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Problem Set 2

1 Let $\varepsilon > 0$. Assume $0 < \sqrt{x^2+y^2} < \delta$.

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Choose $\delta = \frac{\sqrt{\varepsilon}}{10}$.

We see that $\sqrt{x^2+y^2} \geq \sqrt{x^2} = |x|$ and similarly, $\sqrt{x^2+y^2} \geq \sqrt{y^2} = |y|$

Thus, $|f(x,y)| = \left| \frac{xy^3}{x^4+y^4+x^2+0.01y^2} \right| = \frac{|x||y|^3}{|x^4+y^4+x^2+0.01y^2|} = \frac{|x||y||y|^2}{|x^4+y^4+x^2+0.01y^2|} \leq$

$$\frac{|x||y||y|^2}{|x^2+0.01y^2|} \leq \frac{(\sqrt{x^2+y^2})(\sqrt{x^2+y^2})(\sqrt{x^2+y^2})^2}{x^2+0.01y^2} \leq \frac{(x^2+y^2)^2}{0.01x^2+0.01y^2} = 100(x^2+y^2)$$

Simply, $|f(x,y)| \leq 100(x^2+y^2)$

We assumed that

$$0 < \sqrt{x^2+y^2} < \delta = \frac{\sqrt{\varepsilon}}{10}$$

$$\Rightarrow \quad 0 < x^2+y^2 < \frac{\varepsilon}{100}$$

$$\Rightarrow \quad 0 < 100(x^2+y^2) < \varepsilon$$

$$\Rightarrow \quad |f(x,y)| < \varepsilon$$

Thus, $\forall \varepsilon$, choose $\boxed{\delta = \frac{\sqrt{\varepsilon}}{10}}$ s.t. $0 < \sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y)| < \varepsilon$

2 d) $\lim\limits_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = \lim\limits_{\Delta y \to 0} \frac{\frac{|0|^{3.1}}{\Delta x^2 + 0^2} - 0}{\Delta x^2} = 0$

$\Rightarrow \boxed{\frac{\partial f}{\partial x}(0,0) = 0}$

$\lim\limits_{\Delta y \to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \frac{\frac{|\Delta y|^{3.1}}{0^2 + \Delta y^2} - 0}{\Delta y} = \lim\limits_{\Delta y \to 0} \frac{|\Delta y|^{1.1}}{\Delta y^2} = 0$

$\Rightarrow \boxed{\frac{\partial f}{\partial y}(0,0) = 0}$

3 a) Figure 3: We can see that at $(1,2)$, fig. 3 has value ≈ 5. Shifting 1 unit right, we get a value of ≈ 8 and 1 unit left a value ≈ 2. The slope between these points ≈ 3, thus $\frac{\partial f}{\partial x}(1,2) \approx 3$. Furthermore, in fig. 1, $\frac{\partial f}{\partial x}(1,2) < 0$, Fig 2, $\frac{\partial f}{\partial x}(1,2) < 0$, fig. 4, $\frac{\partial f}{\partial x}(1,2) \approx 0$.

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b) Figure 1: Although the values given by the legend don't match up to -1, figure 1 is the only one where $\frac{\partial f}{\partial y}(3,2) < 0$. In fig 2, $\frac{\partial f}{\partial y}(3,2) \approx 0$, fig 3, $\frac{\partial f}{\partial y}(3,2) > 0$,

fig 4, $\frac{\partial f}{\partial y}(3,2) > 0$.

Thus our best answer here is Figure 1.

5 $xy^2 + yz^3 + xyz = 1$

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differentiate both sides w/ respect to x

$\Rightarrow y^2 + y3z^2 \frac{\partial z}{\partial x} + yz + xy \frac{\partial z}{\partial x} = 0$

$\Rightarrow \frac{\partial z}{\partial x}\left(y3z^2 + xy\right) = -y^2 - yz$

$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-y^2 - yz}{y3z^2 + xy}}$

$xy^2 + yz^3 + xyz = 1$

diff. both sides w/ respect to y

$\Rightarrow 2xy + z^3 + 3yz^2 \frac{\partial z}{\partial y} + xz + xy \frac{\partial z}{\partial y} = 0$

$\Rightarrow \frac{\partial z}{\partial y}\left(3yz^2 + xy\right) = -2xy - z^3 - xz$

$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-2xy - z^3 - xz}{3yz^2 + xy}}$

2 a) $\dfrac{f(x,0)-f(0,0)}{x} = \dfrac{\frac{x^2}{x^2+0^2}-0}{x} = \dfrac{1}{x} \underset{x\to 0}{\longrightarrow} \infty$

$\dfrac{\partial f}{\partial x}(0,0)$ does not exist,

$\dfrac{f(0,y)-f(0,0)}{y} = \dfrac{0-0}{y} = 0 \underset{y\to 0}{\longrightarrow} 0$

