

SOME APPLICATIONS OF ITERATED ULTRAPOWERS IN SET THEORY

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§0. Introduction

Let \mathcal{U} be a κ -complete ultrafilter on the measurable cardinal κ . Scott [13] proved $V \neq L$ by using \mathcal{U} to take the ultrapower of V . Gaifman [2] considered iterated ultrapowers of V by \mathcal{U} to conclude even stronger results; for example, that $L \cap \mathcal{P}(\omega)$ is countable. In this paper we discuss some new applications of iterated ultra-powers.

In §§1–4, we develop a straightforward generalization of Gaifman's method which is needed for some of the results in §§6–11. Namely, we consider iterated ultrapowers of a sub-model, M , of the universe by an ultrafilter which need not be in M . §5 discusses some known results within our present framework.

In §6, we investigate the universe constructed from a normal ultrafilter on the measurable cardinal κ , and show that in this universe the normal ultrafilter is unique. In §7, we obtain a characterization of arbitrary κ -complete free ultrafilters in this universe, and in §8, we show that this universe has some pathological model-theoretic properties.

§9 uses methods of §6 to discuss the problem of GCH at a measurable cardinal. We show that in the theory

ZFC + $\exists \kappa$ [κ measurable and $2^\kappa > \kappa^+$]

one can prove the consistency of

ZFC + $\exists \kappa$ [κ measurable] .

§ 10 shows that the assumption of the existence of a strongly compact cardinal is more powerful than had been realized. We use an idea of Vopěnka and Hrbáček [19] to prove from this assumption the existence of inner models with many measurable cardinals.

§ 11 uses methods developed in § 10 to show that if κ carries a κ^+ -saturated κ -complete non-trivial ideal, it is measurable in some inner model.

We shall use without comment standard set-theoretical notation and results. For less well-known items, we often refer the reader to the survey by Mathias [9].

Technically speaking, the development of this paper is done within Morse-Kelley set theory (see the appendix to Kelley [6]), since we often talk about arbitrary classes being models for ZFC. However, by the usual metamathematical circumlocutions, all of the results can be reformulated within ZFC. We shall comment further on this in the body of the paper in places where the reformulation is not immediately apparent.

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§ 1. *M*-ultrafilters

To avoid excessive repetition, in §§ 1–4 we use the following conventions: *M* will be a transitive model of ZFC (possibly a proper class), ρ an ordinal in *M*, and \mathcal{U} an *M*-ultrafilter on ρ , where

1.1. Definition. \mathcal{U} is an *M*-ultrafilter on ρ iff $\rho > \omega$ and

- (i) \mathcal{U} is a proper subset of $\mathcal{P}(\rho) \cap M$ containing no singletons;
- (ii) $\forall x, y [x \subseteq y \in \mathcal{P}(\rho) \cap M \wedge x \in \mathcal{U} \rightarrow y \in \mathcal{U}]$;
- (iii) $\forall x \in \mathcal{P}(\rho) \cap M [x \in \mathcal{U} \vee \rho - x \in \mathcal{U}]$;
- (iv) If $\eta < \rho$, the sequence $\langle x_\xi : \xi < \eta \rangle \in M$, and each $x_\xi \in \mathcal{U}$, then $\cap \{x_\xi : \xi < \eta\} \in \mathcal{U}$;
- (v) If the sequence $\langle x_\xi : \xi < \rho \rangle \in M$, then $\{\xi : x_\xi \in \mathcal{U}\} \in M$.

Note that we do not assume $\mathcal{U} \in M$. Standard arguments show that ρ must be weakly compact in *M*. Conditions (i)–(iv) alone imply that ρ is regular in *M*, but, as we shall see in § 10, they do not exclude ρ from being a successor cardinal in *M*.

In the case that $M = V$, ρ is a measurable cardinal and \mathcal{U} is a ρ -complete free ultrafilter on ρ . Scott [13] used \mathcal{U} to take an ultrapower of the universe; he showed that since \mathcal{U} is countably complete, the ultrapower is well-founded. Thus, one can set $N_0 = V$, N_1 = the transitive class isomorphic to V^ρ/\mathcal{U} , and i_{01} the usual elementary embedding from N_0 into N_1 . Now $i_{01}(\mathcal{U})$ is, in N_1 , an ultrafilter on $i_{01}(\rho)$, so, working within N_1 , we can repeat the process and define an ultrapower N_2 of N_1 and an elementary embedding $i_{12} : N_1 \rightarrow N_2$. Clearly, this may be iterated through any finite number of steps. Gaifman [2] shows now in fact this process can be continued through the transfinite. He thus obtained transitive classes, N_α , for all ordinals α , and elementary embeddings $i_{\alpha\beta} : N_\alpha \rightarrow N_\beta$ for $\alpha \leq \beta$, where for each α , $N_{\alpha+1}$ can be defined within N_α as the ultrapower of N_α by $i_{0\alpha}(\mathcal{U})$.

In §§ 2–4, we show that Gaifman's construction can be carried out for *M*, even when $\mathcal{U} \notin M$. N_0 will equal *M*. Roughly, N_1 will be

defined as the collection of equivalence classes of functions *in M* from ρ to *M*, and i_{01} will be as before. Condition (v) of Definition 1.1 enables us to define an ultrafilter, $\mathcal{U}^{(1)}$, on $\mathcal{P}(i_{01}(\rho)) \cap N_1$: Subsets in N_1 of $i_{01}(\rho)$ are equivalence classes of functions, f in *M* from ρ to $\mathcal{P}(\rho) \cap M$; put the equivalence class of f in $\mathcal{U}^{(1)}$ iff $\{\xi : f(\xi) \in \mathcal{U}\} \in \mathcal{U}$; note that (v) says that $\{\xi : f(\xi) \in \mathcal{U}\} \in M$. We could now take the ultrapower of N_1 by $\mathcal{U}^{(1)}$ to form N_2 , and so forth.

For technical reasons, it will be convenient to carry out as much of the construction as is possible within *M*. We thus take a slightly different tack. Elements of N_2 are usually determined by functions in N_1 from $i_{01}(\rho)$ to N_1 , and these are in turn determined by functions from ρ to M^ρ . But M^{ρ^ρ} can be identified with $M^{\rho \times \rho}$ so we can consider N_2 to be made up of equivalence classes of functions from $\rho \times \rho$ into *M*. In general, N_α will be made up of equivalence classes of functions from ρ^α into *M*. The formal development of this will be carried out in the next section, and related to the original idea by Theorem 2.11.

Many of the results of this chapter could be obtained for *M* an arbitrary (not well-founded) model of ZFC and \mathcal{U} any ultrafilter on $\mathcal{P}(\rho) \cap M$ satisfying (v) of Definition 1.1. Furthermore, following Gaifman, N_R could be defined for an arbitrary linear ordering *R*. However, the development here will suffice for the applications in §§5–11.

Our treatment of iterated ultrapowers is very similar to a method developed independently by Keisler to handle iterated ultrapowers in model theory (see Chang-Keisler [1]).

§ 2. Definition of the iterated ultrapower

2.1. Definition

(i) For each α ,

$$\text{Fn}_\alpha(\rho) = \{f \in \mathcal{V}^{\rho^\alpha} : \exists F \subseteq \alpha [F \text{ finite} \wedge \forall s, t \in \rho^\alpha [s \upharpoonright F = t \upharpoonright F \rightarrow f(s) = f(t)]]\};$$

(ii) For each α ,

$$\mathcal{P}_\alpha(\rho) = \{x \in \mathcal{P}(\rho^\alpha) : \exists F \subseteq \alpha [F \text{ finite} \wedge \\ \wedge \forall s, t \in \rho^\alpha [s \upharpoonright F = t \upharpoonright F \rightarrow [s \in x \leftrightarrow t \in x]]]\};$$

(iii) In (i) and (ii), F is called a support of f, x respectively;

(iv) If $x \in \mathcal{P}_{\alpha+\beta}(\rho)$ and $f \in \text{Fn}_{\alpha+\beta}(\rho)$, and $s \in \rho^\alpha$, define $x_{(s)} = \{t \in \rho^\beta : \widehat{s}t \in x\}$ and set $f_{(s)}(t) = f(\widehat{s}t)$ for $t \in \rho^\beta$. For finite β , abbreviate $x_{(\langle \xi_0, \dots, \xi_{\beta-1} \rangle)}$ by $x_{(\xi_0, \dots, \xi_{\beta-1})}$.

