21-241 – Homework assignment week #11

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Reminder

Homework will be given on Fridays and due on the next Friday before 5pm, to me in class or in Andrew Zucker's mailbox in Wean Hall 6113. Late homework will never be accepted without a proper reason. In case of physical absence, electronic submissions by e-mail to both me and Andrew Zucker can be accepted. Please do not forget to write your name, andrew id and section and please use a staple if you have several sheets.

Reading

1. Poole: Sec. 5.1.

Exercises (19 pts)

Exercise 1: diagonalizability of 2×2 matrices in $\mathbb{R}$ (4 pts)

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

1. (2 pts) Prove that if $(a-d)^2 + 4bc > 0$ then A is diagonalizable **in** $\mathbb{R}$ and if $(a-d)^2 + 4bc < 0$ it is not.
2. (2 pts) Find two examples to show that in the case $(a-d)^2 + 4bc = 0$, A may or may not be diagonalizable.

Exercise 2: characterization of 2×2 orthogonal matrices (9 pts)

1. (3 pts) Prove that any 2×2 orthogonal matrix must have the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

   where $a^2 + b^2 = 1$.
2. (1 pt) Deduce that it can be written

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \text{or} \quad S = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

   for some $0 \leq \theta \leq 2\pi$.
3. (1 pt) Compute $S \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$.
4. (4 pts) Draw a picture in the plane with the result of the action of these two matrices on the standard basis, an illutration of the above computation, and give a geometrical interpretation of R and S.

Exercise 3: properties of orthogonal matrices (6 pts)

1. (3 pts) Show that the determinant of a matrix is the product of its eigenvalues (and keep this nice property in mind !).
2. (2 pts) Show that the eigenvalues of an orthogonal matrix can only be $+1$ or -1.
3. (1 pt) Deduce that the determinant of an orthogonal matrix is either $+1$ or -1.