21-268 HW #13
Section A

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1) a) $P = \dfrac{x^3}{x^4+y^2}$, $Q = \frac{1}{2}\dfrac{y}{x^4+y^2}$ $D = \mathbb{E}^2 \setminus \{(0,0)\}$

+6 $\varphi(x,y) = \int P\,dx = \frac{1}{4}\int \dfrac{4x^3}{x^4+y^2}\,dx = \frac{1}{4}\ln(x^4+y^2) + h(y)$

$\Rightarrow \dfrac{\partial\varphi}{dy} = Q \Rightarrow \dfrac{d\left(\frac{1}{4}\ln(x^4+y^2)+h(y)\right)}{dy} = \frac{1}{2}\dfrac{y}{x^4+y^2}$

$\Rightarrow \frac{1}{4}\left(\dfrac{2y}{(x^4+y^2)}\right) + h'(y) = \frac{1}{2}\dfrac{y}{x^4+y^2}$

$\Rightarrow \varphi(x,y) = \frac{1}{4}\ln(x^4+y^2)$

Since the potential function φ is defined on the whole domain D, such that $\vec{F} = \vec{\nabla}\varphi$, $\int P\,dx + Q\,dy$ is path independent.

b) $P = \dfrac{x+y}{x^2+y^2}$, $Q = \dfrac{y-x}{x^2+y^2}$ $D = \mathbb{E}^2 \setminus \{(x,0) \mid x \leq 0\}$

Since D is simply connected, in order for $\int P\,dx + Q\,dy$ to be path independent, we only need that $\dfrac{\partial Q}{dx} = \dfrac{\partial P}{dy}$.

$\dfrac{\partial Q}{\partial x} = \dfrac{-1(x^2+y^2) - 2x(y-x)}{(x^2+y^2)^2} = \dfrac{-x^2-y^2-2xy+2x^2}{(x^2+y^2)^2} = \dfrac{x^2-2xy-y^2}{(x^2+y^2)^2}$ $(x-y)(x-y)$

$\dfrac{\partial P}{\partial y} = \dfrac{(x^2+y^2) - 2y(x+y)}{(x^2+y^2)^2} = \dfrac{x^2-2yx-y^2}{(x^2+y^2)^2} = \dfrac{(x-y)^2}{(x^2+y^2)^2} = \dfrac{\partial Q}{dx}$

So $\int P\,dx + Q\,dy$ is path independent in this situation as well.

c) $P = \frac{x+y}{x^2+y^2}$, $Q = \frac{y-x}{x^2+y^2}$, $D = \mathbb{E}^2 \setminus \{(0,0)\}$

In this case, $f(x,y)$ is not easy to find and the domain is not simply connected, so in order to prove path independence, we have to show that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{AND} \quad \oint_C \vec{F}\cdot d\vec{r} = 0 \ \forall \text{ simple, closed curves in } D.$$

First consider c = unit circle, so $x^2+y^2=1$.

let $\vec{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$, $0 \le \theta < 2\pi$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \begin{pmatrix} \frac{\cos\theta+\sin\theta}{\cos^2\theta+\sin^2\theta} \\ \frac{\sin\theta-\cos\theta}{\cos^2\theta+\sin^2\theta} \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} d\theta$$

$$= \int_0^{2\pi} (\cos\theta+\sin\theta)(-\sin\theta) + (\sin\theta-\cos\theta)\cos\theta \; d\theta$$

$$= \int_0^{2\pi} -\cancel{\cos\theta\sin\theta} - \sin^2\theta + \cancel{\sin\theta\cos\theta} - \cos^2\theta \; d\theta$$

$$= \int_0^{2\pi} -1 \; d\theta = -2\pi \neq 0.$$

Thus, since $\oint P\,dx + Q\,dy \neq 0 \ \forall$ simple, closed curves on the domain, it is <u>not</u> path independent in this situation.

