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Section B

Hw 3

1. $f(x+\Delta x, y+\Delta x) = (x^2 + 2x\Delta x + \Delta x^2)(y^2 + 2y\Delta y + \Delta y^2)$

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$= x^2y^2 + 2x^2 y\Delta y + x^2\Delta y^2 +$
$2xy^2\Delta x + 4xy\Delta x\Delta y + 2x\Delta x\Delta y^2 +$
$y^2\Delta x^2 + 2y\Delta x^2\Delta y + \Delta x^2\Delta y^2$

$= f(x,y) + 2xy^2\Delta x + 2x^2y\Delta y + (x^2\Delta y)\Delta y +$
$+ (4xy\Delta y + 2x\Delta y^2 + y^2\Delta x + 2y\Delta x\Delta y + \Delta x\Delta y^2)\Delta x$

$\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1(x, y, \Delta x, \Delta y) = 0$ by continuity of $+, *$
$\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2(x, y, \Delta x, \Delta y) = 0$ by continuity of $+, *$

$\therefore\ a(x,y) = 2xy^2, \quad b(x,y) = 2x^2y,$
$\varepsilon_1(x,y,\Delta x,\Delta y) = (4xy\Delta y \cdots + x\Delta y^2), \quad \varepsilon_2(x,y,\Delta x,\Delta y) = x^2\Delta y$

2. a.)

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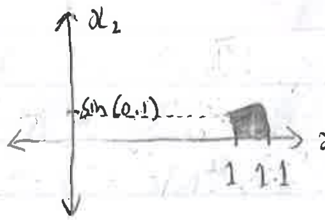
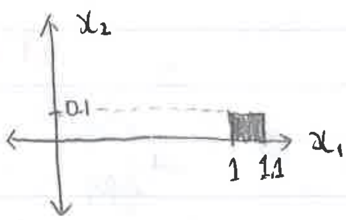


The area is $\frac{0.1}{2}\cdot(1.1^2 - 1)$
$= 0.0105$

b.) $\begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} \cos(x_2) & -x_1\sin(x_2) \\ \sin(x_2) & x_1\cos(x_2) \end{bmatrix} = J(x_1, x_2)$

c.)



$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$

The area is $0.1 \cdot 0.1 = 0.01$

3. a.) $z|_{t=1} = f(0,1) \qquad z|_{t=0} = f(1,0)$

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Use chain rule b/c total differentiability

$\partial z/\partial t\,|_{t=1} = \partial f/\partial x\,(0,1)\cdot \partial/\partial t\,(1-t^2)|_{t=1} + \partial f/\partial y\,(0,1)\cdot \partial/\partial t\,(t^3)|_{t=1}$
$= 8\cdot(-2) + 9\cdot(3) = 11$

$\partial z/\partial t\,|_{t=0} = \partial f/\partial x\,(1,0)\cdot \partial/\partial t\,(1-t^2)|_{t=0} + \partial f/\partial y\,(1,0)\ \partial/\partial t\,(t^3)|_{t=0}$
$= 0 \qquad$ (the $\partial/\partial t$ terms $= 0$)

4. $r\cos\theta$ and $r\sin\theta$ are totally differentiable at all (r,θ)

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$$\partial z/\partial r = \partial z/\partial x \cdot \partial x/\partial r + \partial z/\partial y \cdot \partial y/\partial r$$
$$= \partial z/\partial x \cdot (\cos\theta) + \partial z/\partial y \cdot (\sin\theta)$$
$$\partial z/\partial \theta = \partial z/\partial x \cdot \partial x/\partial \theta + \partial z/\partial y \quad \partial y/\partial\theta$$
$$= \partial z/\partial x \cdot (-r\sin\theta) + \partial z/\partial y \cdot (r\cos\theta)$$
$$= r \cdot (-\partial z/\partial x \sin\theta + \partial z/\partial y \cos\theta)$$
$$(\partial z/\partial r)^2 + 1/r^2 (\partial z/\partial\theta)^2 = (\partial z/\partial x)^2(\cos^2\theta + \sin^2\theta) + (\partial z/\partial y)^2(\sin^2\theta + \cos^2\theta)$$
$$= (\partial z/\partial x)^2 + (\partial z/\partial y)^2$$

5. a.) $$\vec{y}_{\vec{x}} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{bmatrix}$$

