to be established : for this we refer to Berestycki-Nirenberg [20] which gives the existence of an increasing (in x) travelling front $\psi(x,y)$ with exponential convergence towards 0 and $p(y)$ as $x \to \pm\infty$.

Now we notice that the subsolution argument in Theorem 3.3.1 can be used but in a simpler fashion for the Robin homogeneous boundary value problem (3.36) : one the one hand, the structure of the problem is simpler than the one studied in Theorem 3.3.1 since here we deal with a single equation : the original construction of [42] with $q_v = \varepsilon e^{-\omega t}$ will suffice. On the other hand, 1 is not a steady state for the problem so one has to replace 1 by $p(y)$ in the computations. Nonetheless, one can check that the above computations still hold with the adequate subsolution

$$\psi + \tilde{\psi} - p - q_v \min(\Gamma, \tilde{\Gamma})$$

As a consequence, just as in Theorem 3.3.1, provided v_0 is above an initial shift of a pair of waves – hence the existence of δ' and M' – its level lines will be pushed by below by the pair of waves travelling as $\pm c_p t \mp O(1)$. This implies the desired bound. □

End of the proof of Theorem 3.4.1. Let (u^D, v^D) be the solution of (3.2) starting from compactly supported $0 \leq \mu u_0, v_0 \leq 1$ and let v_0 satisfy the rescaled assumptions of Theorem 3.4.1. First, let $T_D = D^{1/2} \ln D$. We now show the following

$$\liminf_{t\to+\infty}\inf_{D>d}\min_{(x,y)\in\overline{\Omega_{L,M}}}\{\mu u^D(T_D+t,x),v^D(T_D+t,x,y)\}\geq 1-\delta''>1-\delta \quad (3.43)$$

where $\Omega_{L,M}=(-M,M)\times(-L,0)$. First, it is an easy but tedious exercise to see that the left hand-side of (3.43) can be characterised as the limit as $n\to+\infty$ of some $\mu u^{D_n}(T_{D_n}+t_n,x_n)$ or $v^{D_n}(T_{D_n}+t_n,x_n,y_n)$ where $t_n\to+\infty$, $D_n>d$, $(x_n,y_n)\in\overline{\Omega_{L,M}}$. We then extract from (t_n,D_n,x_n,y_n) a subsequence so that $x_n\to x_\infty$ and $y_n\to y_\infty$. Our objective is to extract from (u,v) a subsequence converging to some limiting (u_∞,v_∞) to which the maximum principle will apply and force the above limit to be $\geq 1-\delta''$. The difficulty comes from the fact that (D_n) might be unbounded and so that standard parabolic estimates and the usual maximum principle might fall at the limit. Two cases can appear :

i) (D_n) is unbounded. Then we extract again so that $D_n\to+\infty$. Let

$$u_n(t,x):=u^{D_n}(t_{D_n}+t_n+t,x_\infty+x)$$
$$v_n(t,x,y):=v^{D_n}(t_{D_n}+t_n+t,x_\infty+x,y)$$

Since $f\geq 0$ and by Lemma 3.4.2 above, by the comparison principle we have $(u_n,v_n)\geq(\underline{u}_n,\underline{v}_n)$ the solution of

$$\begin{array}{c}\partial_t\underline{u}_n-\partial^2_{xx}\underline{u}_n=\underline{v}_n-\mu\underline{u}_n\\ \hline d\partial_y\underline{v}_n=\mu\underline{u}_n-\underline{v}_n\\ \\ \partial_t\underline{v}_n-\frac{d}{D_n}\partial^2_{xx}\underline{v}_n-d\partial^2_{yy}v_n=0\\ \\ \underline{v}_n=(1-\delta'')\varphi_{T_{D_n}+t_n}(x_\infty+x)-Ce^{-b(t_n+t)}\\ \hline\end{array}\quad (3.44)$$

Since $d/D_n\to 0$, the standard parabolic estimates applied on v_n will fall concerning the x-derivatives. We overcome this difficulty since equation (3.44) is linear and the boundary data $\underline{v}_n(t,x,-L)$ is bounded in $\mathcal{C}^3$: the maximum principle applied on x-derivatives of (u_n,v_n) up to order 3 gives that they are all bounded independantly of n :

$$|\partial^2_{xx}\underline{u}_n|_\infty,|\partial^3_{xxx}\underline{u}_n|_\infty,|\partial^2_{xx}\underline{v}_n|_\infty,|\partial^3_{xxx}\underline{v}_n|_\infty\leq C_1$$