Here, $\widehat{s}t$ is the concatenation of the sequences s, t . Note that for n finite, $\mathcal{P}_n(\rho) = \mathcal{P}(\rho^n)$, and $\text{Fn}_n(\rho) = \text{V}^{\rho^n}$.

2.2. Definition. Let j be a 1–1 order preserving map from α into β .

- (i) j_β^* is the function from ρ^β to ρ^α defined by $(j_\beta^*(s))(\gamma) = s(j(\gamma))$;
- (ii) $j_{*\beta}$ will be used for the function either from $\text{Fn}_\alpha(\rho)$ to $\text{Fn}_\beta(\rho)$ or from $\mathcal{P}_\alpha(\rho)$ to $\mathcal{P}_\beta(\rho)$, where $(j_{*\beta}(f))(s) = f(j_\beta^*(s))$ or $j_{*\beta}(x) = \{s \in \rho^\beta : j_\beta^*(s) \in x\}$;
- (iii) $i_{\alpha\beta} = j_{*\beta}$, where j is the identity on α .

For the rest of §§2–4, j will always denote a 1–1 order preserving function on ordinals. Any $f \in \text{Fn}_\beta(\rho)(x \in \mathcal{P}_\beta(\rho))$, with support F , equals $j_{*\beta}(g)(j_{*\beta}(y))$ for a suitable $g \in \text{Fn}_n(\rho)(y \in \mathcal{P}_n(\rho))$ and $j : n \rightarrow \beta$, where $n = \overline{F}$. We use the subscript, β , since β cannot be determined from j , but this will be dropped if no confusion could result.

2.3. Definition

- (i) $\text{Fn}_\alpha(M, \rho)$ is the set of all $f \in \text{Fn}_\alpha(\rho)$ such that $f = j_{*\alpha}(g)$ for some $g \in \text{Fn}_n(\rho) \cap M$, where n is finite and $j : n \rightarrow \alpha$;
- (ii) $\mathcal{P}_\alpha(M, \rho)$ is the set of all $x \in \mathcal{P}_\alpha(\rho)$ such that $x = j_{*\alpha}(y)$ for some $y \in \mathcal{P}_n(\rho) \cap M$, where n is finite and $j : n \rightarrow \alpha$.

Note that for n finite, $\mathcal{P}_n(M, \rho) = \mathcal{P}(\rho^n) \cap M$, and $\text{Fn}_n(M, \rho) = \text{V}^{\rho^n} \cap M$.

If $f, g \in \text{Fn}_\alpha(M, \rho)$, then $\{s \in \rho^\alpha : f(s) = g(s)\}$ and $\{s \in \rho^\alpha : f(s) \in g(s)\}$ are in $\mathcal{P}_\alpha(M, \rho)$. As indicated in §1, $\text{Fn}_\alpha(M, \rho)$ will become

N_α upon dividing out by a suitable ultrafilter, to be defined below. Since $\rho^0 = \{0\}$, $\text{Fn}_0(M, \rho)$ can be identified with M .

Condition (v) of Definition 1.1 implies trivially that if the double sequence, $\langle x_{\xi\eta} : \xi, \eta < \rho \rangle \in M$, then $\{\langle \xi, \eta \rangle : x_{\xi\eta} \in \mathcal{U}\} \in M$. With this remark in mind,

2.4. Definition. We define inductively $\mathcal{U}_n \subseteq \mathcal{P}_n(M, \rho)$, and show inductively that if $\langle x_\eta : \eta < \rho \rangle$ is a sequence in M , then $\{\eta : x_\eta \in \mathcal{U}_n\} \in M$.

(i) $\mathcal{U}_0 = \{\{0\}\}$ (note $\mathcal{P}_0(M, \rho) = \{0, \{0\}\}$); $\mathcal{U}_1 = \mathcal{U}$;

(ii) Assuming the inductive hypothesis for \mathcal{U}_n , define $x \in \mathcal{U}_{n+1}$ iff $\{\xi : x_{(\xi)} \in \mathcal{U}_n\} \in \mathcal{U}$. Then if $\langle x_\eta : \eta < \rho \rangle \in M$, $\{\langle \eta, \xi \rangle : x_{\eta(\xi)} \in \mathcal{U}_n\} \in M$, so $\{\eta : x_\eta \in \mathcal{U}_{n+1}\} \in M$.

We can easily check that each \mathcal{U}_n satisfies conditions (i)–(v) of Definition 1.1 (substituting \mathcal{U}_n for \mathcal{U} and ρ^n for ρ). Also, by induction,

$$\{\langle \xi_0, \dots, \xi_{n-1} \rangle : \xi_0 < \dots < \xi_{n-1} < \rho\} \in \mathcal{U}_n .$$

The following lemma is proved by unraveling the inductive Definition 2.4. Thus, for example, $x \in \mathcal{U}_4$ iff

$$\{\xi_0 : \{\xi_1 : \{\xi_2 : x_{(\xi_0\xi_1\xi_2)} \in \mathcal{U}\} \in \mathcal{U}\} \in \mathcal{U}\} \in \mathcal{U} .$$

2.5. Lemma

(i) Let $j : m \rightarrow n$, $x \in \mathcal{P}_m(M, \rho)$. Then $x \in \mathcal{U}_m$ iff $j_{*n}(x) \in \mathcal{U}_n$;

(ii) Let $x \in \mathcal{P}_{m+n}(M, \rho)$. Then $x \in \mathcal{U}_{m+n}$ iff $\{s \in \rho^m : x_{(s)} \in \mathcal{U}_n\} \in \mathcal{U}_m$;

The definition of j_* could, of course, have been given for j not order preserving, but Lemma 2.5 (i) would not hold, since

$$\{\langle \xi_0, \xi_1 \rangle : \xi_0 < \xi_1 < \rho\} \in \mathcal{U}_2 ,$$

but

$$\{(\xi_0, \xi_1) : \xi_1 < \xi_0 < \rho\} \notin \mathcal{U}_2.$$

2.6. Definition. Define $\mathcal{U}_\alpha \subseteq \mathcal{P}_\alpha(M, \rho)$ as follows: If $x \in \mathcal{P}_\alpha(M, \rho)$ and $x = j_*(y)$ for some y in some $\mathcal{P}_n(M, \rho)$ and $j: n \rightarrow \alpha$, then $x \in \mathcal{U}_\alpha$ iff $y \in \mathcal{U}_n$.

Lemma 2.5 (i) shows this definition to be meaningful and establishes part (i) of the following; similarly (ii) follows from 2.5 (ii).

2.7. Lemma

- (i) Let $j: \alpha \rightarrow \beta$, $x \in \mathcal{P}_\alpha(M, \rho)$. Then $x \in \mathcal{U}_\alpha$ iff $j_*(x) \in \mathcal{U}_\beta$;
- (ii) Let $x \in \mathcal{P}_{\alpha+\beta}(M, \rho)$. Then $x \in \mathcal{U}_{\alpha+\beta}$ iff $\{s \in \rho^\alpha : x_{(s)} \in \mathcal{U}_\beta\} \in \mathcal{U}_\alpha$.

Now that we have \mathcal{U}_α we can use it to divide out $\text{Fn}_\alpha(M, \rho)$. Thus,

2.8. Definition

- (i) If $f, g \in \text{Fn}_\alpha(M, \rho)$,

$$f \approx_\alpha g \text{ iff } \{s \in \rho^\alpha : f(s) = g(s)\} \in \mathcal{U}_\alpha.$$

$$[f]_\alpha = \{g : g \approx_\alpha f \wedge \forall h [h \approx_\alpha f \rightarrow \text{rank}(h) \geq \text{rank}(g)]\}.$$

The subscript, α , will often be dropped;

- (ii) $\text{Ult}_\alpha(M, \mathcal{U})$ is the pair (N_α, E_α) , where

$$N_\alpha = \{[f]_\alpha : f \in \text{Fn}_\alpha(M, \rho)\},$$

and E_α is the relation on N_α defined by:

$$[f] E_\alpha [g] \text{ iff } \{s \in \rho^\alpha : f(s) \in g(s)\} \in \mathcal{U}_\alpha;$$

- (iii) *In the case that E_α is well-founded, we shall always identify N_α with the transitive class isomorphic to (N_α, E_α) ;*
- (iv) *If $j: \alpha \rightarrow \beta$, define $j_{*\beta}: N_\alpha \rightarrow N_\beta$ by $j_{*\beta}([f]_\alpha) = [j_{*\beta}(f)]_\beta$. $i_{\alpha\beta} = j_{*\beta}$, where j is the identity on α .*

Note that each $\{g: g \approx f\}$ is a proper class if M is, and in (i) we employed Scott's trick for handling a class of equivalence classes.