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21-268 HW #13
Section A

1) a) $\vec{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$ $\vec{F}$ is conservative / path independent

$P = \frac{x^3}{x^4+y^2}$, $Q = \frac{1}{2}\frac{y}{x^4+y^2}$ $\Longleftrightarrow$ $\oint_C \vec{F}\cdot d\vec{r} = 0$ around any pole in the graph

AND $\vec{\nabla} \times \vec{F} = 0$.

pole $= (0,0)$

curve around pole = circle of radius 1 (for ease)

$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$

c) $f(x,y) = \int \frac{x+y}{x^2+y^2}\,dx = \frac{1}{2}\int \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2}\,dx$

$= \frac{1}{2}\ln(x^2+y^2$

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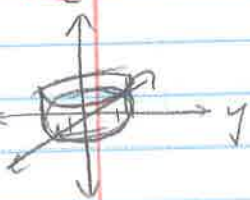
2) $\vec{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ $\quad S = \{(x,y,z) \mid x^2+y^2=4, \ 0 \le z \le 1\}$



$$\iint_S \vec{F}\cdot d\vec{\sigma} = \iint_S (\vec{F}\cdot\vec{n})\, d\sigma$$

$r(\theta,z) = 2\cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath} + z\hat{k}$

$0 \le \theta < 2\pi, \quad 0 \le z \le 1$

$$\vec{n} = \frac{\frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial z}}{\left|\frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial z}\right|} = \begin{pmatrix} -2\sin\theta \\ 2\cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{-(2\cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath})}{\left|\frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial z}\right|}$$

*note: negative n because it's oriented inwards

$$\Rightarrow \iint_S (\vec{F}\cdot\vec{n})\, d\sigma = \iint_R \vec{F}(\vec{r}(\theta,z)) \cdot \begin{pmatrix} -2\cos\theta \\ -2\sin\theta \\ 0 \end{pmatrix} dA_{\theta,z}$$

$$\Rightarrow \iint \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \\ z \end{pmatrix}\begin{pmatrix} -2\cos\theta \\ -2\sin\theta \\ 0 \end{pmatrix} dA_{\theta,z} = \iint_R -4\cos^2\theta - 4\sin^2\theta \ dA_{\theta,z}$$

$$= \iint_R -4\, d\theta\, dz = \int_0^1\int_0^{2\pi} -4\, d\theta\, dz = \int_0^1 -4\theta\Big]_0^{2\pi} dz$$

$$= \int_0^1 -8\pi\, dz = \boxed{-8\pi}$$

3) $R = \{(x,y,z) : x^2+y^2+z^2 \le R_0^2\}$ ✶ sphere w/ radius R_0

$S = \{(x,y,z) : x^2+y^2+z^2 = R_0^2\}$

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$\vec{F} = x\hat{\imath} + y\hat{\jmath}$

A) $\iint_S \vec{F} \cdot d\vec{\sigma} = \iint_S (\vec{F}\cdot\vec{n})\, d\sigma$

$\vec{r}(\theta,\phi) = R_0\cos\theta\sin\phi\,\hat{\imath} + R_0\sin\theta\sin\phi\,\hat{\jmath} + R_0\cos\phi\,\hat{k}$

$0 \le \theta < 2\pi$

$0 \le \phi < \pi$

$$n = \underbrace{\begin{pmatrix} -R_0\sin\phi\sin\theta \\ R_0\cos\theta\sin\phi \\ 0 \end{pmatrix}}_{\frac{\partial r}{\partial\theta}} \times \underbrace{\begin{pmatrix} R_0\cos\theta\cos\phi \\ R_0\sin\theta\cos\phi \\ -R_0\sin\phi \end{pmatrix}}_{\frac{\partial r}{\partial\phi}}$$

$$= \begin{pmatrix} -R_0^2\cos\theta\sin^2\phi \\ -R_0^2\sin^2\phi\sin\theta \\ -R_0^2\sin\phi\cos\phi\sin^2\theta - R_0^2\cos^2\theta\sin\phi\cos\phi \end{pmatrix}$$

