21-268 – Homework assignment week #3

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Reminder

Homework will be given on Wednesdays and due on the next Wednesday before 5pm, to me in class or in Christopher Cox's mailbox in Wean Hall 6113 (pay attention to the arrow !). Late homework will never be accepted without a proper reason. In case of physical absence, electronic submissions by e-mail to both me your TA can be accepted. Please do not forget to write your name, andrew id and section and please use a staple if you have several sheets.

Reading

1. Proofs in the textbook that I did not do in class.

Exercises (28 pts)

1. (2 pts) Consider $f(x,y) = x^2y^2$. Find some $a(x,y)$, $b(x,y)$, $\epsilon_1(x,y,\Delta x,\Delta y)$ and $\epsilon_2(x,y,\Delta x,\Delta y)$ that satisfy the definition of total differential for f.

2. Consider $\vec{y} = \vec{f}(\vec{x})$ where $f_1(x_1,x_2) = x_1\cos(x_2)$ and $f_2(x_1,x_2) = x_1\sin(x_2)$.
   a) (2 pts) Draw the region
   $$\left\{\vec{f}(x_1,x_2) \;\middle|\; 1 < x_1 < 1.1,\ 0 < x_2 < 0.1\right\}$$
   and compute its area.
   Hint: first think about fixing x_1 to 1 and vary x_2.
   b) (2 pts) Compute $J(x_1,x_2)$, the Jacobian matrix of $\vec{f}$ evaluated at (x_1,x_2).
   c) (1 pt) For (x_1,x_2) near $(1,0)$, $\vec{f}(x_1,x_2)$ is approximated (at first order) by
   $$\vec{L}(x_1,x_2) = \vec{f}(1,0) + J(1,0)\vec{dx} \quad \text{where} \quad \vec{dx} = \begin{pmatrix} x_1 - 1 \\ x_2 - 0 \end{pmatrix}.$$
   Draw the region
   $$\left\{\vec{L}(x_1,x_2) \;\middle|\; 1 < x_1 < 1.1, 0 < x_2 < 0.1\right\}$$
   and compute its area. Your drawing should be similar to that from part a), but not the same.

3. (2+1 pts) Assume that $f(x,y)$ defines a function differentiable at all points and that $\frac{\partial f}{\partial x}(1,0) = 5$, $\frac{\partial f}{\partial y}(1,0) = 6$, $\frac{\partial f}{\partial x}(0,1) = 8$, $\frac{\partial f}{\partial y}(0,1) = 9$, and $\frac{\partial f}{\partial x}(1,1) = 11$, $\frac{\partial f}{\partial y}(1,1) = 12$. Let $z = f(1-t^2, t^3)$ and find
   $$\left.\frac{dz}{dt}\right|_{t=1} \quad \text{and} \quad \left.\frac{dz}{dt}\right|_{t=0}$$
   The vertical bars on the right mean "evaluated at".

4. (4 pts) Let $f(x,y)$ define a differentiable function and let $z = f(x,y)$ with $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + r^{-2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

5. a) (2+1 pts) Let

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \vec{y} = \vec{f}(\vec{x}) = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{pmatrix} \quad \text{where} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

   Find $\vec{y}_{\vec{x}}$ and $\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)} = \det(\vec{y}_{\vec{x}})$.

   b) (2+1+1 pts) Define $\vec{x}$ as a function of $\vec{y}$ implicitly in $\vec{y} = \vec{f}(\vec{x})$. Find some equations that $\frac{\partial x_1}{\partial y_1}$ and $\frac{\partial x_2}{\partial y_1}$ must satisfy in that way and then solve them for $\frac{\partial x_1}{\partial y_1}$ and $\frac{\partial x_2}{\partial y_1}$. In the same manner find $\frac{\partial x_1}{\partial y_2}$ and $\frac{\partial x_2}{\partial y_2}$. Find $\vec{x}_{\vec{y}}$ and $\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}$.

   c) (1 pt) Substitute your results from parts a and b and find $\vec{y}_{\vec{x}}\vec{x}_{\vec{y}}$ (matrix multiplication) and $\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}$.

6. Let

$$F(x,y) = x^2 + y^2 - \cos(y)$$

   and consider solving $F(x,y) = 0$ near the point $(x,y) = (1,0)$.

   a) (2 pts) For solving for x as a function of y, show that the hypotheses of the implicit function theorem are satisfied.

   b) (2 pt) For solving for y as a function of x, show that the hypotheses of the implicit function theorem are not satisfied.

   c) (2 pt) Suppose that there is a differentiable function $Y(x)$ with $Y(1) = 0$ and

$$F(x, Y(x)) = 0$$

   for x near 1. Derive a contradiction from this.