Even in the case that E_α is not well-founded, we shall often abuse notation and say " N_α " when we mean " (N_α, E_α) ", or " $\text{Ult}_\alpha(M, \mathcal{U})$ " when we mean " N_α ". In §§ 1–4, N_α will be understood to have been constructed from the ultrafilter \mathcal{U} and model M under discussion. By our conventions, N_0 is always M .

In later sections, we shall sometimes simultaneously consider more than one ultrafilter on ρ . In that case, we shall write $i_{\alpha\beta}^{\mathcal{U}}$ for the embedding defined using the ultrafilter \mathcal{U} .

By the usual arguments with ultrapowers, using the fact that M satisfies the axiom of choice, we have

2.9. Lemma

- (i) N_α satisfies ZFC;
- (ii) For each formula $\varphi(v_0, \dots, v_{n-1})$,

$$\begin{aligned} &\varphi^{(N_\alpha)}([f_0], \dots, [f_{n-1}]) \\ &\text{iff } \{s \in \rho^\alpha : \varphi^{(M)}(f_0(s), \dots, f_{n-1}(s))\} \in \mathcal{U}_\alpha ; \end{aligned}$$

- (iii) *Each j_* is an elementary embedding;*
- (iv) *If β is a limit ordinal, N_β is isomorphic to the direct limit of the systems N_α ($\alpha < \beta$) and the embeddings $i_{\alpha\gamma}$ ($\alpha < \gamma < \beta$).*

Gaifman [2] first defines N_n for $n \in \omega$, and then obtains N_α as a direct limit of these, using the directed system $\{F: F \subseteq \alpha \wedge \wedge F \text{ finite}\}$ and embeddings j_* for inclusions, j .

Finally, we connect our construction with the original idea (expressed in § 1) of iterating ultrapowers. $i_{0\alpha}(\rho)$ is an ordinal of N_α ,

and there is a natural way of defining an ultrafilter on $\mathcal{P}(i_{0\alpha}(\rho)) \cap N_\alpha$.

2.10. Definition

$$\mathcal{U}^{(\alpha)} = \{ [f]_\alpha \in \mathcal{P}^{(N_\alpha)}(i_{0\alpha}(\rho)) : \{s \in \rho^\alpha : f(s) \in \mathcal{U}\} \in \mathcal{U}_\alpha \}.$$

We can easily check that the definition is independent of the choice of f from $[f]_\alpha$, and, by Lemma 2.7 (ii),

$$[f]_\alpha \in \mathcal{U}^{(\alpha)} \text{ iff } \{s(\widehat{\xi}) : s \in \rho^\alpha \wedge \xi \in f(s) \cap \rho\} \in \mathcal{U}_{\alpha+1}.$$

2.11. Theorem. *Suppose N_α is well-founded.*

- (i) $\mathcal{U}^{(\alpha)}$ is an N_α -ultrafilter on $i_{0\alpha}(\rho)$;
- (ii) For any β , there is an isomorphism $e_{\alpha\beta}$ from $N_{\alpha+\beta}$ onto $\text{Ult}_\beta(N_\alpha, \mathcal{U}^{(\alpha)})$ such that $e_{\alpha\beta} \circ i_{\alpha, \alpha+\beta} = i_{0\beta}^{(\alpha)}$, where $i_{0\beta}^{(\alpha)}$ is the embedding: $N_\alpha \rightarrow \text{Ult}_\beta(N_\alpha, \mathcal{U}^{(\alpha)})$ defined from $\mathcal{U}^{(\alpha)}$.

Proof of (ii). Define $e_{\alpha\beta}$ as follows: Let $f \in \text{Fn}_{\alpha+\beta}(M, \rho)$ with support $\{\gamma_0, \dots, \gamma_{n-1}, \alpha + \delta_0, \dots, \alpha + \delta_{m-1}\}$, where $\gamma_0 < \dots < \gamma_{n-1} < \alpha$, and $\delta_0 < \dots < \delta_{m-1} < \beta$. $f = j_*(f')$, where $f' \in \text{Fn}_{\alpha+m}(M, \rho)$, $j: \alpha + m \rightarrow \alpha + \beta$, j is the identity on α , and $j(k) = \alpha + \delta_k$ for each $k < m$. Let $g \in \text{Fn}_\alpha(M, \rho)$, where for each $s \in \rho^\alpha$, $g(s) = f'(s) \in \text{Fn}_m(M, \rho)$. Then $[g]_\alpha = h' \in \text{Fn}_m(N_\alpha, i_{0\alpha}(\rho))$. Let $\tilde{j}: m \rightarrow \beta$ be such that $\tilde{j}(k) = \delta_k$ for each $k < m$. Let $h = \tilde{j}_*(h') \in \text{Fn}_\beta(N_\alpha, i_{0\alpha}(\rho))$. Then set $e_{\alpha\beta}([f]_{\alpha+\beta}) = [h]_\beta$.

Note that the isomorphism is with $\text{Ult}_\beta(N_\alpha, \mathcal{U}^{(\alpha)})$, not $\text{Ult}_{i_{0\alpha}(\beta)}(N_\alpha, \mathcal{U}^{(\alpha)})$. If we had defined the construction for non-well-founded models, then we would not have needed to assume that N_α is well-founded.

§3. Well-founded ultrapowers

We now prove some additional theorems for N_β well-founded. Of course, since $i_{\alpha\beta}$ is 1-1, if N_β is well-founded and $\alpha < \beta$ then N_α is also well-founded.

3.1. Theorem. *Suppose N_β is well-founded and $\alpha < \beta$.*

- (i) *If $\xi < i_{0\alpha}(\rho)$ then $i_{\alpha\beta}(\xi) = \xi$;*
- (ii) *$i_{\alpha\beta}(i_{0\alpha}(\rho)) = i_{0\beta}(\rho) > i_{0\alpha}(\rho)$;*
- (iii) *If β is a limit ordinal, $i_{0\beta}(\rho) = \sup \{i_{0\gamma}(\rho) : \gamma < \beta\}$;*
- (iv) *$\mathcal{P}(i_{0\alpha}(\rho)) \cap N_\alpha = \mathcal{P}(i_{0\alpha}(\rho)) \cap N_\beta$.*

Proof

(i) By Theorem 2.11, we can assume $\alpha = 0$. Now prove (i) by induction on β , using Theorem 2.11 for successor stages and Lemma 2.9 (iv) for limit stages.

(ii) Again we may take $\alpha = 0$, and, since $i_{0\beta}(\rho) \geq i_{01}(\rho)$, take $\beta = 1$. If id is the identity function on ρ then $\xi < [\text{id}]_1 < i_{01}(\rho)$ for each $\xi < \rho$, so $\rho \leq [\text{id}]_1 < i_{01}(\rho)$.

(iii) Suppose $\xi < i_{0\beta}(\rho)$. $\xi = i_{\gamma\beta}(\eta)$ for some $\gamma < \beta$ and $\eta < i_{0\gamma}(\rho)$. But then (i) implies $\xi = \eta$, so $\xi < i_{0\gamma}(\rho)$.

(iv) Again we may take $\alpha = 0$, $\beta = 1$. Now use condition (v) of Definition 1.1.

We remark that if N_1 is well-founded, then standard arguments show (using Theorem 3.1 (iv) for $\alpha = 0$, $\beta = 1$) that ρ is Π_n^1 indescribable in M for all n . However, it is also easy to check that if ρ is weakly compact in M and $\mathcal{P}(\rho) \cap M$ is countable, then there is an M -ultrafilter on ρ . Thus, N_1 need not in general be well-founded.

If $\alpha < \beta$ and N_β is well-founded, we can ask what function in $\text{Fn}_\beta(M, \rho)$ has equivalence class $i_{0\alpha}(\rho)$. Of course, this question is meaningless if N_β is not well-founded.

3.2. Lemma. *Suppose N_β is well-founded, and $\alpha < \beta$. Let*

$f \in \text{Fn}_1(M, \rho)$ be such that $[f]_1 = \rho$. Define $j: 1 \rightarrow \beta$ by $j(0) = \alpha$. Then $[j_{*\beta}(f)]_\beta = i_{0\alpha}(\rho)$.

