21-241 – Solution to Homework assignment week #6

Laurent Dietrich
Carnegie Mellon University, Fall 2016, Sec. F and G

1 Standard basis of $\mathcal{M}_{nn}(\mathbb{R})$

1. The product formula yields

$$(E_{ij}E_{kl})_{rs} = \sum_{q=1}^{n} (E_{ij})_{rq}(E_{kl})_{qs}$$

   If $r \neq i$, then in each term of this sum the first factor is zero, so the sum is zero. If $s \neq l$, the same happens with the second factor.

2. So the only possibly non-zero entry of $E_{ij}E_{kl}$ is

$$(E_{ij}E_{kl})_{il} = \sum_{q=1}^{n} (E_{ij})_{iq}(E_{kl})_{ql}$$

   But each term of this sum is zero except when $q = j = k$. So if $j \neq k$, this entry is 0 as well and if $j = k$, there is only one non-zero term in this sum which is $1 \times 1 = 1$. So in the end,

$$(E_{ij}E_{kl})_{il} = \delta_{jk}$$

3. In the end we can conclude that the matrix $E_{ij}E_{kl}$ has only one possible non-zero entry at index (i, l) and that it is 1 iff $j = k$ and 0 otherwise. That is exactly :

$$E_{ij}E_{kl} = \delta_{jk}E_{il}$$

2 The inverse of an elementary matrix

We assume that $i \neq j$, otherwise C is just I_n and B is just the same as A with λ replaced by $\lambda + 1$. We also assume $\lambda \neq 0$ otherwise A is not invertible.

1.

$$C = I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji} = \left(\sum_{k=1, k\neq i, k\neq j}^{n} E_{kk} \right) + E_{ij} + E_{ji}$$

2. A corresponds to $R_i \leftarrow \lambda R_i$. B corresponds to $R_j \leftarrow R_j + \lambda R_i$. C corresponds to $R_i \leftrightarrow R_j$.

3. In the light of elementary row operations and their reverse operations, I propose

$$A' = I_n + (1/\lambda - 1)E_{ii}$$

Indeed, observe that

$$\begin{aligned} AA' &= (I_n + (\lambda - 1)E_{ii})(I_n + (1/\lambda - 1)E_{ii}) \\ &= I_n + (1/\lambda - 1)E_{ii} + (\lambda - 1)E_{ii} + (\lambda - 1)(1/\lambda - 1)E_{ii}E_{ii} \\ &= I_n + 1/\lambda E_{ii} + \lambda E_{ii} - 2E_{ii} + E_{ii} - \lambda E_{ii} - 1/\lambda E_{ii} + E_{ii} \\ &= I_n \end{aligned}$$

since $E_{ii}E_{ii} = E_{ii}$ by the formula proved in Exercise 1. Since $AA' = I_n$, A' is the inverse of A.

4. Again, in the light of elementary operations I propose $B' = I_n - \lambda E_{ij}$ and $C' = C$. Indeed,

$$\begin{aligned} BB' &= (I_n + \lambda E_{ij})(I_n - \lambda E_{ij}) \\ &= I_n - \lambda E_{ij} + \lambda E_{ij} - \lambda^2 E_{ij}E_{ij} \\ &= I_n \end{aligned}$$

since $E_{ij}E_{ij} = 0$ by the formula proved in Exercise 1, since $i \neq j$.

$$CC = \left(\left(\sum_{k=1,k\neq i,k\neq j}^{n} E_{kk}\right) + E_{ij} + E_{ji}\right)\left(\left(\sum_{k=1,k\neq i,k\neq j}^{n} E_{kk}\right) + E_{ij} + E_{ji}\right)$$

Since $k \neq i$ and $k \neq j$ in the sums, there is no cross terms between the terms in the sum and the two other terms. Moreover since $i \neq j$ just like above, $E_{ij}^2 = E_{ji}^2 = 0$. So the only terms left are

$$\begin{aligned} CC &= \left(\sum_{k=1,k\neq i,k\neq j}^{n} E_{kk}\right) + E_{ij}E_{ji} + E_{ji}E_{ij} \\ &= \left(\sum_{k=1,k\neq i,k\neq j}^{n} E_{kk}\right) + E_{ii} + E_{jj} \\ &= I_n. \end{aligned}$$