Now concerning the y-derivatives, even though $d/D_n\to 0$ the standard estimates hold : indeed since $\underline{v}_n\leq 1$, standard L^p parabolic estimates with p large enough applied on $\underline{u}_n$ give that $\underline{u}_n$ is bounded in $\mathcal{C}^{\alpha,1+\alpha}$ by some C_2. Now rescale by $x\leftarrow x\sqrt{D_n}$ so that $|\underline{u}_n(t,\frac{x}{\sqrt{D_n}})|_{\mathcal{C}^{\alpha,1+\alpha}}\leq C_3$ (the semi-norms of the derivatives even go to zero since $1/D_n\to 0$). Moreover, under this rescaling $-d/D_n\partial^2_{xx}-d\partial^2_{yy}$ becomes $-d\Delta$ so that standard parabolic estimates up to the Robin boundary apply and give that $|\underline{v}_n(t, \frac{x}{\sqrt{D_n}}, y)|_{\mathcal{C}^{1+\alpha/2,2+\alpha}} \leq C_4$. Since this rescaling does not impact ∂_y or ∂_t, this gives

$$|\partial_t \underline{v}_n|_{\alpha/2}, |\partial_y \underline{v}_n|_\alpha, |\partial^2_{yy} \underline{v}_n|_\alpha \leq C_4$$

The bound on $\partial^2_{xy} \underline{v}_n$ follows also by combining the two arguments above, and finally by plugging the estimate on v in the equation for u, standard Schauder estimates yield that $\underline{u}_n$ is bounded in $\mathcal{C}^{1+\alpha/2,2+\alpha}$. In the end one can extract from $(\underline{u}_n, \underline{v}_n)$ some subsequence converging in $\mathcal{C}^{1,2}_{loc}$ to some (u_∞, v_∞) global in time (since $t_n \to +\infty$) solving

$$\frac{\partial_t u_\infty - \partial^2_{xx} u_\infty = v_\infty - \mu u_\infty}{d\partial_y v_\infty = \mu u_\infty - v_\infty}$$

$$\frac{\begin{array}{c}\partial_t v_\infty - d\partial^2_{yy} v_\infty = 0 \\ \\ v_\infty = (1 - \delta'')\end{array}}{} \tag{3.45}$$

Indeed, $v_\infty(t, x, -L) \equiv 1 - \delta''$ since

$$1 - \delta'' \geq \underline{v}_n(t, x, -L) \geq 1 - \delta'' - Ce^{-b(t_{D_n} + t_n + t)}$$

for $x \in (-\frac{c_p}{2} \ln D_n, \frac{c_p}{2} \ln D_n)$ by Lemma 3.4.2 above and by use of T_{D_n}.

Since (u_∞, v_∞) are global in time, there is no initial data anymore and the maximum principle applies to give

$$\mu u_\infty, v_\infty \equiv 1 - \delta''$$

Indeed, no value different than $1 - \delta''$ can be reached, because then (u, v) would have an infimum smaller or a supremum larger than $1 - \delta''$. By translating over time (which is possible since the solution is global) this infimum or supremum would become a minimum or maximum, that cannot be reached by u because of the strong parabolic maximum principle, and neither by v by the strong parabolic maximum principle and Hopf's lemma applied on the suitable y-slice.

ii) (D_n) is bounded. Then one extracts so that $D_n \to D_\infty > d$ and the above proof is much simpler since standard regularity and maximum principle apply. Moreover T_D is not necessary.

In any case, the lim inf above is $\geq u_\infty(0, 0) = 1 - \delta''$ or $\geq v_\infty(0, 0, y_\infty) = 1 - \delta''$, thus (3.43) holds. Theorem 3.4.1 follows easily : indeed, there exists t_1 independant of D such that after $t_D = T_D + t_1$, $\mu u, v > 1 - \delta$ on $(-M, M)$.

□

Remark 3.4.1. Observe that $T_D = D^{1/2} \ln D$ could be replaced by any $D^{1/2} h(D)$ with $h(D) \to +\infty$ as $D \to +\infty$.

3.5 Initial data supported on the road only

In this section we investigate the behaviour of solutions starting from $(u_0, v_0) = (\mathbf{1}_{(-a,a)}, 0)$. From now on, in order to alleviate the notations let $\varepsilon := 1/\sqrt{D}$.

3.5.1 a is small

Theorem 3.5.1. *There exists $a_0 > 0$ such that for $a < a_0$, the solution of (3.2) starting from $(\mathbf{1}_{(-a,a)}, 0)$ decays to 0 uniformly.*

The proof relies on a suitable reformulation of equation (3.1) and a crude linear bound on f.

Lemma 3.5.1. *Replace v by its even extension on $\mathbb{R} \times [-L, L]$. Then (u, v) satisfies*

$$\begin{cases} \partial_t v - d\varepsilon^2 \partial^2_{xx} v - d\partial^2_{yy} v = f(v) + 2d(\mu u - v)(x, 0)\mathrm{d}\lambda_{y=0} \\ \partial_t u - \partial^2_{xx} u + \mu u = v(x, 0) \end{cases} \tag{3.46}$$

where $\mathrm{d}\lambda_{y=0}$ denotes the Lebesgue measure on the line $\{y = 0\}$.