Proof. By Theorem 3.1 (i) we may assume $\beta = \alpha + 1$.

If $[j_*(f)]_{\alpha+1} = [h]_\alpha < i_{0\alpha}(\rho)$, then, since $[i_{\alpha, \alpha+1}(h)]_{\alpha+1} = [h]_\alpha$, $\{s(\widehat{\xi}) : s \in \rho^\alpha \wedge f(\xi) = h(s)\} \in \mathcal{U}_{\alpha+1}$, so for some s , $\{\xi : f(\xi) = h(s)\} \in \mathcal{U}$, which is impossible.

But suppose $[j_*(f)]_{\alpha+1} > i_{0\alpha}(\rho) = [g]_{\alpha+1}$. Then

$$\{s \in \rho^\alpha : \{\xi : f(\xi) > g(s(\widehat{\xi}))\} \in \mathcal{U}\} \in \mathcal{U}_\alpha.$$

Since $[f]_1 = \rho$, there is a $h \in \text{Fn}_\alpha(M, \rho)$ with $\text{range}(h) \subseteq \rho$ such that

$$\{s \in \rho^\alpha : \{\xi : h(s) = g(s(\widehat{\xi}))\} \in \mathcal{U}\} \in \mathcal{U}_\alpha.$$

Then $i_{0\alpha}(\rho) > [h]_\alpha = [i_{\alpha, \alpha+1}(h)]_{\alpha+1} = [g]_{\alpha+1}$, a contradiction.

We can use this to get a result on indiscernibles.

3.3. Lemma. If $\varphi(v_0, \dots, v_n)$ is any formula of set theory, $\alpha \leq \gamma_0 < \gamma_1 < \dots < \gamma_{n-1} < \beta$, $\alpha \leq \delta_0 < \delta_1 < \dots < \delta_{n-1} < \beta$, $a \in N_\alpha$ and N_β is well-founded, then N_β satisfies

$$\varphi(i_{0\gamma_0}(\rho), \dots, i_{0\gamma_{n-1}}(\rho), i_{\alpha\beta}(a)) \leftrightarrow$$

$$\leftrightarrow \varphi(i_{0\delta_0}(\rho), \dots, i_{0\delta_{n-1}}(\rho), i_{\alpha\beta}(a)).$$

Proof. Define $j: \alpha + n \rightarrow \beta$ by $j(\xi) = \xi$ for $\xi < \alpha$ and $j(\alpha + k) = \gamma_k$ for $k < n$. $j_{*\beta}(i_{\alpha, \alpha+n}(a)) = i_{\alpha\beta}(a)$. By Lemma 3.2, $j_{*\beta}(i_{0, \alpha+k}(\rho)) = i_{0\gamma_k}(\rho)$. By Lemma 2.9 (iii), N_β satisfies $\varphi(i_{0\gamma_0}(\rho), \dots, i_{0\gamma_{n-1}}(\rho), i_{\alpha\beta}(a))$ iff $N_{\alpha+n}$ satisfies $\varphi(i_{0\alpha}(\rho), \dots, i_{0, \alpha+n-1}(\rho), i_{\alpha, \alpha+n}(a))$. Doing the same with the δ_k gives the lemma.

3.4. Theorem. If N_β is well-founded, $\beta < \omega_1^{(M)}$, and $\xi < \rho$, then M satisfies $[\rho \rightarrow (\beta)_r^{<\omega}]$.

Proof. Suppose $P = \langle P_n \rangle_{n < \omega} \in M$, where $P_n : [\rho]^n \rightarrow \xi$. By Lemma 3.3, $\{i_{0\gamma}(\cdot) : \gamma < \beta\}$ is a homogeneous set for $i_{0\beta}(P)$. Since $\beta < \omega_1^{(M)} = \omega_1^{(N_\beta)}$, an argument due to Silver [15] and Vaught shows that $i_{0\beta}(P)$ has a homogeneous set of order type β in N_β . $i_{0\beta}(\beta) = \beta$, so P has a homogeneous set of type β in M .

The next two theorems give some sufficient conditions for N_α to be well-founded for all α . The first is essentially due to Gaifman for the case $M = V$. The second was done independently by Keisler in a slightly different context (see Chang-Keisler [1]).

3.5. Theorem. *If N_α is well-founded for all $\alpha < \omega_1$, it is well-founded for all α .*

Proof. If N_α is not well-founded, let

$$\dots [f_n]_\alpha E_\alpha \dots E_\alpha [f_1]_\alpha E_\alpha [f_0]_\alpha,$$

where $f_n \in \text{Fn}_\alpha(M, \rho)$ with support F_n . Let $G = \cup_n F_n$. G is of some order type $\beta < \omega_1$. If j is the 1-1 order preserving map from β onto G , then there are $g_n \in \text{Fn}_\beta(M, \rho)$ such that $j_*(g_n) = f_n$. Then

$$\dots [g_n]_\beta E_\beta \dots E_\beta [g_1]_\beta E_\beta [g_0]_\beta,$$

so N_β is not well-founded.

3.6. Theorem. *If arbitrary countable intersections of elements of \mathcal{U} are non-empty, then N_α is well-founded for all α .*

Proof. Suppose

$$\dots [f_n]_\alpha E_\alpha \dots E_\alpha [f_1]_\alpha E_\alpha [f_0]_\alpha,$$

where $f_n \in \text{Fn}_\alpha(M, \rho)$.

Let $x_n = \{s \in \rho^\alpha : f_{n+1}(s) \in f_n(s)\}$. We shall derive a contradiction

by finding an $s \in \cap \{x_n : n < \omega\}$, since this would mean

$$\dots f_n(s) \in \dots \in f_1(s) \in f_0(s).$$

We define s inductively, assuming inductively that for each n , $x_{n(s \upharpoonright \gamma)} \in \mathcal{U}_{\alpha-\gamma}$. If $s \upharpoonright \gamma$ is defined and satisfies this, choose $s(\gamma)$ in $\cap_{n < \omega} \{\xi < \rho : x_{n((s \upharpoonright \gamma) \cup \{\xi\})} \in \mathcal{U}_{\alpha-(\gamma+1)}\}$. It is easy to verify that $s \in \cap_{n < \omega} x_n$.

Note that the assumption of Theorem 3.6 implies that $\text{cf}(\rho) > \omega$ and that arbitrary countable intersections of elements of \mathcal{U} are uncountable. By Theorem 3.1 (iii), $\text{cf}(i_{0\omega}(\rho)) = \omega$, so that Theorem 2.11 (i) shows the condition of Theorem 3.6 to be not necessary.

Finally, we give a bound on the size of $i_{0\beta}(\rho)$, and of $i_{0\beta}(\delta)$ for other δ .

3.7. Theorem. *If N_γ is well-founded and $\gamma \geq 1$, then*

$$i_{0\gamma}(\rho) < ((2^{\rho(M)})^\omega \cdot \bar{\gamma})^+.$$

Proof. This follows from the fact that the cardinal on the right is greater than the number of elements in $\text{Fn}_\gamma(M, \rho)$ with range ρ .

3.8. Corollary. *If β is any cardinal larger than $2^{\rho(M)}$ and N_β is well-founded, then $i_{0\beta}(\rho) = \beta$.*

Proof. Use Theorems 3.1 (iii) and 3.7.

Similarly,

3.9. Theorem. *Suppose $\gamma \geq 1$ and N_γ is well-founded. Then*

- (i) *For any δ , $i_{0\gamma}(\delta) < ((\delta^{\rho(M)})^\omega \cdot \bar{\gamma})^+$;*
- (ii) *If δ is a limit ordinal and $\text{cf}^{(M)}(\delta) \neq \rho$, then $i_{0\gamma}(\delta) = \sup \{i_{0\gamma}(\xi) : \xi < \delta\}$;*
- (iii) *If $\text{cf}^{(M)}(\delta) \neq \rho$, δ is a cardinal $> \gamma$, and for all $\xi < \delta$, $(\xi^{\rho(M)})^\omega < \delta$, then $i_{0\gamma}(\delta) = \delta$.*

§4. Normal ultrafilters

4.1. **Definition.** \mathcal{U} is normal iff whenever $\langle x_\xi : \xi < \rho \rangle \in M$, each $x_\xi \in \mathcal{U}$, and $x_\xi = \bigcap \{x_\eta : \eta < \xi\}$ for all limit ξ , then $\{\xi : \xi \in x_\xi\} \in \mathcal{U}$.