$$\vec{n} = \begin{pmatrix} +R_0^2\cos\theta\sin^2\phi \\ +R_0^2\sin\theta\sin^2\phi \\ +R_0^2\sin\phi\cos\phi \end{pmatrix}$$ ← but make positive to have outwards orientation

$$\Rightarrow \iint_S (\vec{F}\cdot\vec{n})\,d\sigma = \iint_{R_{\theta,\phi}} \vec{F}(\vec{r}(\theta,\phi))\cdot\vec{n}\; dA_{\theta,\phi}$$

$$\Rightarrow \iint_{R_{\theta,\phi}} \begin{pmatrix} R_0\cos\theta\sin\phi \\ R_0\sin\theta\sin\phi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} +R_0^2\cos\theta\sin^2\phi \\ +R_0^2\sin\theta\sin^2\phi \\ +R_0^2\sin\phi\cos\phi \end{pmatrix} dA_{\theta,\phi}$$

$$\Rightarrow \iint_{R_{\theta,\phi}} R_0^3\cos^2\theta\sin^3\phi + R_0^3\sin^2\theta\sin^3\phi \;\; dA_{\theta,\phi}$$

$$\Rightarrow \iint_{R_{\theta,\phi}} R_0^3\sin^3\phi \;\; dA_{\theta,\phi} = \int_0^{2\pi}\int_0^{\pi} R_0^3\sin\phi(1-\cos^2\phi)\,d\phi\,d\theta$$

$$\Rightarrow R_0^3\int_0^{2\pi} -\cos\phi + \tfrac{1}{3}\cos^3\phi\Big]_0^{\pi} d\theta = R_0^3\int_0^{2\pi}\Big[\underbrace{1 + \tfrac{1}{3} + 1 - \tfrac{1}{3}}_{= 4/3}\Big]\,d\theta$$

$$= \tfrac{4}{3}R_0^3\,\theta\Big]_0^{2\pi} = \boxed{\tfrac{8\pi}{3}R_0^3}$$

3) continued...

B) $\iiint_R \vec{\nabla} \cdot \vec{F} \, dV$

$\vec{r}(\theta, \phi, \rho) = \rho\cos\theta\sin\phi\,\hat{\imath} + \rho\sin\theta\sin\phi\,\hat{\jmath} + \rho\cos\phi\,\hat{k}$

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$$\Rightarrow \vec{\nabla} \cdot \vec{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = 2$$

$$\Rightarrow \iiint_R 2 \, dV = 2 \cdot \left(\frac{4}{3}\pi (R_0)^3\right) = \boxed{\frac{8}{3}\pi R_0^3} \checkmark$$

so YES! methods A) & B) got the same results!

4) $$\iiint_R \vec{\nabla} f \, dV = \iiint_R \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} dV$$

$$= \left(\iiint_R f_x \, dV\right)\hat{\imath} + \left(\iiint_R f_y \, dV\right)\hat{\jmath} + \left(\iiint_R f_z \, dV\right)\hat{k}$$

$$\Rightarrow \iiint_R f_x \, dV = \iiint_R \vec{\nabla} \cdot \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} dV \xrightarrow[\text{theorem}]{\text{by div}} = \iint_S \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} \cdot d\vec{\sigma}$$

$$= \iint_S \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} \cdot \vec{n} \, d\sigma$$

let $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

$\leftarrow$ same logic applies for f_y and f_z

SO: $$\iiint_R \vec{\nabla} f \, dV = \Bigg(\iint_S \underbrace{\begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} \cdot \vec{n}}_{= fn_1} d\sigma\Bigg)\hat{\imath} + \Bigg(\iint_S \underbrace{\begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix} \cdot \vec{n}}_{= fn_2} d\sigma\Bigg)\hat{\jmath} + \Bigg(\iint_S \underbrace{\begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} \cdot \vec{n}}_{= fn_3} d\sigma\Bigg)\hat{k}$$

$$= \iint_S \begin{pmatrix} fn_1 \\ fn_2 \\ fn_3 \end{pmatrix} d\sigma = \iint_S f\vec{n} \, d\sigma \quad \square$$