21-268 – Partial derivatives, additional example

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January 27, 2017

Abstract

Since partial derivatives are nothing more than classical derivatives while freezing all the other variables, the theorems about sum, product, quotient (chain rule) for existence and formulas of derivatives that you already know how to use is also valid for partial derivatives. Nonetheless, there are some functions that one needs to analyse "by hand". Here is an example, where f is defined in several pieces.

Let

$$f(x,y) = \begin{cases} \dfrac{x^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

- At any $(x,y) \neq (0,0)$, f has a partial derivative in x as a quotient of such functions whose denominator does not cancel. Moreover

$$f_x(x,y) = \frac{(x^2+y^2)4y^3 - x^4(2x)}{(x^2+y^2)^2}$$

- No such observation can be made at $(0,0)$ and one has to compute the limit of the rate of increase by hand.

$$\lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \frac{\frac{\Delta x^4}{\Delta x^2} - 0}{\Delta x} = \lim_{\Delta x \to 0} \Delta x = 0$$

  so $f_x(0,0) = 0$ exists.

So even though this function looks singular at the origin at first glance, it actually is very nice: it is continuous (check it), and has partial derivatives in x (proved above) and in y (check it). Here is the 3D-graph of that function.