As in the usual theory of normal ultrafilters,

4.2. **Lemma.** \mathcal{U} is normal iff N_1 has a ρ th ordinal and this ordinal is $[\text{id}]_1$, where id is the identity function on ρ .

Also, we get normal ultrafilters from ordinary ones by

4.3. **Lemma.** Suppose N_1 has a ρ th ordinal, $[f]_1$, where $f: \rho \rightarrow \rho$. Define \mathcal{V} by $x \in \mathcal{V}$ iff $x \in \mathcal{P}(\rho) \cap M$ and $[f]E_1 i_{01}(x)$ (iff $x \in \mathcal{P}(\rho) \cap M$ and $f^{-1}(x) \in \mathcal{U}$). Then

- (i) \mathcal{V} is a normal M -ultrafilter on ρ ;
- (ii) If arbitrary countable intersections of elements of \mathcal{U} are non-empty, the same is true for \mathcal{V} ;
- (iii) If \mathcal{U} is normal, $\mathcal{V} = \mathcal{U}$;
- (iv) If $\text{Ult}_\alpha(M, \mathcal{U})$ is well-founded, so is $\text{Ult}_\alpha(M, \mathcal{V})$.

Proof. The proofs of (i)–(iii) are standard. For (iv), define an embedding, $e: \text{Fn}_\alpha(M, \rho) \rightarrow \text{Fn}_\alpha(M, \rho)$ by $(e(g))(s) = g(f \circ s)$. This defines an embedding: $\text{Ult}_\alpha(M, \mathcal{V}) \rightarrow \text{Ult}_\alpha(M, \mathcal{U})$, so the existence of a descending ϵ -chain in $\text{Ult}_\alpha(M, \mathcal{V})$ would imply the existence of one in $\text{Ult}_\alpha(M, \mathcal{U})$.

To go along with Theorem 2.11 we have

4.4. **Lemma.** Suppose \mathcal{U} is normal and N_α is well-founded. Then $\mathcal{U}^{(\alpha)}$ is normal.

Proof. Use Lemma 4.2, along with Lemma 3.2 and the proof of Theorem 2.11.

The next lemma will show that $\mathcal{U}^{(\omega)}$ can be defined from a single countable set, $\{i_{0n}(\rho) : n < \omega\}$. This idea will be very useful in later sections.

4.5. Lemma. *Suppose \mathcal{U} is normal, α a limit ordinal, and N_α is well-founded. Then for all $x \in \mathcal{P}(i_{0\alpha}(\rho)) \cap N_\alpha$,*

$$x \in \mathcal{U}^{(\alpha)} \text{ iff } \exists \beta < \alpha \{ \{i_{0\gamma}(\rho) : \beta \leq \gamma < \alpha\} \subseteq x \} .$$

Proof. It is clearly only necessary to prove the implication from left to right. Let $x \in \mathcal{U}^{(\alpha)}$. $x = i_{\beta\alpha}(y)$ for some $\beta < \alpha$ and $y \in \mathcal{U}^{(\beta)}$. Then for all γ such that $\beta \leq \gamma < \alpha$, $i_{\beta\gamma}(y) \in \mathcal{U}^{(\gamma)}$, so that $i_{0\gamma}(\rho) \in i_{\beta,\gamma+1}(y) \subseteq x$.

Ultrafilters give rise to elementary embeddings. Conversely, we can get ultrafilters from elementary embeddings. Thus,

4.6. Lemma. *Let N be a transitive model such that $\mathcal{P}(\rho) \cap N = \mathcal{P}(\rho) \cap M$, and suppose i is an elementary embedding from M into N such that $i(\rho) > \rho$ and i is the identity on ρ . Then $\{x \in \mathcal{P}^{(M)}(\rho) : \rho \in i(x)\}$ is a normal M -ultrafilter on ρ .*

§5. Measurable cardinals

A special case of the situation discussed in §§1–4 occurs when \mathcal{U} is an M -ultrafilter on ρ and \mathcal{U} is actually a member of M , i.e., ρ is a measurable cardinal in M . Now $\text{Ult}_\alpha(M, \mathcal{U})$ can, for $\alpha \in M$, be constructed completely within M , and is essentially the same as $O^\alpha M$ in Gaifman [2]. Theorem 3.6 (relativized to M) shows that $\text{Ult}_\alpha(M, \mathcal{U})$ is well-founded, for $\alpha \in M$, and hence, by Theorem 3.5, for all α when $\omega_1 \subseteq M$.

The only non-trivial part of the next lemma is due to Scott.

5.1. Lemma. *Let \mathcal{U} be an M -ultrafilter on ρ such that $\mathcal{U} \in M$,*

$\alpha < \beta \in M$. Then (using the notation of Definition 2.8)

- (i) $\mathcal{U}^{(\alpha)} = i_{0\alpha}(\mathcal{U})$ (see Definition 2.10);
- (ii) (Scott) $\mathcal{U}^{(\alpha)} \notin N_{\alpha+1}$;
- (iii) $N_\beta \subseteq N_\alpha$ and $N_\beta \neq N_\alpha$.

Proof. (i) follows immediately from the definition of $\mathcal{U}^{(\alpha)}$. By Theorem 2.11, we need only prove (ii) and (iii) for $\alpha = 0$. $N_\beta \subseteq N_0 = M$ since the definition of N_β is made completely within M . That $N_\beta \neq N_0$ follows from (ii).

Suppose $\mathcal{U} \in N_1$. By Theorem 3.1, $\mathcal{P}(\rho) \cap M = \mathcal{P}(\rho) \cap N_1$. Furthermore, there is a map from ρ^ρ onto $i_{01}(\rho)$ definable from $\mathcal{P}(\rho) \cap M$ and \mathcal{U} . Thus, N_1 satisfies $[i_{01}(\rho) < (2^\rho)^+]$. But $\rho < i_{01}(\rho)$ and $i_{01}(\rho)$ is inaccessible in N_1 , a contradiction.

When $M = V$, a V -ultrafilter on ρ is the same as a ρ -complete free ultrafilter on ρ , and a normal V -ultrafilter on ρ is the same as a normal ultrafilter on ρ in the usual sense.

Many of the results of this paper deal with the universe constructed from a normal ultrafilter on a measurable cardinal, and we shall define now our notation regarding this universe. These results usually have rather trivial generalizations to the universe constructed from a sequence of normal ultrafilters on a sequence of measurable cardinals (see e.g. [7]). We shall not bother with these generalizations here. However, for § 10 we shall need some of the basic notation for construction from such sequences, so we shall define our notation in suitable generality. To simplify notation, we shall often apply* terms and formulas to sequences coordinate-wise. Thus, if $\vec{\rho} = \langle \rho_\mu : \mu < \pi \rangle$, then $\mathcal{P}(\vec{\rho}) = \langle \mathcal{P}(\rho_\mu) : \mu < \pi \rangle$; $\vec{\mathcal{J}} \subseteq \mathcal{P}(\vec{\rho})$ means that $\vec{\mathcal{J}}$ is a sequence, $\langle \mathcal{J}_\mu : \mu < \pi \rangle$, and $\forall \mu < \pi [\mathcal{J}_\mu \subseteq \mathcal{P}(\rho_\mu)]$; $\vec{\mathcal{J}} \cap x = \langle \mathcal{J}_\mu \cap x : \mu < \pi \rangle$; etc.

The following definition is a specialization of a more general notion of construction discovered by Lévy and others:

5.2. **Definition.** Let $\vec{\mathcal{D}} \subseteq \mathcal{P}(\vec{\rho})$, $\vec{\rho}$ a sequence of length π .

- (i) $L_0[\vec{\mathcal{D}}] = 0$;
- (ii) $L_\alpha[\vec{\mathcal{D}}] = \cup \{L_\beta[\vec{\mathcal{D}}] : \beta < \alpha\}$ if α is a limit ordinal;
- (iii) $L_{\alpha+1}[\vec{\mathcal{D}}] = \{a \subseteq L_\alpha[\vec{\mathcal{D}}] : a \text{ is first order definable from elements of } L_\alpha[\vec{\mathcal{D}}] \text{ in the relational system,}$

$$\langle L_\alpha[\vec{\mathcal{D}}]; \epsilon, \{(\mu, \rho_\mu) : \mu < \pi \cap \alpha \wedge \rho_\mu < \pi \cap \alpha\},$$

$$\{(\mu, x) : \mu < \pi \cap \alpha \wedge x \in \delta_\mu \cap L_\alpha[\vec{\mathcal{D}}]\} \rangle;$$

- (iv) $L[\vec{\mathcal{D}}] = \cup \{L_\alpha[\vec{\mathcal{D}}] : \alpha \text{ is an ordinal}\}$.