Proof. Outside $\{y = 0\}$ it is trivial. Along $\{y = 0\}$ just use that $\partial_y v$ has a jump discontinuity $2d(\mu u - v)(x, 0)$. □

Lemma 3.5.2. *Let $C = \max(Lipf, 2d)$. Then*

$$v(t, x, y) \leq C(t + 2C'\sqrt{t})$$

where C' is a constant that depends only on d and L.

Proof. The proof relies on a Aronson type inequality (see [2]) that we compute explicitly here. By Duhamel's formula,

$$v(t, x, y) = \int_0^t e^{s\Delta d,\varepsilon^N} [f(v(t - s, x, y)) + 2d(\mu u - v)(t - s, x, 0)\mathrm{d}\lambda_{y=0}]\mathrm{d}s \tag{3.47}$$

where $\Delta^N_{d,\varepsilon} = d\varepsilon^2\partial^2_{xx} + d\partial^2_{yy}$ endowed with Neumann boundary conditions on $y = \pm L$. Since $\mathbb{R} \times (-L, L)$ is a product domain and since ∂^2_{xx} and ∂^2_{yy} commute, we can compute this heat kernel as follows.

Denote $\lambda_k = d(k\pi/(2L))^2$ the eigenvalues of $d\partial^2_{yy}$ on $(-L, L)$ with Neumann conditions and ϕ_k the associated eigenfunctions. Then the associated heat Kernel is

$$K_2(t, y, y') = \sum_{k \geq 0} e^{-\lambda_k t}\phi_k(y)\phi_k(y')$$

For $d\varepsilon^2\partial^2_{xx}$ on $\mathbb{R}$ the heat Kernel is trivially

$$K_1(t, x, x') = \frac{1}{\sqrt{4\pi d\varepsilon^2 t}} e^{-(x-x')^2/(4d\varepsilon^2 t)}$$

As a consequence,

$$\begin{aligned} e^{s\Delta d,\varepsilon^N} \mathrm{d}\lambda_{y=0} &= \int_{\mathbb{R}} K_1(s,x,x')K_2(s,y,0)\mathrm{d}x' \\ &= \sum_{k\geq 0} e^{-\lambda_k s}\phi_k(y)\phi_k(0) \end{aligned}$$

which of course depends only on y and is even in y (the ϕ_k being even or odd). Observe that this is nothing more than the fundamental solution of the diffusion equation in y on $(-L, L)$. Now using that the ϕ_k are uniformly bounded by a C' depending only on d and L one gets

$$e^{s\Delta d,\varepsilon^N} \mathrm{d}\lambda_{y=0} \leq C' \sum_{k\geq 0} e^{-\lambda_k s} \leq C'/\sqrt{s}$$

for another constant C'. The last inequality comes from the growth of λ_k as Ck^2.

Going back to (3.47) and using $f(v) \leq \mathrm{Lip} f$ as well as $\mu u - v \leq 1$ and the positivity of the integral, one gets

$$v(t,x,y) \leq C \int_0^t (1 + C'/\sqrt{s})\mathrm{d}s \leq C(t + 2C'\sqrt{t})$$

□

Lemma 3.5.3.

$$u(t,x) \leq \frac{2e^{-\mu t}}{\sqrt{4\pi t}} a + \frac{Ct^2}{2} + \frac{4CC'}{3} t^{3/2}$$

Proof. We inject the previous estimate on $v(t, x, 0)$ in the equation satisfied by u and solve it using Duhamel's formula. By the maximum principle, this gives the following upper bound :

$$\begin{aligned} u(t,x) &\leq e^{-\mu t} e^{t\partial^2_{xx}} u_0 + C \int_0^t e^{-\mu(t-s)} e^{(t-s)\partial^2_{xx}} (s + 2C'\sqrt{s})\mathrm{d}s \\ &= e^{-\mu t} \int_{-a}^{a} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x')^2}{4t}} \mathrm{d}x' + C\left(\frac{t^2}{2} + \frac{4}{3}C' t^{3/2}\right) \end{aligned}$$

which gives the desired result. □

Proof of Theorem 3.5.1. Chose t_1' such that $C(t_1'^2 + 2C' t_1'^{3/2}) = \frac{\theta}{2}$ and set $t_1 = \max(1, t_1')$. As a consequence, at time $t = t_1$ one has $v \leq \frac{\theta}{2}$ and

$$\mu u(t,x) \leq 2e^{-\mu t_1} \frac{1}{\sqrt{4\pi t_1}} a + \frac{\theta}{2} \leq \frac{2\theta}{3}$$

if $a < a_0$ for some a_0. Then the maximum principle yields that from this time $\mu u, v$ will always stay below the constant solution $2\theta/3$ of (3.1). As a consequence, the equation is then linear, L^1 mass is preserved, and μu and v will decay to 0.

