+27

Homework 10

1. +6

a.) $Y(x) = x, \quad Z(x) = 2x^2$

b.) $Y^2(\theta) = 4 - 4\sin^2\theta = 4\cos^2\theta \Rightarrow Y(\theta) = 2\cos\theta, \quad Z(\theta) = 4$ ↙ multiple choices

c.) $Y(\theta,\varphi) = \sin\varphi\sin\theta, \quad Z(\theta,\varphi) = 2\cos\varphi$

d.) $Y(\theta,y) = y; \quad Z(\theta,y) = 2\sin\theta$

2. +4

a.) Double Containment, assuming $0 \leq a \leq b$.

$\Rightarrow \forall r \in [a,b], \theta \in [0,2\pi]: f(\sqrt{x^2+y^2}) = f(\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}) = f(r) = z$.

$\sqrt{x^2+y^2} = r$, so $a \leq \sqrt{x^2+y^2} \leq b$.

$\Leftarrow$ If $a \leq \sqrt{x^2+y^2} \leq b$, then setting $r = \sqrt{x^2+y^2}$, $\theta = \arg(x+yi)$ (or $\tan^{-1}2(y,x)$)

we get $\vec{R}(r,\theta) = x\vec{i} + y\vec{j} + z\vec{k}$, $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$

b.) $R_{r,\theta} = \begin{bmatrix} \cos\theta & \sin\theta & f'(r) \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix}, \quad \left\| \begin{bmatrix} \cos\theta \\ \sin\theta \\ f'(r) \end{bmatrix} \times \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \right\| = \cdots$

$\cdots = \left\| [-rf'(r)\cos\theta, -rf'(r)\sin\theta, r\cos^2\theta + r\sin^2\theta]^T \right\|$

$= \sqrt{r^2 f'^2(r)(\cos^2\theta + \sin^2\theta) + r^2} = r\sqrt{f'^2(r)+1}$

$\int_b^a \int_0^{2\pi} r\sqrt{f'^2(r)+1}\, d\theta\, dr = 2\pi \int_b^a r\sqrt{f'^2(r)+1}\, dr$

3. +5

$x^{-p}y^{-q}$ is continuous & therefore integrable in $\{(x,y) : x,y \in (0,1]\}$

This set is bounded. Consider $\varepsilon > 0$. Let $0 < \eta < \sqrt{\delta}$.

Let $P = \{(x,y) : x,y \in [\eta,1]\}$;

$R \setminus P = \{(x,y) : x,y \in (0,\eta)\}$ has area $\eta^2 < \delta$

$\iint_P f\,dA = \int_\eta^1 \int_\eta^1 x^{-p}y^{-q}\,dy\,dx = X_\varepsilon Y_\varepsilon$

Where $X_\varepsilon = \begin{cases} \frac{1}{1-p}(1-\eta^{1-p}) & p \neq 1, p > 0 \\ \ln 1/\eta & p = 1 \end{cases}$ and $Y_\varepsilon = \begin{cases} \frac{1}{1-q}(1-\eta^{1-q}) & q \neq 1, q > 0 \\ \ln 1/\eta & q = 1 \end{cases}$

If $0<p<1$ and $0<q<1$, then:

$\therefore$ $X_\varepsilon Y_\varepsilon = 1/(1-p)(1-q) \; (1-\eta^{1-p})(1-\eta^{1-q})$

This is continuous in $\eta \geq 0$, with the value $1/(1-p)(1-q)$ for $\eta = 0$. So $\lim_{\eta\to 0} X_\varepsilon Y_\varepsilon = 1/(1-p)(1-q)$

η is bounded between 0 and $\sqrt{\delta}$,

so since $\lim_{\eta\to 0} X_\varepsilon Y_\varepsilon = 1/(1-p)(1-q)$, there exists $\delta > 0$ such that $|X_\varepsilon Y_\varepsilon - 1/(1-p)(1-q)| < \varepsilon$ for all $\eta < \sqrt{\delta}$.

If $p=1$, $0<q<1$, then $X_\varepsilon Y_\varepsilon = 1/1-q \; (1-\eta^{1-q}) \ln 1/\eta$

So $X_\varepsilon Y_\varepsilon > 1/(1-q) \cdot 1/2 \cdot \ln 1/\eta$ for $0 < \eta \leq \sqrt[(1-q)]{1/2}$

Can make $X_\varepsilon Y_\varepsilon$ arbitrarily large for small $\eta > 0$, since $\lim_{\eta\to 0^+} \ln 1/\eta$ is infinite. So improper integral doesn't exist.

Similar proof for $q=1$, $0<p<1$ by swapping the roles of p & q

If $p \geq 1$ then X_ε grows arbitrarily large for small $\eta > 0$

since $X_\varepsilon = \ln 1/\eta$ or $X_\varepsilon = 1/(p-1)\left((1/\eta)^{p-1} - 1\right)$ $(p-1>0)$

Similarly if $q \geq 1$ then Y_ε grows arbitrarily large for small $\eta > 0$

So $X_\varepsilon Y_\varepsilon$ grows arbitrarily large for small $\eta > 0$ if $p \geq 1$ and $q \geq 1$.

$\therefore$ The improper integral only converges if $0<p<1$ and $0<q<1$.

It diverges if $p \geq 1$ or $q \geq 1$.

+6

4. a.) For any x, $A_x(t) = a(x,t)$, $B_x(t) = b(x,t)$, $F_x(t,y) = f(t,x,y)$ are all C^1 functions. So Leibniz's rule applies.

