21-241 – Solution to Homework assignment week #11

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Exercises

Ex 1

1. The characteristic polynomial of A is

$$\det(A - \lambda I_2) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc.$$

The discriminant of this second order polynomial is

$$\Delta = (a + d)^2 - 4(ad - bc) = a^2 + 2ad + d^2 - 4ad + 4bc = (a - d)^2 + 4bc$$

If this quantity is positive, we have two distinct eigenvalues, so A is diagonalizable. If it is negative, we have no real eigenvalue so A is not.

2. $A = I_2$ satisfies $\Delta = 0$ and is diagonalizable (it is diagonal !).

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ satisfies $\Delta = 0$ but is not diagonalizable (it has only 1 as eigenvalue so it would be similar to I_2, but we know that only I_2 is similar to I_2. Actually what happens is that 1 is of geometrical multiplicity only 1 but algebraic multiplicity 2).

Ex 2

(a) Let us write $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ a 2×2 orthogonal matrix. We know that $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ are orthogonal and

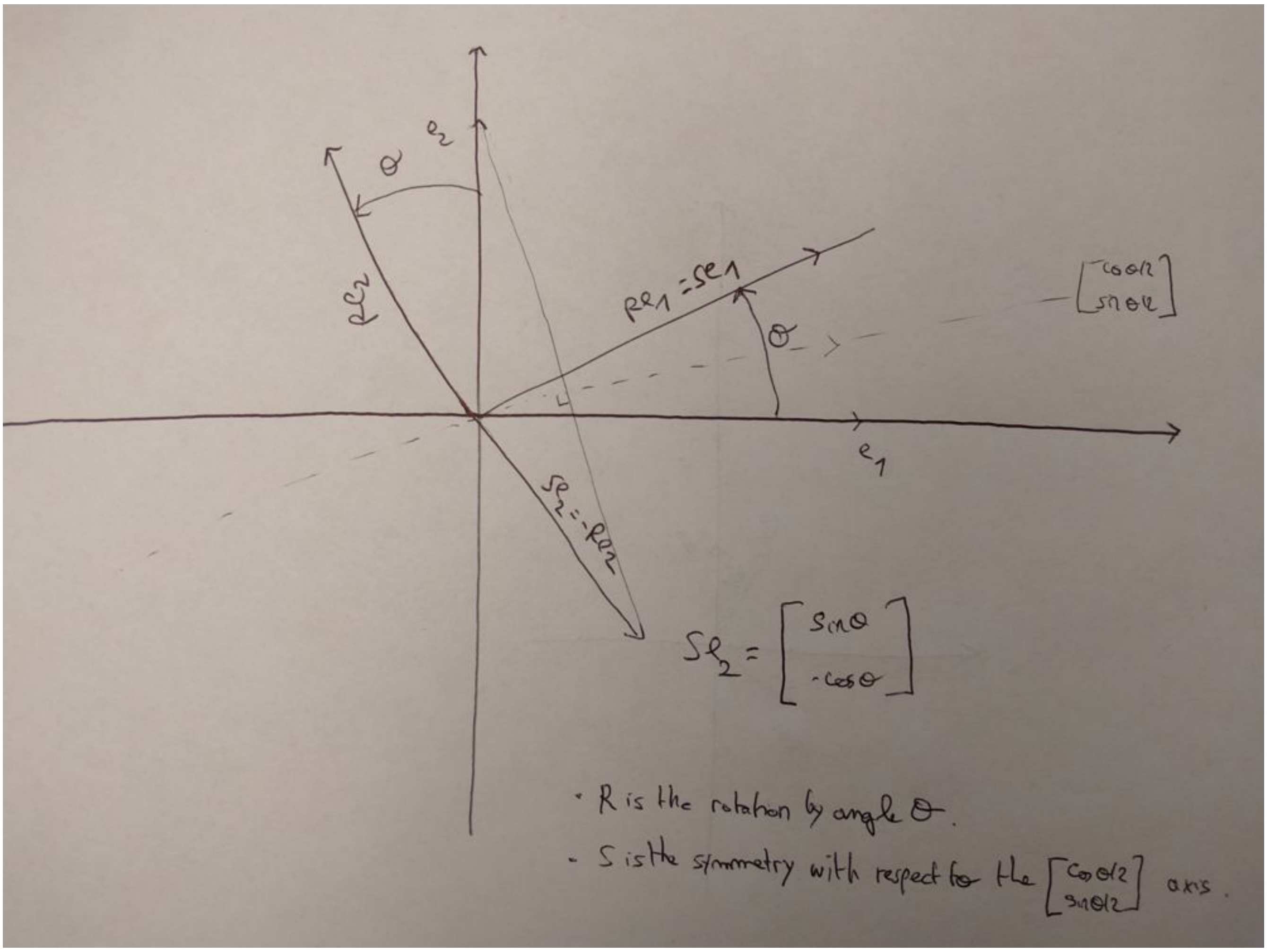
$$a^2 + b^2 = 1 \qquad (1)$$
$$c^2 + d^2 = 1 \qquad (2)$$

In $\mathbb{R}^2$, the direction orthogonal to $\begin{bmatrix} a \\ b \end{bmatrix}$ is spanned by $\begin{bmatrix} -b \\ a \end{bmatrix}$, so there exists λ such that $\begin{bmatrix} c \\ d \end{bmatrix} = \lambda \begin{bmatrix} -b \\ a \end{bmatrix}$. But by using (1)-(2) we see that $\lambda^2 = 1$ so $\lambda = \pm 1$.

(b) (1) says that $\begin{bmatrix} a \\ b \end{bmatrix}$ lies on the trigonometric circle, i.e. there exists $0 \le \theta < 2\pi$ such that $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. This precisely gives that $A = R$ or $A = S$.

3. $S \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$, so $\begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$ is an eigenvector associated to the eigenvalue 1 for S : S does not change $\begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$.

4.


e_2
θ
Re_2
$Re_1 = Se_1$
θ
$\begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix}$
e_1
$Se_2 = -Re_2$
$Se_2 = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$
• R is the rotation by angle θ.
• S is the symmetry with respect to the $\begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix}$ axis.