□

3.5.2 Best case scenario : $a = +\infty$

In this subsection we take $\mu u_0 \equiv 1$. Since both the initial data and equation (3.2) enjoy here a translation invariance in the x direction, u and v do not depend on x. We prove the following

Theorem 3.5.2. *There exists $\mu^\pm > 0$ such that :*

a) If $\mu > \mu^+$, μu and v converge uniformly to $1/(\mu(L + 1/\mu))$ as $t \to +\infty$.

b) If $\mu < \mu^-$, μu and v converge uniformly to 1 as $t \to +\infty$.

Proof of point a). Using Lemma 3.5.2 one gets that for $t \leq 1$

$$v(t, x, y) \leq C\sqrt{t}$$

(for some constant C different than the C in the afore mentioned Lemma). Using this in the equation for u, one gets

$$\mu u(t, x) \leq e^{-\mu t} + C\mu \int_0^t e^{-\mu(t-s)}\sqrt{s}\ \mathrm{d}s \leq e^{-\mu t} + C\mu t^{3/2}$$

So that at

$$t_\mu = \left(\frac{\theta/(2C)}{\mu}\right)^{2/3}$$

provided μ is large enough so that $e^{-\mu t_\mu} \leq \theta/2$ and $v \leq \theta$, i.e.

$$\mu \geq \max\left(\left(\frac{2C}{\theta}\right)^2 |\ln(\theta/2)|^3, \frac{1}{2}\left(\frac{C}{\theta}\right)^2\right) =: \mu^+$$

one has $\mu u, v \leq \theta$. By the comparison principle, this will hold for all $t > t_\mu$ and v never gets above θ anywhere.

As a consequence, L^1 mass is preserved

$$\int_{-L}^0 v(t, y)\mathrm{d}y + u(t) = 1/\mu$$

and $\mu u(t), v(t, y)$ converge to a common limit $l \leq \theta$ satisfying $(L + 1/\mu)l = 1/\mu$ □

Remark 3.5.1. Making $l = \theta$ yields the existence of a threshold value

$$\mu_\theta = \frac{1-\theta}{\theta L}$$

that could give interesting properties. For instance, due to $l \leq \theta$ above we know that our threshold μ^+ satisfies $\mu^+ > \mu_\theta$. Following the philosophy of Du and Matano [40] we think that μ_θ might play the role of a sharp threshold since when $\mu = \mu_\theta$ the solution converges to θ. It would be interesting to see if Theorem 1.3 of [40] applies to our system.

Proof of point b). The idea of the proof is simple : we investigate whether the sole diffusion is able to transfer enough mass from u to v so that in finite time v is above θ on a large enough interval $(-L_0, 0)$. The quantity L_0 is linked to Kanel$'$ and Aronson-Weinberger [1, 53].

Using $v \geq 0$ and the strong parabolic maximum principle one gets

$$\mu u \geq e^{-\mu t}$$

So that setting $\theta' = (1+\theta)/2$ and $t_M = \frac{1}{\mu} \ln\left(\frac{1}{\theta'}\right)$ one has $\mu u \geq \theta'$ while $t \leq t_M$ so that, by the maximum principle, Hopf's lemma and the positivity of f, up to time t_M we have $v \geq \underline{v}$ the solution of

$$d\partial_y \underline{v} + \underline{v} = \theta'$$

$$\partial_t \underline{v} - d\partial^2_{yy}\underline{v} - d\varepsilon^2 \partial^2_{xx}\underline{v} = 0$$

$$\partial_y \underline{v} = 0 \tag{3.48}$$

starting from $\underline{v}_0 = 0$. Observe that this equation and its data do not depend on x so actually $\underline{v}$ is independent of x and we will call it $\underline{v}(t, y)$ from now on. Setting $w = \theta' - \underline{v}$ one sees that $w(t, y)$ satisfies

$$\begin{cases} \partial_t w - d\partial^2_{yy} w = 0 \\ \partial_y w = 0 \text{ on } y = -L \\ \partial_y w + w = 0 \text{ on } y = 0 \\ w(0, y) = \theta' \end{cases} \tag{3.49}$$

By decomposing $w(t, \cdot)$ on a basis of $L^2(-L, 0)$ of eigenvectors for $d\partial^2_{yy}$ endowed with the boundary conditions from (3.49) one gets

$$w(t, y) = \sum_{k \geq 0} e^{-\lambda_k t} \tilde{w}_k(0) \cos\left(\sqrt{\frac{\lambda_k}{d}}(y + L)\right)$$

where the $\frac{\lambda_k}{d} > 0$ are the solutions of $\sqrt{x} = \operatorname{cotan}(\sqrt{x}L)$ and

$$\sum \tilde{w}_k(0) \cos\left(\sqrt{\frac{\lambda_k}{d}}(y + L)\right) = \theta'$$

