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Section F
21-241
Problem Set #3

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1) Prove Vector Spaces:

To Prove $\forall v, u \in W.\ v+u \in W \wedge \forall \lambda \in \mathbb{R}\ \lambda v \in W \wedge 0 \in \underline{W}$ whe $W \in \mathbb{R}^n$

a) W = set of solutions to $2x-y-z=5$

<u>False</u> because $(0,0,0) \notin W$ Show ✓

b) W: set of 3x3 real valued matrices s.t. sum of diagonal terms = 0

- $\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix} \in W$ so $0 \in W$ holds because $0+0+0=0 \wedge 0+0+0=0$ ✓

- Let $u, v \in W$ be arbitrary, and $U = \begin{bmatrix} U_1 & U_2 & U_3 \\ U_4 & U_5 & U_6 \\ U_7 & U_8 & U_9 \end{bmatrix} \wedge V = \begin{bmatrix} V_1 & V_2 & V_3 \\ V_4 & V_5 & V_6 \\ V_7 & V_8 & V_9 \end{bmatrix}$

To show: $U+V \in W$

$U+V = \begin{bmatrix} U_1 & U_2 & U_3 \\ U_4 & U_5 & U_6 \\ U_7 & U_8 & U_9 \end{bmatrix} + \begin{bmatrix} V_1 & V_2 & V_3 \\ V_4 & V_5 & V_6 \\ V_7 & V_8 & V_9 \end{bmatrix} = \begin{bmatrix} U_1+V_1 & U_2+V_2 & U_3+V_3 \\ U_4+V_4 & U_5+V_5 & U_6+V_6 \\ U_7+V_7 & U_8+V_8 & U_9+V_9 \end{bmatrix}$ = sum of diagonal terms = ✓

$(U_1+V_1)+(U_5+V_5)+(U_9+V_9) = (U_1+U_5+U_9)+(V_1+V_5+V_9) = 0+0=0$

Thus $u+v \in W$

- To show: $\forall \lambda \in \mathbb{R}.\ \forall u \in W.\ \lambda u \in W$.

Let $u \in W$ be arbitrary. call $u = \begin{bmatrix} U_1 & U_2 & U_3 \\ U_4 & U_5 & U_6 \\ U_7 & U_8 & U_9 \end{bmatrix}$ and let $\lambda \in \mathbb{R}$ be arbitrary.

$\lambda u = \begin{bmatrix} \lambda U_1 & \lambda U_2 & \lambda U_3 \\ \lambda U_4 & \lambda U_5 & \lambda U_6 \\ \lambda U_7 & \lambda U_8 & \lambda U_9 \end{bmatrix} \rightarrow \lambda U_1 + \lambda U_5 + \lambda U_9 = \lambda(U_1+U_5+U_9) = 0$ ✓

so $\lambda u \in W$.

Thus it is True, it is a subspace of $(\mathbb{R}^3)^3$ so it is also a vector space

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c) $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x, y \in \mathbb{R}, y \leq x \right\}$

- To Show: $0 \in W \rightarrow$ that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$

  $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$ because $0 \leq 0$

- To shows $\forall u, v \in W,\ v + u = W$s

  Let $u, v \in W$ be arbitrary. let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \wedge v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_1 + v_2 \end{bmatrix}$. We know $u_2 \leq u_1 \wedge v_2 \leq v_1$ so $u_2 + v_2 \leq u_1 + v_1$, which satisfies property. Thus $u + v \in W$

- to show: $\forall v \in W \wedge \forall \lambda \in \mathbb{R}\ \lambda v \in W$

  let $v \in W$ be arbitrary and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\lambda \in \mathbb{R}$ be arbitrary.

  $\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}$ we know that $v_2 \leq v_1$ but $\lambda v_2 \leq \lambda v_1$ is not always true. Let $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda = -1$. Then $(-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $0 \nleq -1$ so the proposition fails

  This it is False ✓

very nice ! You actually show way more than what was requested, but this is indeed an important property.
A quick way to answer d) : the intersection is either the line itself (if it is contained in the plane), or just the origin. I
we know that this is a subspace since it contains the origin.

d) $\mathbb{R}^3$, intersection of plane passing through origin and line passing through origin

First I will show that the intersection of two subspaces is a subspace: Interesting!