$\vec{\mathcal{D}}$ need not be in $L[\vec{\mathcal{D}}]$. For example, if $\vec{\mathcal{D}} = \mathcal{P}(\vec{\rho})$ and $\vec{\rho} \in L$, then $L[\vec{\mathcal{D}}] = L$. However, we have

5.3. **Lemma.** If $\vec{\mathcal{D}} \subseteq \mathcal{P}(\vec{\rho})$,

- (i) $\vec{\mathcal{D}} \cap L[\vec{\mathcal{D}}] \in L[\vec{\mathcal{D}}]$, and $\vec{\rho} \in L[\vec{\mathcal{D}}]$;
- (ii) $L[\vec{\mathcal{D}}] = L[\vec{\mathcal{D}} \cap L[\vec{\mathcal{D}}]]$;
- (iii) $L[\vec{\mathcal{D}}]$ satisfies ZFC;
- (iv) $L[\vec{\mathcal{D}}]$ has a well-ordering definable in $L[\vec{\mathcal{D}}]$ from $\vec{\mathcal{D}} \cap L[\vec{\mathcal{D}}]$;
- (v) If M is a transitive model for ZF containing all the ordinals, and $\vec{\mathcal{D}} \cap M \in M$, then $L[\vec{\mathcal{D}}] \subseteq M$;
- (vi) If $\vec{\rho}$ are measurable and $\vec{\mathcal{D}}$ are $\vec{\rho}$ -complete free ultrafilters on $\vec{\rho}$, then, in $L[\vec{\mathcal{D}}]$, $\vec{\rho}$ are measurable and $\vec{\mathcal{D}} \cap L[\vec{\mathcal{D}}]$ are $\vec{\rho}$ -complete free ultrafilters on $\vec{\rho}$. If $\vec{\mathcal{D}}$ are normal, so are $\vec{\mathcal{D}} \cap L[\vec{\mathcal{D}}]$ in $L[\vec{\mathcal{D}}]$.

In (iv), the definable well-ordering is the analog of the usual well-ordering for L , and will be called "the order of construction from $\vec{\mathcal{D}}$ ".

The original intent of Definition 5.2 was that $\vec{\mathcal{D}}$ be ultrafilters on $\vec{\rho}$. However, it may turn out that $\vec{\mathcal{D}}$ are merely filters on $\vec{\rho}$, but that in $L[\vec{\mathcal{D}}]$, $\vec{\mathcal{D}} \cap L[\vec{\mathcal{D}}]$ are ultrafilters.

5.4. Definition. $\vec{\mathcal{F}}$ is a strong sequence of filters on $\vec{\rho}$ iff $\vec{\mathcal{F}}$ are filters on $\vec{\rho}$, and, in addition, in $L[\vec{\mathcal{F}}]$, $\vec{\rho}$ are measurable and $\vec{\mathcal{F}} \cap L[\vec{\mathcal{F}}]$ are normal ultrafilters on $\vec{\rho}$.

There are two natural candidates for being strong filters:

5.5. Definition

(i) If $\text{cf}(\rho) > \omega$, the closed unbounded filter on ρ is

$$\{x : x \subseteq \rho \wedge \exists y [y \subseteq x \wedge y \text{ is a closed, unbounded subset of } \rho]\};$$

(ii) If ρ is a limit cardinal, the cardinal filter on ρ is

$$\{x : x \subseteq \rho \wedge \exists \xi < \rho \forall \eta [\xi < \eta < \rho \wedge \eta \text{ a cardinal} \rightarrow \eta \in x]\}.$$

Note that if ρ is a limit cardinal and $\text{cf}(\rho) > \omega$, the closed unbounded filter on ρ is an extension of the cardinal filter.

We do not need the following theorem for future work, but cite it to show what is possible.

5.6. Theorem (Solovay). Suppose the class of measurable cardinals is of order type at least $\pi + 1$. Let $\vec{\rho}$ be an increasing sequence of cardinals of length π such that $\rho_0 > \pi$ and $\rho_\mu > \sup\{\rho_\nu : \nu < \mu\}$ for all $\mu < \pi$. Let $\vec{\mathcal{F}}$ be filters on $\vec{\rho}$ such that for each $\mu < \pi$, either
 (i) $\text{cf}(\rho_\mu) > \omega$ and \mathcal{F}_μ is the closed unbounded filter on ρ_μ , or
 (ii) ρ_μ is a limit cardinal and \mathcal{F}_μ is the cardinal filter on ρ_μ .
 Then $\vec{\mathcal{F}}$ is a strong sequence of filters on $\vec{\rho}$.

In Solovay's proof, the $\pi + 1$ measurable cardinals are used to construct $\pi + 1$ sets of indiscernibles for the universe constructed from normal ultrafilters on the first π measurable cardinals. An alternate proof can be given using iterated ultrapowers.

For the rest of this paper, except in §10, we shall restrict ourselves to construction from sequences of length 1. We shall write, e.g., $L[\mathcal{U}]$ for $L[\langle \mathcal{U} \rangle]$.

The universe constructed from a normal ultrafilter on a measurable cardinal is in many ways analogous to L . For example,

5.7. Theorem (Silver [16]). *Suppose ρ is measurable, \mathcal{U} a normal ultrafilter on ρ , and $V = L[\mathcal{U}]$. Then GCH holds. Furthermore, if $\alpha \geq \rho$ and $x \subseteq \alpha$, then $x \in L_\xi[\mathcal{U}]$ for some $\xi < \alpha^+$.*

Silver also shows that there is a Δ_3^1 well-ordering of the continuum in $L[\mathcal{U}]$. Silver used methods of Rowbottom to get his results, although alternate proofs can be constructed using iterated ultrapowers.

We shall need an analog of Theorem 5.6 for the case where we only know that there is 1 measurable cardinal of perhaps 1 measurable cardinal in some sub-model of the universe.

5.8. Theorem. *Suppose M is a transitive model for ZFC containing all the ordinals, $\mathcal{U} \in M$ is a normal M -ultrafilter on ρ , $M = L[\mathcal{U}]$, and σ is a cardinal greater than $\rho^{+(M)}$. Let \mathfrak{F} be either the closed unbounded filter on σ (assuming $\text{cf}(\sigma) > \omega$) or the cardinal filter on σ (assuming σ is a limit cardinal). Then*

(i) \mathfrak{F} is a strong filter on σ .

(ii) $L[\mathfrak{F}] = \text{Ult}_\sigma(M, \mathcal{U})$, $i_{0_\sigma}(\rho) = \sigma$, and $i_{0_\sigma}(\mathcal{U}) = \mathfrak{F} \cap L[\mathfrak{F}]$.

Proof. We start by proving (ii). That $i_{0_\sigma}(\rho) = \sigma$ follows from Corollary 3.8 and Theorem 5.7.

If $x \in i_{0_\sigma}(\mathcal{U})$, Lemmas 4.5 and 5.1 (i) imply that $x \in \mathfrak{F} \cap \text{Ult}_\sigma(M, \mathcal{U})$, so $i_{0_\sigma}(\mathcal{U}) = \mathfrak{F} \cap \text{Ult}_\sigma(M, \mathcal{U})$. Then $\text{Ult}_\sigma(M, \mathcal{U}) = L[i_{0_\sigma}(\mathcal{U})] = L[\mathfrak{F}]$. This also establishes (i).

This result is best possible in the sense that if $V = L[\mathcal{U}]$, it can be shown that (i) is false whenever $\sigma \leq \rho^+$.

5.9. Theorem. *Suppose M, \mathcal{U}, ρ and M, \mathcal{U}', ρ' both satisfy the hypothesis of Theorem 5.8 for M, \mathcal{U}, ρ . Then*

$$\mathcal{P}(\rho \cap \rho') \cap M = \mathcal{P}(\rho \cap \rho') \cap M'.$$

Proof. By Theorem 5.8 (ii), these are both equal to $\mathcal{P}(\rho \cap \rho') \cap \mathcal{L}[\mathfrak{F}]$ for a suitable \mathfrak{F} .