Since the first eigenfunction does not change sign, we also know by the maximum principle that, when $y \in (-L_0, 0)$:

$$w \leq \theta' e^{-\lambda_1 t} \frac{\cos\left(\sqrt{\lambda_1/d}\ (L - L_0)\right)}{\cos\left(\sqrt{\lambda_1/d}\ L\right)} =: K e^{-\lambda_1 t}$$

so that for $y \in (-L_0, 0), v(t_M, y) \geq \theta' - Ke^{-\lambda_1 t_M} \geq \frac{1+3\theta}{4}$ provided

$$\mu \leq \lambda_1 \frac{\ln(1/\theta')}{\ln\left(\frac{4K}{1-\theta}\right)} =: \mu^- \tag{3.50}$$

Chose L_0 large enough in the beginning so that an initial condition

$$\mu u(t_M), v(t_M, y) \geq (1+3\theta)/4$$

for all $y \in (-L_0, 0)$ leads to invasion : $\mu u, v \to 1$ as $t \to \infty$. The existence of such an L_0 can be adapted from Kanel$'$, Aronson-Weinberger [1, 53] on $\mathbb{R}$. In our context (we rewrite equation (3.2) without the dependance in x) :

$$\begin{cases} u_t = v(t,0) - \mu u(t) \\ v_t - dv_{yy} = f(v) \\ v_y(t,0) = \mu u(t) - v(t,0) \\ v_y(t,-L) = 0 \end{cases}$$

it is even simpler since total mass is confined in $(-L, 0)$ and a single point whereas in [1, 53] it can be spread on all $\mathbb{R}$.

Observe that estimate (3.50) is rather crude : since $\lambda_1(L)$ decays as L^{-2} for large L (and $K(L)$ might be large too) this leads to the fact that our threshold is at least in $\mu^- \leq C/L^2$ for large values of L. This might be due to several effects :

- The estimate $v \geq 0$ used for u
- The estimate $e^{-\mu t} \geq \theta'$ while $t \leq t_M$
- The non-sharpness of the first eigenmode as a supersolution and its nonlocal character (we are only interested in $y \in (-L_0, 0)$ but we have to put the first eigenfunction everywhere above θ').

Nonetheless, numerical simulations of

$$\max_{t\in(0,+\infty)} \min_{y\in[-L_0,0]} v(t,y)$$

with $\mu u = e^{-\mu t}$ and solving the associated (3.48) on the first 5 eigenmodes (which gives a pretty decent approximation) let us think that this is not so far from reality.

□

3.5.3 Large $a < +\infty$

We use the best case scenario described above to prove the existence of large but finite a that will lead to invasion. Our proof relies on the fact that $\mathbf{1}_{(-a,a)}$ and $\mathbf{1}_{(-\infty,\infty)}$ are close in L^∞ weighted by some $\rho(x)$ with tails $e^{-|x|}$ and that such a weight preserves the semi-linear parabolic and monotone structure of the system (3.2). In particular, the "weighted equation" will have a locally (in time) Lipschitz continuous flow. Going back to the original solutions, this Lipschitz continuity becomes a uniform continuity on every compact subset. Let us describe this argument precisely.

Lemma 3.5.4. *There exists a $\mathcal{C}^2$ positive weight $\rho(x)$ such that $\rho(x) = e^{-|x|}$ for $|x| > 1$ and such that the following holds:*

Let $\|\cdot\|_X$ denote the product weighted L^∞ norm

$$\|(f,g)\|_X = \max\left(\|\rho f\|_{L^\infty(\mathbb{R})}, \|\rho g\|_{L^\infty(\Omega_L)}\right)$$

For every $T > 0$ and $M > 0$ there exists a constant $C_{T,M} > 0$ that does not depend on D and such that

$$\sup_{0 \le t \le T, x \in (-M,M)} (|u - \tilde{u}| + |v - \tilde{v}|) \le C_{T,M} \|(u_0 - \tilde{u}_0, v_0 - \tilde{v}_0)\|_X$$

for every (u,v) and $(\tilde{u},\tilde{v})$ solutions of (3.2) *starting from respectively (u_0,v_0) and $(\tilde{u}_0,\tilde{v}_0)$.*

Remark 3.5.2. In other words, Lemma 3.5.4 asserts that on every space-time compact, one can control the uniform distance between two solutions of (3.2) by the weighted distance $\|e^{-|x|}\cdot\|_\infty$ between their initial data. As a consequence, initial data that differ only very far away give very close solutions for small times and small x.

Observe also that the above Lemma could be stated for any $\rho_\alpha(x) = e^{-\alpha|x|}$ (with $\alpha > 0$) by changing the constants : this is due to the scaling invariance $(t,x,y) \to (\Lambda t, \sqrt{\Lambda}x, \sqrt{\Lambda}y)$ of equation (3.2) ; indeed, ρ_α becomes ρ_1 in the rescaling by $\Lambda = \alpha^2$.

