21-268 – Review questions #1

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Reminder

The first midterm will be on Feb. 22. It will be closed book, without notes or calculator. These are midterm-like problems inended to help you prepare and will not be collected, but solutions will be posted. I strongly advise you to try to solve them before looking at the solutions.

1. Prove that $S = \{(x, y) \in \mathbb{E}^2 \mid x^2 + y^2 < 4\}$ is open.

2. Find the limits of the following functions as $(x, y) \to (0, 0)$ if they exist or prove that they do not. Justify your answer in either case (not necessarily by finding $\delta(\varepsilon)$, theorems about sums, product, quotient or the squeeze theorem can be way faster).
   (a) $f(x, y) = \frac{x^4}{x^4+y^2}$
   (b) $f(x, y) = \frac{x^4}{x^2+y^2}$
   (c) $f(x, y) = \frac{\sin(x^2+y^2)}{\cos(x^4+y^4)}$
   (d) $f(x, y) = \frac{xy(y^2-x^2)(y^2-4x^2)}{x^4+y^4}$
   (e) $f(x, y) = \frac{xy(y^2-x^2)(y^2-4x^2)}{x^6+y^6}$.

3. (a) Let $f(x, y) = \frac{7x^6}{x^4+y^4}$ for $(x, y) \neq (0, 0)$. For $\varepsilon > 0$, find $\delta > 0$ such that

$$|f(x, y)| < \varepsilon \text{ if } 0 < \sqrt{x^2 + y^2} < \delta.$$

   (b) Same question with $f(x, y) = \frac{x^3y^2}{x^4+y^4}$.

4. Let $F(x, y, z) = \begin{pmatrix} y^2e^{x^2} - z \\ xy\sin(z) \\ \cos(z) + e^y \end{pmatrix}$. What is the domain of F and where are its partial derivatives continuous functions ? Compute the Jacobian matrix of F there.

5. For each of the following, determine if $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist and find them if they do.
   (a) $f(x, y) = \frac{x|y|}{x^4+y^4}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.
   (b) $f(x, y) = \frac{x^4}{x^4+y^4}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.
   (c) $f(x, y) = \frac{|y|^5}{x^4+y^4}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

6. Let $f(x, y) = xy^3$. Find $a(x, y), b(x, y), \varepsilon_1(x, y, \Delta x, \Delta y), \varepsilon_2(x, y, \Delta x, \Delta y)$ that satisfy the definition of differentiability for f.

7. Assume that $f(x, y)$ defines a function differentiable at all points and that $\frac{\partial f}{\partial x}(1, 0) = 5$, $\frac{\partial f}{\partial y}(1, 0) = 6$, $\frac{\partial f}{\partial x}(0, 1) = 8$, $\frac{\partial f}{\partial y}(0, 1) = 9$, and $\frac{\partial f}{\partial x}(1, 1) = 11$, $\frac{\partial f}{\partial y}(1, 1) = 12$. Let $z(s, t) = f(1 - s^2 - t^2, t^3 + s^3)$ and find

$$\frac{\partial z}{\partial t}(0, 1) \quad \text{and} \quad \frac{\partial z}{\partial t}(0, 0)$$

8. Consider the set of equations in $\mathbb{E}^4$

$$\begin{cases} xy + z^2w^2 = 1 \\ xz + y^2w^2 = 1 \end{cases}$$

   (a) Find $\frac{\partial w}{\partial y}$ by implicit differentiation, first by **assuming** that $w = w(y,z),\ x = x(y,z)$.

   (b) Can you do this assumption around the point $(w,x,y,z) = (1,0,1,1)$ (which is solution of the equation) ?

9. Define $f(x,y) = \frac{|x|^5}{x^4+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. At what points (x,y) does $\frac{\partial f}{\partial x}$ exist ? Same question for $\frac{\partial f}{\partial y}$.

10. Let

$$F(w,x,y,z) = w + x + y + z + \sin(wxyz), \quad G(w,x,y,z) = w + x + 2y + 2z + \cos(wxyz) - 1$$

and consider solving

$$\begin{cases} F(w,x,y,z) = 0 \\ G(w,x,y,z) = 0 \end{cases}$$

near $(w,x,y,z) = (0,0,0,0)$. For solving for w,z as functions of x,y are the assumptions of the implicit function theorem satisfied ? Same question for solving for w,x, for x,y, for x,z and for y,z (each time in terms of the remaining variables of course).

11. Let $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2$.

   (a) Find all the critical points of f.

   (b) Determine which ones are relative minima, maxima or saddle points.

   (c) Compute the Laplacian of f at these points.
   Remark: observe that the Laplacian is the trace of the Hessian matrix. As such, it is also the sum of its eigenvalues (this is a general theorem of Linear algebra).