We shall now show that ρ is the only measurable cardinal in M . The proof is essentially the same as Scott's proof [13] that there are no measurable cardinals in L . An earlier proof was given by Solovay using methods of Theorem 5.6.

5.10. Lemma. *Let σ and ρ be measurable, $\sigma \neq \rho, \mathcal{V}$ a σ -complete free ultrafilter on σ, \mathcal{U} a normal ultrafilter on ρ . Let $N_1 = \text{Ult}_1(V, \mathcal{V})$, and $i_{01} : V \rightarrow N_1$ as in Definition 2.8. Then $i_{01}(\rho) = \rho$ and $i_{01}(\mathcal{U}) = \mathcal{U} \cap N_1$.*

Proof. The theorem is trivial if $\sigma > \rho$, so assume $\sigma < \rho$.

That $i_{01}(\rho) = \rho$ follows from Theorem 3.9 (iii).

Let $a = \{ \delta : \sigma < \delta < \rho \wedge \delta \text{ is inaccessible} \}$. 3.9 (iii) also implies that $i_{01}(\delta) = \delta$ for each $\delta \in a$. Furthermore, $a \in \mathcal{U}$ since \mathcal{U} is normal.

Now suppose $x \in i_{01}(\mathcal{U})$. Let $x = [f]$, where $f \in \text{Fn}_1(\sigma)$ and $\text{range}(f) \subseteq \mathcal{U}$. Let $b = \bigcap \{ f(\xi) : \xi < \sigma \}$. $b \in \mathcal{U}$ and $i_{01}(b) \subseteq x$. Also, $b \cap a \in \mathcal{U}$ and $b \cap a \subseteq i_{01}(b)$. Thus $x \in \mathcal{U}$.

5.11. Theorem. *Suppose M, \mathcal{U}, ρ satisfy the hypothesis of Theorem 5.8. Then ρ is the only measurable cardinal in M .*

Proof. Suppose $\mathcal{V} \in M$ is an M -ultrafilter on σ , where $\sigma \neq \rho$. Let $N_1 = \text{Ult}_1(M, \mathcal{V})$.

By Lemma 5.10, $i_{01}(\mathcal{U}) = \mathcal{U} \cap N_1$, so $N_1 = M$, contradicting Lemma 5.1 (ii).

§6. Structure of $L[\mathcal{U}]$

It is well-known that L is the unique transitive model for $ZFC + V = L$ containing all the ordinals. In this section we develop analogous theorems for the universe constructed from a normal ultrafilter on a measurable cardinal.

6.1. Definition. *ZFML is the theory $ZFC + \exists \kappa, \mathcal{U} [\kappa \text{ is measurable} \wedge \mathcal{U} \text{ is a normal ultrafilter on } \kappa \wedge V = L[\mathcal{U}]]$.*

6.2. Definition. *For any ordinal ρ , a ρ -model is a transitive model, M , for ZFML, such that M contains all the ordinals and ρ is the measurable cardinal in M . A constructing ultrafilter for M is a $\mathcal{U} \in M$ such that $M = L[\mathcal{U}]$ and $M \models [\mathcal{U} \text{ is a normal ultrafilter on } \rho]$.*

Note that by Theorem 5.11, ρ is the only measurable cardinal in M . We shall show (Corollary 6.5) that in M , ρ has exactly one normal ultrafilter, but we have not yet ruled out the possibility that there are more than one constructing ultrafilter, or that there is a $\mathcal{V} \in M$ such that

$$M \models [\mathcal{V} \text{ is a normal ultrafilter on } \rho] ,$$

but $L[\mathcal{V}]$ is a proper subset of M .

We shall eventually obtain a complete description of all ρ -models, assuming any exist. We remark here that the discussion can be formulated entirely within ZFC, even though we are talking about arbitrary class models for ZFML. This formulation would talk about sets, \mathcal{U} , such that $\mathcal{U} \cap L[\mathcal{U}]$ is, in $L[\mathcal{U}]$, a normal ultrafilter. Note that $L[\mathcal{U}]$ always satisfies ZFC.

Our main tool is the following lemma.

6.3. Lemma. *Let M be any ρ -model, \mathcal{U} a constructing ultrafilter for*

M. Let γ_μ ($\mu < \delta$) be an increasing sequence of ordinals such that $\gamma_0 > \rho$ and $\delta \geq \rho^{+(M)}$. Let θ be any cardinal greater than all the γ_μ . Then every element of $\mathcal{P}(\rho) \cap M$ is definable in $\langle L_\theta[\mathcal{U}]; \epsilon, \mathcal{U} \rangle$ from a finite number of ordinals in $\{\gamma_\mu : \mu < \delta\} \cup (\rho + 1)$.

Proof. Let A be the set of elements so definable. Then $\{\gamma_\mu : \mu < \delta\} \cup (\rho + 1) \subseteq A$. Furthermore, $L_\theta[\mathcal{U}]$ has a well-ordering definable from \mathcal{U} , so A is an elementary subsystem of $L_\theta[\mathcal{U}]$. Hence, A is isomorphic to some $L_\beta[\mathcal{U}]$, where $\beta \geq \delta \geq \rho^{+(M)}$. By Silver's Theorem 5.7, $L_\beta[\mathcal{U}]$, and hence A , contains all elements of $\mathcal{P}(\rho) \cap M$.

6.4. Theorem. Let M and N both be ρ -models, \mathcal{U} a constructing ultrafilter for M , \mathcal{V} a constructing ultrafilter for N . Then $\mathcal{U} = \mathcal{V}$, and hence $M = N$.

Proof. Let λ be a regular cardinal $> \rho^+$, \mathcal{F} the closed unbounded filter on λ . By Theorem 5.8, $L[\mathcal{F}] = \text{Ult}_\lambda(M, \mathcal{U}) = \text{Ult}_\lambda(N, \mathcal{V})$, and $i_{0\lambda}^{\mathcal{U}}(\mathcal{U}) = i_{0\lambda}^{\mathcal{V}}(\mathcal{V}) = \mathcal{F} \cap L[\mathcal{F}]$.

Let γ_μ ($\mu < \delta$) be an increasing sequence of ordinals such that $\gamma_0 > \lambda$ and $\delta \geq \rho^+$, and let θ be a cardinal greater than all the γ_μ . Furthermore, assume the γ_μ and θ are chosen so as to be fixed by the embeddings $i_{0\lambda}^{\mathcal{U}}$ and $i_{0\lambda}^{\mathcal{V}}$; this is possible by Theorem 3.9 (iii).

We shall show $\mathcal{U} \subseteq \mathcal{V}$. The reverse inclusion is proved in exactly the same manner.

Suppose $x \in \mathcal{U}$. By Lemma 6.3, there is a formula φ (with symbols for $=$, ϵ , and \mathcal{U}), and ordinals $\eta_1, \dots, \eta_m < \rho$ and $\mu_1, \dots, \mu_n < \delta$, such that

$$x = \{ \xi < \rho : \langle L_\theta[\mathcal{U}]; \epsilon, \mathcal{U} \rangle \models \varphi(\xi, \eta_1, \dots, \eta_m, \rho, \gamma_{\mu_1}, \dots, \gamma_{\mu_n}) \} .$$

Let

$$y = \{ \xi < \rho : \langle L_\theta[\mathcal{V}]; \epsilon, \mathcal{V} \rangle \models \varphi(\xi, \eta_1, \dots, \eta_m, \rho, \gamma_{\mu_1}, \dots, \gamma_{\mu_n}) \} .$$

Now $x \in \mathcal{U}$ iff $i_{0\lambda}^{\mathcal{U}}(x) \in \mathcal{F}$, and $y \in \mathcal{V}$ iff $i_{0\lambda}^{\mathcal{V}}(y) \in \mathcal{F}$. But $i_{0\lambda}^{\mathcal{U}}(x) = i_{0\lambda}^{\mathcal{V}}(y)$, since they both equal

$$\{\xi < \lambda : \langle L_{\theta}[\mathcal{F}] ; \epsilon, \mathcal{F} \cap L_{\theta}[\mathcal{F}] \rangle \models \varphi(\xi, \eta_1, \dots, \eta_m, \lambda, \gamma_{\mu_1}, \dots, \gamma_{\mu_n})\} .$$

Hence $y \in \mathcal{V}$. Also

$$x = i_{0\lambda}^{\mathcal{U}}(x) \cap \rho = i_{0\lambda}^{\mathcal{V}}(y) \cap \rho = y ,$$

so $x \in \mathcal{V}$.