End of the proof of Theorem 3.1.4. Once Lemma 3.5.4 is proved, the end of Theorem 3.1.4 follows easily. Indeed, let $u_0 = \mathbf{1}_{(-a,a)}$ and $v_0 = 0$ as well as $\tilde{u}_0 = \mathbf{1}_{(-\infty,+\infty)}$ and $\tilde{v}_0 = 0$. Observe that if $a > 1$,

$$\|(u_0 - \tilde{u}_0, v_0 - \tilde{v}_0)\|_X = e^{-a}$$

Moreover since $\mu < \mu^-$, by Theorem 3.5.2 above there exists $T > 0$ such that

$$\mu\tilde{u}(T,x), \tilde{v}(T,x,y) \ge 1 - \delta/2$$

with δ as in Theorem 3.3.1. By choosing

$$a > \max\{1, -\ln(\delta/(2C_{T,M}))\} =: a_1$$

(which does not depend on D) and applying Lemma 3.5.4 on $[0,T] \times [-M,M]$ (with M as in Theorem 3.3.1) one has

$$|\mu u(T,x) - \mu\tilde{u}(T,x)| + |v(T,x,y) - \tilde{v}(T,x,y)| \le \delta/2 \text{ for all } -M < x < M$$

And as a consequence,

$$\mu u(T,x), v(T,x,y) \ge 1 - \delta \text{ for all } -M < x < M$$

so that (u,v) satisfies Theorem 3.1.3 at time T and invasion follows. □

Proof of Lemma 3.5.4. The proof relies only on the parabolic maximum principle applied to a weighted equation. Let $\rho(x)$ define a positive $\mathcal{C}^2$ function such that $\rho(x) = e^{-|x|}$ for $|x| \geq 1$. Let (u, v) solve system (3.2). Observe that

$$(\mathfrak{u}, \mathfrak{v}) := (\rho u, \rho v)$$

satisfies

$$\begin{array}{c} \partial_t \mathfrak{u} + 2\frac{\rho'}{\rho}\partial_x \mathfrak{u} - \partial^2_{xx} u = \mathfrak{v} - \left(\mu + \frac{\rho''}{\rho} - 2\left(\frac{\rho'}{\rho}\right)^2\right)\mathfrak{u} \\ \hline d\partial_y \mathfrak{v} = \mu \mathfrak{u} - \mathfrak{v} \\ \partial_t \mathfrak{v} + 2\frac{d\rho'}{D\rho}\partial_x \mathfrak{v} - \frac{d}{D}\partial^2_{xx}\mathfrak{v} - d\partial^2_{yy}\mathfrak{v} = \rho f\left(\frac{\mathfrak{v}}{\rho}\right) - \frac{d}{D}\left(\frac{\rho''}{\rho} - 2\left(\frac{\rho'}{\rho}\right)^2\right)\mathfrak{v} \\ \partial_y \mathfrak{v} = 0 \\ \hline \end{array} \tag{3.51}$$

Equation (3.51) is a semilinear parabolic system, and thanks to the definition of ρ, has bounded coefficients. Moreover, the non-linearity $g(\mathfrak{v}) := \rho f\left(\frac{\mathfrak{v}}{\rho}\right)$ is Lipschitz with Lipschitz constant Lip$f$. Let

$$C := \mathrm{Lip} f - \inf_{\mathbb{R}}\left(\frac{\rho''}{\rho} - 2\left(\frac{\rho'}{\rho}\right)^2\right) > \mathrm{Lip} f - \frac{d}{D}\inf_{\mathbb{R}}\left(\frac{\rho''}{\rho} - 2\left(\frac{\rho'}{\rho}\right)^2\right)$$

Now define $(\tilde{\mathfrak{u}}, \tilde{\mathfrak{v}})$ in a similar way and let $\mathfrak{U} := e^{-Ct}\left(\mathfrak{u} - \tilde{\mathfrak{u}}\right), \mathfrak{V} := e^{-Ct}\left(\mathfrak{v} - \tilde{\mathfrak{v}}\right)$. Observe that $(\mathfrak{U}, \mathfrak{V})$ satisfies

$$\begin{array}{c} \partial_t \mathfrak{U} + 2\frac{\rho'}{\rho}\partial_x \mathfrak{U} - \partial^2_{xx}\mathfrak{U} = \mathfrak{V} - \mu\mathfrak{U} - \left(\frac{\rho''}{\rho} - 2\left(\frac{\rho'}{\rho}\right)^2 + C\right)\mathfrak{U} \\ \hline d\partial_y \mathfrak{V} = \mu\mathfrak{U} - \mathfrak{V} \\ \partial_t \mathfrak{V} + 2\frac{d\rho'}{D\rho}\partial_x \mathfrak{V} - \frac{d}{D}\partial^2_{xx}\mathfrak{V} - d\partial^2_{yy}\mathfrak{V} + \left(\frac{d}{D}\left(\frac{\rho''}{\rho} - 2\left(\frac{\rho'}{\rho}\right)^2\right) + \frac{g(\mathfrak{v}) - g(\tilde{\mathfrak{v}})}{\mathfrak{v} - \tilde{\mathfrak{v}}} + C\right)\mathfrak{V} = 0 \\ \partial_y \mathfrak{V} = 0 \\ \hline \end{array} \tag{3.52}$$