So fixing x, $\partial/\partial t \int_{a(t,x)}^{b(t,x)} f(t,x,y)\,dy = d/dt \int_{A_x(t)}^{B_x(t)} F_x(t,y)\,dy$

$= (\partial/\partial t\, b(x,t)) \cdot f(x,t,b(x,t)) - (\partial/\partial t\, a(x,t)) f(x,t,a(x,t))$

$+ \int_{a(x,t)}^{b(x,t)} \partial/\partial t\, f(x,t,y)\,dy = b_t f(x,t,b) - a_t f(x,t,a) + \int_a^b f_t\,dy$

Similarly, $\partial/\partial x \int_{a(t,x)}^{b(t,x)} f(t,x,y)\,dy =$

$(\partial/\partial x\, b(x,t)) f(x,t,b(x,t)) - (\partial/\partial x\, a(x,t)) f(x,t,a(x,t))$

$+ \int_{a(x,t)}^{b(x,t)} \partial/\partial x\, f(x,t,y)\,dy = b_x f(x,t,b) - a_x f(x,t,a) + \int_a^b f_x\,dy$

b.) Here $b_x = 1$, $b_t = 0$, $a_x = 1$, $a_t = -2$.

Let $f(x,t,y) = g(t+(y-x)/2, y)$ for some $g(v1, v2)$.

Then $f_x = -1/2 \cdot g_{v1}(t+(y-x)/2, y)$

$f_t = g_{v1}(t+(y-x)/2, y)$ Note: $f_x = -\frac{1}{2} f_t$.

So $u_t = b_t f(x,t,b) - a_t f(x,t,a) + \int_a^b f_t \, dy$

$= 0 + 2f(x,t,a) + \int_a^b f_t \, dy$

And $u_x = b_x f(x,t,b) - a_x f(x,t,a) + \int_a^b f_x \, dy$

$= g(t+((x)-x)/2, x) - f(x,t,a) - \frac{1}{2}\int_a^b f_t \, dy$

So $u_t + 2u_x = 2g(t+0/2, x) + 2f(x,t,a) - 2f(x,t,a) + \int_a^b f_t \, dy - \frac{2}{2}\int_a^b f_t \, dy$

$= 2g(t,x)$

+4

5. a) $\sum_{i=1}^{n} f(x_i^*, y_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}, 1+(x_i+x_{i-1})\right)(x_i - x_{i-1})$

$= \sum_{i=1}^{n} (x_i - x_{i-1})(1 + x_i + x_{i-1})$

$= \sum_{i=1}^{n} (x_i - x_{i-1})(x_i + x_{i-1}) + (x_i - x_{i-1})$

$= \sum_{i=1}^{n} (x_i^2 - x_{i-1}^2) + (x_i - x_{i-1})$

$= -x_0^2 + x_n^2 - x_0 + x_n = -1 + 9 - 1 + 3 = 10.$

b.) Given $h < \delta$, we have $|x_i^{**} - x_i| < h$ and $|x_i^{**} - x_{i-1}| < h$. So

$\left| \sum_{i=1}^{n} (x_i - x_{i-1}) f(x_i^{**}, 1 + 2x_i^{**}) - 10 \right|$

$= \left| \sum_{i=1}^{n} (x_i - x_{i-1})(1 + 2x_i^{**}) - \sum_{i=1}^{n} (x_i - x_{i-1})(1 + x_i + x_{i-1}) \right|$ ← from a)

$= \left| \sum_{i=1}^{n} (x_i - x_{i-1})(2x_i^{**} - x_i - x_{i-1}) \right|$

$= \left| \sum_{i=1}^{n} (x_i - x_{i-1})((x_i^{**} - x_i) + (x_i^{**} - x_{i-1})) \right|$

Noting that $x_i - x_{i-1} > 0$ and applying triangle inequality:

$\leq \sum_{i=1}^{n} (x_i - x_{i-1}) \left| (x_i^{**} - x_i) + (x_i^{**} - x_{i-1}) \right|$

Triangle inequality again

$\leq \sum_{i=1}^{n} (x_i - x_{i-1})(|x_i^{**} - x_i| + |x_i^{**} - x_{i-1}|)$

$< \sum_{i=1}^{n} (x_i - x_{i-1}) \, 2h$

$= (-x_0 + x_n) \, 2h = 4h < 4\delta.$

Given $\varepsilon > 0$, set $\delta = \varepsilon/4 \Rightarrow |R - 10| < \varepsilon.$

6. $C = \{(x,y) : 1 \le x \le 3 \ , \ y = \frac{3}{2}x - 5/2\}$. Parameterize on x

$ds^2 = dx^2 + dy^2, \quad dy = 3/2\, dx \quad \Rightarrow \quad ds^2 = 13/4\, dx^2$

$\Rightarrow ds = \sqrt{13}/2 \ dx$

x2

$\int_C f\, ds = \sqrt{13}/2 \int_C f\, dx = \sqrt{13}/2 \int_1^3 x\left(\frac{3}{2}x - 5/2\right) dx$

$= \sqrt{13}/2 \left(\frac{1}{2}x^3 - 5/4\, x^2\right)\Big|_1^3 = \sqrt{13}/2 \left(\frac{27-1}{2} - \frac{45-1}{4}\right)$

$= \sqrt{13}/2\,(13 - 10) = 3\sqrt{13}/2$