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$$\partial(y_1, y_2)/\partial(x_1, x_2) = 4x_1^2 + 4x_2^2$$

b.) $$\begin{cases} y_1 = x_1^2 - x_2^2 \\ y_2 = 2x_1 x_2 \end{cases} \Rightarrow \begin{cases} 2x_1\, \partial x_1/\partial y_1 - 2x_2\, \partial x_2/\partial y_1 = 1 \\ 2x_2\, \partial x_1/\partial y_1 + 2x_1\, \partial x_2/\partial y_1 = 0. \end{cases}$$

$$\Rightarrow \begin{cases} x_1 x_2\, \partial x_1/\partial y_1 - x_2^2\, \partial x_2/\partial y_1 = x_2/2 \\ x_1 x_2\, \partial x_1/\partial y_1 + x_1^2\, \partial x_2/\partial y_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1\, \partial x_1/\partial y_1 - x_2\, \partial x_2/\partial y_1 = 1/2 \\ (x_1^2 + x_2^2)\, \partial x_2/\partial y_1 = -x_2/2 \end{cases}$$

$$\Rightarrow \begin{cases} \partial x_2/\partial y_1 = -x_2/2(x_1^2 + x_2^2) \\ x_1\, \partial x_1/\partial y_1 + x_2^2/2(x_1^2 + x_2^2) = 1/2 \end{cases} \Rightarrow \begin{cases} \partial x_1/\partial y_1 = x_1/2(x_1^2 + x_2^2) \\ \partial x_2/\partial y_1 = -x_2/2(x_1^2 + x_2^2) \end{cases}$$

$$\begin{cases} y_1 = x_1^2 - x_2^2 \\ y_2 = 2x_1 x_2 \end{cases} \Rightarrow \begin{cases} 2x_1\, \partial x_1/\partial y_2 - 2x_2\, \partial x_2/\partial y_2 = 0 \\ 2x_2\, \partial x_1/\partial y_2 + 2x_1\, \partial x_2/\partial y_2 = 1. \end{cases}$$

$$\Rightarrow \begin{cases} x_1\, \partial x_1/\partial y_2 - x_2\, \partial x_2/\partial y_2 = 0 \\ (x_1^2 + x_2)^2\, \partial x_2/\partial y_2 = x_1/2 \end{cases}$$

$$\Rightarrow \begin{cases} \partial x_2/\partial y_2 = x_1/2(x_1^2 + x_2^2) \\ x_1\, \partial x_1/\partial y_1 - x_1 x_2/2(x_1^2 + x_2^2) = 0 \end{cases} \Rightarrow \begin{cases} \partial x_1/\partial y_2 = x_2/2(x_1^2 + x_2^2) \\ \partial x_2/\partial y_2 = x_1/2(x_1^2 + x_2^2) \end{cases}$$

$$\vec{x}_{\vec{y}} = \begin{bmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} \cdot \frac{1}{2(x_1^2+x_2^2)}$$

$$\partial(x_1, x_2)/\partial(y_1, y_2) = \frac{1}{4(x_1^2+x_2^2)^2} \cdot (x_1^2 + x_2^2) = \frac{1}{4(x_1^2+x_2^2)}$$

c.) $$\begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} \cdot 2 \cdot \frac{1}{2(x_1^2+x_2^2)}$$

$$= \begin{bmatrix} x_1^2+x_2^2 & 0 \\ 0 & x_2^2+x_1^2 \end{bmatrix} \cdot \frac{1}{(x_1^2+x_2^2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(\vec{y}_{\vec{x}}) \cdot \det(\vec{x}_{\vec{y}}) = 4(x_1^2+x_2^2) \cdot \frac{1}{4(x_1^2+x_2^2)} = 1$$

6. F is continuous because it is composed from the sums of products of continuous functions x, y, $\cos(y)$, -1, $\sin(y)$, 2

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$\partial F/\partial x = 2x$ is continuous (same reason)

$\partial F/\partial y = 2y + \sin(y)$ is continuous

a.) $\partial F/\partial x\,(1,0) = 2 \neq 0$. So the conditions for the implicit function theorem are satisfied.

b.) $\partial F/\partial y\,(1,0) = 0 + 0 = 0$, which is prohibited. So the conditions are not satisfied.

c.) Consider $d/dx\, F(x, Y(x))$ at $x = 1$.

On the one hand, since there is an open region around $x=1$ where Y is defined and $F(x, Y(x)) = 0$, $\exists d > 0$ s.t. Y is defined for $x \in (1-d, 1+d)$. Fix d.

So $F(x, Y(x)) = 0$ for all $x \in (1-d, 1+d) = U$.

$$d/dx\, F(x, Y(x))\,|_{x=1} = \lim_{\Delta x \to 0} \frac{F(1+\Delta x, Y(1+\Delta x)) - F(1, Y(1))}{\Delta x}$$

$= 0$ within the restricted domain

On the other hand... $$d/dx\, F(x, Y(x))\,|_{x=1} = \partial F/\partial x\,(1, 0)\, \partial x/\partial x + \partial F/\partial y\,(1, 0)\, \partial Y/\partial x$$ ← differential Y assumption.

$$= \partial F/\partial x\,(1, 0) + 0 \cdot \partial Y/\partial x = 0.$$

$$\Rightarrow \quad 2x \big|_{x=1} = 0 \quad \Rightarrow \quad 2 = 0 \quad ※$$

A differentrable Y containing $Y(1) = 0$ and
$F(x, Y(x)) = 0$ in an open region near $x = 1$
cannot exist