21-241 – Handout on bases and dimension

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1 Introduction

We saw in class how to reduce a spanning set to a basis by using row reduction. Thus, provided that we have a spanning set with finitely many vectors, we can find a basis for any subspace of $\mathbb{R}^n$. The aim of this handout is twofold:

1. prove that every subspace of $\mathbb{R}^n$ can be spanned by finitely many vectors
2. prove that any two bases of a same subspace of $\mathbb{R}^n$ have the same number of vectors: we call it the *dimension* of the subspace.

2 Existence of bases

In this section we prove item 1. More precisely, we give an algorithm that finds a basis for any subspace of $\mathbb{R}^n$.

Lemma 2.1. *A linearly independent set of vectors in $\mathbb{R}^n$ cannot have more than n vectors.*

Proof. Let $\{v_1, \cdots, v_m\}$ be linearly independent and assume $m > n$. Then

$$\begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}$$

has more columns than rows, so when reducing

$$\left[\begin{array}{ccc|c} & & & 0 \\ v_1 & \cdots & v_m & \vdots \\ & & & 0 \end{array}\right]$$

to reduced row echelon form one necessarily gets at least one free variable (i.e. one column with no leading 1, since we cannot have m leading 1 because there are only n rows). Thus, the above system has (infinitely many) non-zero solutions. This is in contradiction with the linear independence. So we have $m \leq n$. □

Theorem 2.2. *Let $S \subseteq \mathbb{R}^n$ be a subspace. Then there exists a basis for S with at most n vectors.*

Proof. We first evacuate the case of the zero space. If $S = \{0\}$, then $B = \varnothing$ spans S in the sense that 0 (the only vector in S) is the linear combination of nothing (the sum of nothing is 0). It is clearly linearly independent.

Now if $S \neq \{0\}$, pick a non-zero $v_1 \in S$. Since $v_1 \neq 0$, $\{v_1\}$ is linearly independent. If it spans S, it is a basis and we stop here. Otherwise we can find some $v_2 \in S \setminus \text{span}\{v_1\}$. Since $v_2 \notin \text{span}\{v_1\}$, $\{v_1, v_2\}$ is linearly independent. If it spans S, it is a basis and we stop here. Otherwise we can find a $v_3 \in S \setminus \text{span}\{v_1, v_2\}$... In this *while* loop, $\{v_1, \cdots, v_k\}$ being linearly independent is a *loop invariant*, i.e. it is always true. Thus the algorithm has to stop before or at n steps, thanks to Lemma 2.1. □

3 Dimension

Theorem 3.1. *Let $S \subseteq \mathbb{R}^n$ be a subspace. Any two bases for S have the same number of vectors. We call this number the dimension of S, $\dim S$.*

Proof. Let $B = \{u_1, \cdots u_r\}$ and $C = \{v_1, \cdots v_s\}$ be bases for S. Assume without loss of generality that $r < s$. We look for a contradiction by exhibiting a linear dependance relation in C.

$$\begin{aligned} & c_1 v_1 + \cdots + c_s v_s = 0 \\ \Leftrightarrow\ & c_1(a_{11}u_1 + \cdots + a_{1r}u_r) + \cdots + c_s(a_{s1}u_1 + \cdots + a_{sr}u_r) = 0 \end{aligned}$$

for some family of scalars (a_{ij}) since B spans S. We regroup these terms along the u_i :

$$\begin{aligned} & c_1 v_1 + \cdots + c_s v_s = 0 \\ \Leftrightarrow\ & (a_{11}c_1 + \cdots + a_{s1}c_s)u_1 + \cdots + (a_{1r}c_1 + \cdots + a_{sr}c_s)u_r = 0 \end{aligned}$$

And since B is also linearly independent, the above equality happens if and only if each parentheses equals to zero:

$$c_1 v_1 + \cdots + c_s v_s = 0 \Leftrightarrow \begin{cases} a_{11}c_1 + \cdots + a_{s1}c_s = 0 \\ \vdots \\ a_{1r}c_1 + \cdots + a_{sr}c_s = 0 \end{cases}$$

This is a system of linear equations with unknowns the c_i, moreover since $s > r$ it has more columns than rows, so there is necessarily a free variable and thus (infinitely many) non-zero solutions, i.e. non-trivial dependence relations in C. This is a contradiction with the fact that C is linearly independent. Thus, $r \geq s$. Conversely by exhanging the role of B and C in the first place we get $s \geq r$, so that $s = r$. □

Remark. In addition to Lemma 2.1 recalled below, here are a few good things to know that become clearer now:

- a linearly independent set in a subspace of dimension k cannot have more than k vectors
- a spanning set for a subspace of dimension k cannot have less than k vectors
- a linearly independent set cannot contain the 0 vector.