By choice of C, the 0-order terms in parentheses in equation (3.52) are positive, thus equation (3.52) enjoys the maximum principle and the maximum and minimum values of $(\mu\mathfrak{U}, \mathfrak{V})$ are reached at initial time. Indeed, as usual if the maximum is reached by $\mathfrak{V}$, then either it is reached at initial time or it has to be reached on $y = 0$ but there the Hopf's lemma gives the contradiction $\mu\mathfrak{U} > \mathfrak{V}$. It $\mathfrak{U}$ reaches it, a contradiction is obtained at this point by seeing that the left-hand side in the equation satisfied by $\mathfrak{U}$ is non-negative : thus $\mathfrak{V} > \mu\mathfrak{U}$. In the end we have for all

$0 \le t < T$:

$$\begin{cases} |(\mathfrak{u} - \tilde{\mathfrak{u}})(t)|_{L^\infty(\mathbb{R})} \le e^{CT} \max\Big(|(\mathfrak{u} - \tilde{\mathfrak{u}})(0)|_{L^\infty(\mathbb{R})}, |(\mathfrak{v} - \tilde{\mathfrak{v}})(0)|_{L^\infty(\Omega_L)}\Big) \\ |(\mathfrak{v} - \tilde{\mathfrak{v}})(t)|_{L^\infty(\Omega_L)} \le e^{CT} \max\Big(|(\mathfrak{u} - \tilde{\mathfrak{u}})(0)|_{L^\infty(\mathbb{R})}, |(\mathfrak{v} - \tilde{\mathfrak{v}})(0)|_{L^\infty(\Omega_L)}\Big) \end{cases} \tag{3.53}$$

i.e.for all $t < T$, $x \in \mathbb{R}$, $y \in [-L, 0]$:

$$\rho(x) \left(|u - \tilde{u}|(t, x) + |v - \tilde{v}|(t, x, y)\right) \le 2e^{CT} \|(u_0 - \tilde{u}_0, v_0 - \tilde{v}_0)\|_X$$

and Lemma 3.5.4 follows by taking $C_{T,M} = 2e^{CT} \sup_{x\in(-M,M)} \frac{1}{\rho(x)}$, which is $2e^{CT}e^M$ when M is large. Observe that $C_{T,M}$ depends only on T, M and Lip$f$. $\square$

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Chapter 4

Perspectives

“ I refuse to answer that question on the grounds that I don’t know the answer ”

—Douglas Adams

To finish this manuscript, we present some natural continuations of the above works and some longer term projects.

Q1 **Precise study of the finite time t_1 of Theorem 3.1.3**

We already mentioned that that $t_D = D^{1/2} \ln D + O(1)$. Nonetheless, we believe that t_1 might be smaller. We believe that a lower bound $t_D \geq cD^{1/6}$ or $t_1 \geq cD^{1/4}$ might be found, but the precise behaviour of t_D seems to be a complicated and interesting question. We plan to carry a quantitative study of the heat kernel of (3.1) in order to get precise estimates.

Q2 **Including transport and reaction on the road.**

$$
\begin{array}{c}
\partial_t u - D\partial_{xx}u + q\partial_x u = v - \mu u + g(u) \\
\hline
d\partial_y v = \mu u - v \\
\\
\partial_t v - d\Delta v = f(v) \\
\\
\partial_y v = 0 \\
\hline
\end{array}
\tag{4.1}
$$

As in [21] it would be interesting to check the influence of these new parameters on the travelling waves and their speed, and to see how would the system (2.4) be modified. Here there is a natural scaling $q \sim q_\infty \sqrt{D}$ which seems to open the way to interesting properties.

Q3 **Sharpness of the thresholds a and M'**

Is it possible to find a sharp M' as in Theorem 3.1.3 ? And a sharp a playing the role of a_0 and a_1 in Theorem 3.1.4 ? One possible track is to adapt the ideas of Du and Matano [40].

Q4 **Transition fronts.**

Can we use the previous results and the theory of transition fronts (see [10]) to describe solutions as $D \to +\infty$ when $D = D(x), \mu = \mu(x), q = q(x)$?

Non-trivial transition fronts were already exhibited by Berestycki-Hamel-Matano [12] to describe propagation around an obstacle or Mellet-Roquejoffre-Sire [64] and Nolen-Roquejoffre-Ryzhik-Zlatoš [67] to study

$$\partial_t u - \partial_{xx} u = f(x, u)$$

in various situations.

