21-241 Matrices and Linear Transformations Lecture 3
Midterm #1 Solution

October 3, 2016

1. (a) (8 points) Given $a, b, c \in \mathbb{R}$, solve

$$(S)\begin{cases} x + \ \ y + z = a \\ x - \ \ y + z = b \\ \quad\ \ 2y - z = c \end{cases}$$

by Gauß-Jordan elimination (that is, using reduced row echelon form).

Proof. The augmented matrix of (S) is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 1 & b \\ 0 & 2 & -1 & c \end{array}\right]$$

which can be row reduced to

$$\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & (3b+2c-a)/2 \\ 0 & \mathbf{1} & 0 & (a-b)/2 \\ 0 & 0 & \mathbf{1} & a-b-c \end{array}\right]$$

This means that there is one and only one solution :

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (3b+2c-a)/2 \\ (a-b)/2 \\ a-b-c \end{bmatrix}$$

□

(b) (4 points) Show that

$$\mathbb{R}^3 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Proof. Let $a, b, c \in \mathbb{R}^3$. We wish to find a linear combination

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

This is exactly the column-by-column approach of the above system. We just saw that there is a solution. This is valid for all $a, b, c \in \mathbb{R}$, which means that any vector in $\mathbb{R}^3$ can be expressed as a combination of $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}$, i.e.

$$\mathbb{R}^3 = \operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$$

□

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2. Let $(\mathcal{P}_1)$ resp. $(\mathcal{P}_2)$ be planes in $\mathbb{R}^3$ defined by equations in normal forms

$$n_1 \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = 1 \qquad n_2 \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = -2$$

where $n_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}$ and $n_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

(a) (4 points) Without doing any computation, discuss what the intersection $(\mathcal{P}_1) \cap (\mathcal{P}_2)$ is geometrically (justify your answer).

Proof. As n_1 and n_2 are not scalar multiples of one another, the planes do not have the same orientation. Therefore they have to intersect along a line. □

(b) (8 points) Compute this intersection and write it as an affine space.

Proof. The vectors $\begin{bmatrix}x\\y\\z\end{bmatrix}$ lying on the intersection have to satisfy both plane equations. The augmented matrix associated to that system is

$$\left[\begin{array}{ccc|c}1 & 2 & 0 & 1\\1 & 1 & 1 & -2\end{array}\right]$$

which row reduces to

$$\left[\begin{array}{ccc|c}\mathbf{1} & 0 & 2 & -5\\0 & \mathbf{1} & -1 & 3\end{array}\right]$$

meaning that the set of solution writes

$$\left\{\begin{bmatrix}-5-2z\\3+z\\z\end{bmatrix} \;\middle|\; z \in \mathbb{R}\right\} = \begin{bmatrix}-5\\3\\0\end{bmatrix} + \operatorname{span}\left\{\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}$$

□

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3. (8 points) Find all possible linear combinations of $\begin{bmatrix}3\\2\\4\end{bmatrix}$ and $\begin{bmatrix}-2\\-1\\-3\end{bmatrix}$ that are equal to $\begin{bmatrix}-1\\-1\\-1\end{bmatrix}$.

Proof. We are looking for $x, y \in \mathbb{R}$ such that

$$x \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + y \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

This is a linear system whose augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{array}\right]$$

It row reduces to

$$\left[\begin{array}{cc|c} \mathbf{1} & 0 & -1 \\ 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 \end{array}\right]$$

meaning that there is one and only one solution : $x = y = -1$. □

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4. We define the *trace* of a square $n \times n$ matrix to be the sum of the diagonal terms :

$$\text{tr} : \mathcal{M}_{nn}(\mathbb{R}) \to \mathbb{R} \qquad A \mapsto \sum_{i=1}^{n} a_{ii}$$

(a) (5 points) Prove that tr is a linear transformation.

Proof. Let $A, B \in \mathcal{M}_{nn}(\mathbb{R})$ and $\lambda \in \mathbb{R}$.

$$\text{tr}(A+B) = \sum_{i=1}^{n} (A+B)_{ii} = \sum_{i=1}^{n} a_{ii} + b_{ii} = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \text{tr}(A) + \text{tr}(B)$$

$$\text{tr}(\lambda A) = \sum_{i=1}^{n} (\lambda A)_{ii} = \sum_{i=1}^{n} \lambda a_{ii} = \lambda \sum_{i=1}^{n} a_{ii} = \lambda \ \text{tr}(A)$$

□

(b) (2 points) Let (α_{ij}) denote a family of real numbers indexed by $1 \leq i, j \leq n$. Explain briefly why

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \alpha_{ij} \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \alpha_{ij} \right).$$

Proof. By commutativity of the addition, the order of the two Σ symbols does not matter since in the end all of the n^2 α_{ij} terms are being summed up. □

(c) (6 points) Prove that for all $A, B \in \mathcal{M}_{nn}(\mathbb{R})$, $\text{tr}(AB) = \text{tr}(BA)$.
Hint: use the above identity.

Proof. Let $1 \leq i \leq n$. By the product formula

$$(AB)_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$$

so that

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_{ki} \right)$$

Calling $\alpha_{ik} = a_{ik} b_{ki}$ we can apply the above identity to get

$$\operatorname{tr}(AB) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} a_{ik} b_{ki} \right) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} b_{ki} a_{ik} \right) = \sum_{k=1}^{n} (BA)_{kk} = \operatorname{tr}(BA)$$

□

(d) (5 points) Find examples of matrices such that $\operatorname{tr}(ABC) \neq \operatorname{tr}(ACB)$.

Proof. The idea is to find matrices such that ABC and ACB are different enough so that the sum of their diagonal terms differ. We saw some examples of non-commutative matrix products in class. Inspired by these we can cook:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

We have $ABC = A$ which has trace 1 but $ACB = 0$ which has trace 0. □

Remark. One can prove that *tr* is invariant under cyclic permutations, that is

$$\operatorname{tr}(A_1 A_2 \cdots A_n) = \operatorname{tr}(A_k A_{k+1} \cdots A_n A_1 \cdots A_{k-1})$$

for all $1 \leq k \leq n$. Observe that ACB is not a cyclic permutation of ABC.