$\dfrac{\partial f}{\partial y}(0,0) = 0.$

b) $\dfrac{f(x,0)-f(0,0)}{x} = \dfrac{0-0}{x} = 0 \underset{x\to 0}{\longrightarrow} 0$ so $\dfrac{\partial f}{\partial x}(0,0) = 0$

$\dfrac{f(0,y)-f(0,0)}{y} = \dfrac{0-0}{y} = 0 \underset{y\to 0}{\longrightarrow} 0$ so $\dfrac{\partial f}{\partial y}(0,0) = 0.$

c) $\dfrac{f(x,0)-f(0,0)}{x} = \dfrac{0-0}{x} = 0 \underset{x\to 0}{\longrightarrow} 0$ so $\dfrac{\partial f}{\partial x}(0,0) = 0$

$\dfrac{f(0,y)-f(0,0)}{y} = \dfrac{\frac{|y|^3}{0^2+y^2}-0}{y} = \dfrac{|y|^3}{y^3} = \begin{cases} +1 & \text{if } y>0 \\ -1 & \text{if } y<0 \end{cases}$

This has no limit as $y\to 0$ so $\dfrac{\partial f}{\partial y}(0,0)$ does not exist.

d) $\dfrac{f(x,0)-f(0,0)}{x} = \dfrac{0-0}{x} = 0 \underset{x\to 0}{\longrightarrow} 0$ so $\dfrac{\partial f}{\partial x}(0,0) = 0$

$\dfrac{f(0,y)-f(0,0)}{y} = \dfrac{\frac{|y|^{3.1}}{0^2+y^2}-0}{y} = \dfrac{|y|^{3.1}}{y^3} = \pm|y|^{.1} \underset{y\to 0}{\longrightarrow} 0$

so

$\dfrac{\partial f}{\partial y}(0,0) = 0$

4. $$f(x,y) = \begin{cases} \dfrac{x^3y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{else} \end{cases}$$

For $(x,y) \neq (0,0)$

a) $$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)3x^2y - 2x(x^3y)}{(x^2+y^2)^2}$$

and

b) $$\frac{\partial f}{\partial y} = \frac{(x^2+y^2)x^3 - 2y(x^3y)}{(x^2+y^2)^2}.$$

Also

c) $$\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

and

d) $$\frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0.$$

Hence

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{\Delta y \to 0} \frac{\frac{\partial f}{\partial x}(0,\Delta y) - \frac{\partial f}{\partial x}(0,0)}{\Delta y}$$

from a) from c)

$$= \lim_{\Delta y \to 0} \frac{\frac{0-0}{(\Delta y^2)^2} - 0}{\Delta y} = 0.$$

Also

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{\Delta x \to 0} \frac{\frac{\partial f}{\partial y}(\Delta x, 0) - \frac{\partial f}{\partial y}(0,0)}{\Delta x}$$

from b) $\longrightarrow$ $\longleftarrow$ from d)

$$= \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^2 (\Delta x)^3 - 0}{(\Delta x)^4} - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^5}{(\Delta x)^5} = 1.$$

6. $uvw^2xy^3 = 1$

$u^2v^3wx^2y = 1$

a) Find $\left(\frac{\partial u}{\partial x}\right)_{vw}$ by implicit differentiation

$u = u(x,v,w) \quad y = y(x,v,w)$

$$0 = \frac{\partial u}{\partial x} vw^2xy^3 + uvw^2y^3 + uvw^2x3y^2\frac{\partial y}{\partial x}$$

$$0 = 2u\frac{\partial u}{\partial x} v^3wx^2y + u^2v^3w2xy + u^2v^3wx^2\frac{\partial y}{\partial x}$$

$$0 = xy\frac{\partial u}{\partial x} + uy + 3ux\frac{\partial y}{\partial x}$$

$$0 = 2xy\frac{\partial u}{\partial x} + 2uy + ux\frac{\partial y}{\partial x}$$

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$$0 = -5xy\frac{\partial u}{\partial x} - 5uy$$

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_{vw} = \frac{5uy}{-5xy} = \frac{-u}{x}$$

b) Find $\left(\frac{\partial u}{\partial x}\right)_{vw}$ by explicit diff:

$u^2 v^3 w x^2 y = 1$ so $y = u^{-2} v^{-3} w^{-1} x^{-2}$ and

$$1 = u v w^2 x y^3 = u v w^2 x \left(u^{-2} v^{-3} w^{-1} x^{-2}\right)^3$$
$$= u^{-5} v^{-8} w^{-1} x^{-5}.$$

Hence

$$u = u(x, v, w) = \left(v^{-8} w^{-1} x^{-5}\right)^{1/5} = v^{-8/5} w^{-1/5} x^{-1}$$

So

$$\left(\frac{\partial u}{\partial x}\right)_{vw} = - v^{-8/5} w^{-1/5} x^{-2}.$$

Note that $u = v^{-8/5} w^{-1} x^{-1}$ so

$$\left(\frac{\partial u}{\partial x}\right)_{vw} = - \frac{u}{x}$$

as in part a.