Let U, V be subspaces of W. Prove $U \cap V$ is subspace of W

Closed Under Addition:

Let $a, b \in U \cap V$ be arbitrary. We know $a, b \in U$ which means that $a + b \in U$. Also we know that $a, b \in V$ which means that $a + b \in V$. Thus $a + b \in U \cap V$ ✓

Closed Under Scalar Multiplication

let $a \in U \cap V$ be arbitrary and $\lambda \in \mathbb{R}$ be arbitrary. We know that $\lambda a \in U$ but also that $\lambda a \in V$. Thus $\lambda a \in U \cap V$

Has the Origin:

$(0,0,0) \in U \wedge (0,0,0) \in V$ so $(0,0,0) \in U \cap V$.

Thus the intersection of two subspaces is a subspace!

Now to prove a plane passing through the origin is a subspace of $\mathbb{R}^3$:

call the plane $W: ax+by+cz=0$.

<u>has the origins</u>

$a(0)+b(0)+c(0)=0=0$ So plane passes through origin

<u>Closed Under Addition</u>

let $u,v \in W$ be arbitrary prove $u+v \in W$

let $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$ and both still in W

Then $au_1+bu_2+cu_3=0$ and $av_1+av_2+av_3=0$

The set $u+v$ is $(u_1+v_1, u_2+v_2, u_3+v_3)$

$a(u_1+v_1)+b(u_2+v_2)+c(u_3+v_3)=au_1+bu_2+cu_3+av_1+bv_2+cv_3=0+0=0$

Thus $u+v \in W$

<u>Closed Under Multiplication</u>

let $v \in W$, $\lambda \in \mathbb{R}$ where both v and λ are arbitrary

let $v=(v_1,v_2,v_3) \in W$.

We know that $av_1+bv_2+cv_3=0$

$\lambda v=(\lambda v_1, \lambda v_2, \lambda v_3) \to a(\lambda v_1)+b(\lambda v_2)+c(\lambda v_3)=0$

$\lambda(av_1+bv_2+cv_3)=\lambda(0)=0$.

Thus $\lambda v \in W$

Therefore a plane passing through the origin is a subspace of $\mathbb{R}^3$.

Why?

Now we know a line passing through the origin is the intersection of two planes passing through the origin. We proved that a plane passing through the origin is a subspace of $\mathbb{R}^3$ and the intersection of two subspaces is a subspace. Thus the line is a subspace of $\mathbb{R}^3$. Now the intersection of a line passing through the origin (a subspace), and a plane passing through the origin (a subspace), is a subspace, so the intersection is a subspace.

I'll give benefit of doubt because your proof is cool.

Thus this is a subspace which means it is a vector space.

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2) $2x+y-3z=0$. Vector normal to the plane is $\langle 2,1,23\rangle$ so all planes parallel to $2x+y-3=0$ must share the same coefficients as the original plane. So it must be of the form $2x+y-3z=d$ where d is a constant $\in \mathbb{R}$. we plug $(9,2,-1)$ in to and get $2(9)+(2)-3(1)=7$ so the plane has the equation $2x+3y-z=7$

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3)

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a) $x\begin{bmatrix}3\\1\\-1\end{bmatrix}+y\begin{bmatrix}2\\3\\-2\end{bmatrix}+z\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}a\\6\\c\end{bmatrix}$

$$\begin{aligned}3x+2y+z&=a\\ x+3y+z&=6\\ -x-2y&=c\end{aligned} \rightarrow \left[\begin{array}{ccc|c}3&2&1&a\\1&3&1&6\\-1&-2&0&c\end{array}\right] \xrightarrow[R_1=R_1-R_2]{} \left[\begin{array}{ccc|c}2&-1&0&a-6\\1&3&1&6\\-1&-2&0&c\end{array}\right]$$

$$\xrightarrow[R_2=R_3+R_2]{} \left[\begin{array}{ccc|c}2&-1&0&a-6\\0&1&1&6+c\\-1&-2&0&c\end{array}\right] \xrightarrow[2R_3=R_3]{} \left[\begin{array}{ccc|c}2&-1&0&a-6\\0&1&1&6+c\\-2&-4&0&2c\end{array}\right] \xrightarrow[R_3=R_1+R_3]{} \left[\begin{array}{ccc|c}2&-1&0&a-6\\0&1&1&6+c\\0&-5&0&2c+a-6\end{array}\right]$$