Q5 **Including integral dispersion on the road.**

Do the travelling wave persist in presence of a fractional laplacian on the road, and if yes, do transition fronts persist ?

NB : existence of transition fronts for the following simple equation is still open

$$\partial_t u + (-\partial_{xx})^\alpha u = a(x) f(u)$$

with f of ignition type. This is an occasion to review the case a constant : especially when $\alpha \leq 1/2$ there is no travelling wave (see [51]). How does the spreading look like in this case ?

Q6 **A Harnack inequality for systems of the type** (3.2) **or** (2.4)**.**

As stated in Remark 3.1.1, we did not give a full proof of convergence towards travelling waves but only that the solutions are trapped between translates of the waves. The technical reason of this is the lack of a Harnack inequality up to the boundary concerning (3.2).

Indeed, the Harnack inequality is quite convenient in the study of stability of travelling waves : one starts with the stability among front-like initial data than can be trapped between two translates of the front, separated by a distance d_0. Then, a comparison principle and a Harnack inequality can be used to show that one time step later, the solution can be trapped between two translates of the front separated by a distance qd_0 where $0 < q < 1$ is a constant (maybe very close to 1). This argument is very much in the same spirit as “Harnack inequality implies Hölder regularity”. See [65].

In the case of (3.2), no such Harnack inequality has been given yet. The above argument can be replaced by a maximum principle and compactness argument, but finding a full Harnack inequality near the boundary for the system (3.2) would be both interesting and have surely many applications in the study of the system. Arguments supporting the existence of such a Harnack inequality can be found in Luo [62] for Wentzell boundary value problems, which are, in some sense, a reduced version of the travelling waves problem associated to (3.2) (see [39]).

Moreover, in the case of non-flat boundaries and in presence of a degeneracy as in (2.4), the study of transmission of regularity to the interior of the domain seems interesting and non-trivial.

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Accélération de la propagation dans les équations de réaction-diffusion par une ligne de diffusion rapide

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Résumé : l'objet de cette thèse est l'étude de l'accélération de la propagation dans les équations de réaction-diffusion par un nouveau mécanisme d'échange avec une ligne de diffusion rapide. On répondra à la question de l'influence de ce couplage avec forte diffusivité sur la propagation en généralisant un résultat de Berestycki, Roquejoffre et Rossi de 2013. Le système d'équations étudié a été proposé pour donner une explication mathématique de l'influence des réseaux de transports sur les invasions biologiques. Dans un premier chapitre, on étudiera l'existence et l'unicité de solutions de type ondes progressives via une méthode de continuation. La transition se fait par l'intermédiaire d'une perturbation singulière qui paraît nouvelle dans ce contexte, connectant le système initial à un problème au bord de type Wentzell. Le second chapitre s'intéresse à la vitesse des ondes sus-mentionnées. On y démontre qu'elle croît comme la racine carrée de la diffusivité de l'espèce sur la route, ce qui généralise et démontre la robustesse du résultat de Berestycki, Roquejoffre et Rossi. De plus, on caractérise précisément le ratio de croissance comme unique vitesse admissible pour les ondes d'un système hypoelliptique a priori dégénéré. Enfin dans une dernière partie on s'intéresse à la dynamique. On y montre que ces ondes attirent une large classe de données initiales. En particulier on met en lumière un nouveau mécanisme d'attraction qui permet aux ondes d'attirer des données dont la taille est indépendante de la diffusivité sur la route ; c'est un résultat nouveau au sens où usuellement, l'accélération de fronts de réaction-diffusion se paie en renforçant les hypothèses nécessaires sur la taille des données initiales attirées.

Mots-clés : réaction-diffusion, ondes progressives, fronts, propagation, vitesse.

Abstract : the aim of the thesis is the study of enhancement of propagation in reaction-diffusion equations, through a new mechanism involving a line with fast diffusion. We answer the question of the influence of such a coupling with strong diffusion on propagation by generalizing a result of Berestycki, Roquejoffre and Rossi (2013). The model under study was proposed to give a mathematical understanding of the influence of transportation networks on biological invasions. The first chapter shows existence and uniqueness of travelling waves solutions with a continuation method. The transition occurs through a singular perturbation – new in this context – connecting the system with a Wentzell boundary value problem. The second chapter is concerned with the speed of the waves : we show that it grows as the square root of the diffusivity on the line, generalizing and showing the robustness of the result by Berestycki, Roquejoffre and Rossi. Moreover, the growth ratio is characterized as the unique admissible velocity for the waves of an hypoelliptic a priori degenerate system. The last part is about the dynamics : we show that the waves attract a large class of initial data. In particular, we shed light on a new mechanism of attraction which enables the waves to attract initial data with size independent of the diffusivity on the line : this is a new result, in the sense than usually, enhancement of propagation has to be paid by strengthening the assumptions on the initial data for invasion to happen.

Keywords : reaction-diffusion, travelling waves, fronts, propagation, speed.