$$\xrightarrow[R_3=\frac{1}{5}R_3]{} \left[\begin{array}{ccc|c}2&-1&0&a-6\\0&1&1&6+c\\0&-1&0&-\frac{1}{5}(2c+c-6)\end{array}\right] \xrightarrow[R_1=R_1+R_3]{} \left[\begin{array}{ccc|c}2&0&0&a-6+\frac{1}{5}(2a+c-6)\\0&1&1&6+c\\0&1&0&-\frac{1}{5}(2a+c-6\end{array}\right] =$$

$$a-6+\frac{2a}{5}+\frac{c}{5}+\frac{6}{5}$$

$$= \left[\begin{array}{ccc|c}2&0&0&\frac{3a}{5}-\frac{4b}{5}+\frac{c}{5}\\0&1&1&6+c\\0&1&0&-\frac{2a}{5}+\frac{c}{5}+\frac{6}{5}\end{array}\right] \xrightarrow[R_1/2=R_1,\ R_2=R_2-R_3]{} \left[\begin{array}{ccc|c}1&0&0&\frac{3a}{10}-\frac{2b}{5}-\frac{c}{10}\\0&0&1&6+c+\frac{2a}{5}+\frac{c}{5}-\frac{6}{5}\\0&1&0&\frac{2a}{5}+\frac{c}{5}+\frac{6}{5}\end{array}\right]$$

$$= \left[\begin{array}{ccc|c}1&0&0&\frac{7a}{10}-\frac{3b}{5}+\frac{c}{10}\\0&1&0&-\frac{2a}{5}-\frac{c}{5}+\frac{6}{5}\\0&0&1&\frac{2a}{5}+\frac{6c}{5}+\frac{4b}{5}\end{array}\right]$$

3)

a) $$x\begin{bmatrix}3\\1\\-1\end{bmatrix} + y\begin{bmatrix}2\\3\\-2\end{bmatrix} + z\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}3&2&1&a\\1&3&1&b\\-1&-2&0&c\end{array}\right] \xrightarrow{R_3=3R_3} \left[\begin{array}{ccc|c}3&2&1&a\\1&3&1&b\\-3&-6&0&3c\end{array}\right] \xrightarrow[R_3=R_3/3]{R_1=R_1+R_3} \left[\begin{array}{ccc|c}0&-4&1&a+3c\\1&3&1&b\\-1&-2&0&c\end{array}\right]$$

$$\xrightarrow{R_2=R_2+R_3} \left[\begin{array}{ccc|c}0&-4&1&a+3c\\0&1&1&b+c\\-1&-2&0&c\end{array}\right] \xrightarrow{R_1=R_1-R_2} \left[\begin{array}{ccc|c}0&-5&0&a+2c-b\\0&1&1&b+c\\-1&-2&0&c\end{array}\right]$$

$$y = \frac{b-a-2c}{5}$$

$$y+z = b+c \rightarrow z = b+c-\frac{b-a-2c}{5} = \frac{4b}{5}+\frac{a}{5}+\frac{7c}{5}$$

$$-x-2y=c \rightarrow x=-2y-c = -2\left(\frac{b-a-2c}{5}\right)-c = \frac{2a}{5}-\frac{2b}{5}-\frac{c}{5}$$

$$\boxed{\text{Linear Combination: } \frac{2a-2b-c}{5}\begin{bmatrix}3\\1\\-1\end{bmatrix} + \frac{-a+b-2c}{5}\begin{bmatrix}2\\3\\-2\end{bmatrix} + \frac{a+4b+7c}{5}\begin{bmatrix}1\\1\\0\end{bmatrix}}$$

b) In terms of spanning sets this means that we can represent any vector in $\mathbb{R}^3$ as a linear combination of $\begin{bmatrix}3\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\3\\-2\end{bmatrix}$, and $\begin{bmatrix}1\\1\\0\end{bmatrix}$ so these vectors span $\mathbb{R}^3$

c) This is the only solution - linear Combination - that exists.