6.5. Corollary. *If $V = L[\mathcal{U}]$, where \mathcal{U} is a normal ultrafilter on κ , then \mathcal{U} is the only normal ultrafilter on κ .*

Proof. Let \mathcal{V} be any normal ultrafilter on κ . Then $L[\mathcal{V}]$ is a κ -model with constructing ultrafilter $\mathcal{V} \cap L[\mathcal{V}]$. By Theorem 6.4, $\mathcal{V} \cap L[\mathcal{V}] = \mathcal{U}$, so $\mathcal{V} = \mathcal{U}$.

This corollary shows that it is consistent that a measurable cardinal have a unique normal ultrafilter. It is also consistent that a measurable cardinal have more than one normal ultrafilter. For example, Solovay has shown that if κ is super-compact, κ has at least $(2^\kappa)^+$ distinct normal ultrafilters (see [18]). Also, Jeffrey Paris [10] and the author [7] have shown by a Cohen-style independence proof that if $ZFC + \exists \kappa [\kappa \text{ measurable}]$ is consistent, so is

$$ZFC + \exists \kappa [\kappa \text{ measurable} \wedge \kappa \text{ has } 2^{2^\kappa} \text{ normal ultrafilters}] .$$

We now proceed to get a better description of all ρ -models for varying ρ .

6.6. Lemma. *If M is a ρ -model, and \mathcal{U} is the normal ultrafilter on ρ in M , then there are no σ -models for any σ such that $\rho < \sigma < i_{01}^{\mathcal{U}}(\rho)$.*

Proof. Suppose there is such a σ -model, N , with \mathcal{V} the normal ultrafilter on σ in N . Let λ be a regular cardinal $> \sigma^+$. Let \mathcal{F} , γ_μ ($\mu < \delta$), and θ be as in the proof of Theorem 6.4.

Since $\sigma < i_{01}^{\mathcal{U}}(\rho)$, there is an $f \in \rho^\rho \cap M$ such that $[f] = \sigma$ in $\text{Ult}_1(M, \mathcal{U})$. Then $(i_{0\lambda}^{\mathcal{U}}(f))(\rho) = \sigma$. By Lemma 6.3, f is definable in $\langle L_\theta[\mathcal{U}]; \epsilon, \mathcal{U} \rangle$ from elements of $\{\gamma_\mu : \mu < \delta\} \cup (\rho + 1)$. Thus $i_{0\lambda}^{\mathcal{U}}(f)$, and hence also σ , is definable in $\langle L_\theta[\mathcal{F}]; \epsilon, \mathcal{F} \cap L_\theta[\mathcal{F}] \rangle$ from elements of $\{\gamma_\mu : \mu < \delta\} \cup (\rho + 1) \cup \{\lambda\}$. Now define $j : \lambda \rightarrow \lambda$ by $j(\alpha) = \alpha + 1$. $j_*^{\mathcal{V}} : L[\mathcal{F}] \rightarrow L[\mathcal{F}]$ and fixes θ and every ordinal in $\{\gamma_\mu : \mu < \delta\} \cup (\rho + 1) \cup \{\lambda\}$, so it also fixes σ . But this contradicts Lemma 3.2, which implies $j_*^{\mathcal{V}}(\sigma) = i_{01}^{\mathcal{V}}(\sigma) > \sigma$.

6.7. Theorem. *If M is a ρ -model, \mathcal{U} in M the normal ultrafilter on ρ , and N is a σ -model with $\sigma > \rho$, then, for some α , $N = \text{Ult}_\alpha(M, \mathcal{U})$.*

Proof. If for some α , $i_{0\alpha}^{\mathcal{U}}(\rho) = \sigma$, then $N = \text{Ult}_\alpha(M, \mathcal{U})$ by Theorem 6.4.

If not, then by Theorem 3.1 (iii), there is an α such that $i_{0\alpha}^{\mathcal{U}}(\rho) < \sigma < i_{0, \alpha+1}^{\mathcal{U}}(\rho)$. But this contradicts Lemma 6.6 (by Theorem 2.11).

6.8. Corollary. *If ρ is the least ordinal for which there is a ρ -model, M , and \mathcal{U} is the normal ultrafilter on ρ in M , then all transitive models for ZFML containing all the ordinals are of the form $\text{Ult}_\alpha(M, \mathcal{U})$ for some α .*

The above methods give the following rather technical result which will be useful in §11.

6.9. Theorem. *Let M be a ρ -model. Suppose that for some ordinal $\sigma < \rho$ there is a normal M -ultrafilter, \mathcal{W} , on σ , with the property that arbitrary countable intersections of elements of \mathcal{W} are non-empty. Then there is a σ -model, N , such that $\mathcal{W} \in N$ and \mathcal{W} is the normal ultrafilter on σ in N .*

proof. Let \mathcal{U} be the normal ultrafilter on ρ . By iterating ultrapowers by \mathcal{U} , we can assume that ρ is a regular cardinal such that $\forall \lambda < \rho, (\lambda^\sigma)^\rho < \rho$. Furthermore, we can assume that $\mathcal{U} = M \cap \mathcal{F}$, where \mathcal{F} is the closed unbounded filter on ρ .

Throughout this proof, $i_{0\beta}$ will be $i_{0\beta}^{\mathcal{W}} : M \rightarrow \text{Ult}_\beta(M, \mathcal{W})$. These ultrapowers are well-founded by Theorem 3.6. Note that $i_{0\rho}(c) = \rho$, and $i_{0\beta}(\sigma) < \rho$ for $\beta < \rho$. Also, $\text{Ult}_\rho(M, \mathcal{W}) = M$ when $\beta < \rho$.

Let γ_ξ ($\xi \in \text{ORD}$) be a strictly increasing sequence of ordinals $> \rho$ such that each γ_ξ is fixed by $i_{0\rho}$. Let $S \subseteq M$ be the Skolem hull in M of $\{\gamma_\xi : \xi \in \text{ORD}\} \cup \sigma \cup \{\rho\}$. Then every element of S is fixed by all $i_{0\beta}$ for $\beta < \rho$, so S contains no ordinals, α , such that $\rho \leq \alpha < \rho$. Hence, the transitive model, N , isomorphic to S , is a κ -model.

Now \mathcal{W} is also an N -ultrafilter on σ by Theorem 5.9. Let \mathcal{V} be the normal ultrafilter on σ in N . Let j be the embedding $i_{0\rho}^{\mathcal{W}} : N \rightarrow \text{Ult}_\rho(N, \mathcal{W})$. Note that $\text{Ult}_\rho(N, \mathcal{W}) = M$ by Theorem 6.4. If $c \in \mathcal{W}$, then $j(c) \in \mathcal{F}$, so $j(c) \in \mathcal{U}$, and hence $c \in \mathcal{V}$. Thus $\mathcal{W} = \mathcal{V}$.

In order to formalize the above proof in ZFC, one would take the Skolem hull in a suitable bounded segment of M .

§ 7. Non-normal ultrafilters

We have been talking so far about construction from normal ultrafilters. In this section we explore non-normal ones. Our first result is:

7.1. Theorem. *Suppose κ is a measurable cardinal, \mathcal{U} a normal ultrafilter on κ , and \mathcal{V} an arbitrary κ -complete free ultrafilter on κ . Then $L[\mathcal{U}] = L[\mathcal{V}]$.*

Proof. Since κ is still measurable in $L[\mathcal{V}]$, there is a κ -model which is a sub-class of $L[\mathcal{V}]$. Thus, by Theorem 6.4, $L[\mathcal{U}] \subseteq L[\mathcal{V}]$, so we need only show that $\mathcal{V} \cap L[\mathcal{U}] \in L[\mathcal{U}]$.

Using \mathcal{V} , we obtain the elementary embedding $i_{01}^{\mathcal{V}} : V \rightarrow \text{Ult}_1(V, \mathcal{V})$. Let $\sigma = i_{01}^{\mathcal{V}}(\kappa)$. Let j be the restriction of $i_{01}^{\mathcal{V}}$ to $L[\mathcal{U}]$. By elementarity of $i_{01}^{\mathcal{V}}$ and Theorem 6.4, j is an elementary embedding from $L[\mathcal{U}]$ into *the* σ -model. Call this model N .

By Theorem 6.7, $N = \text{Ult}_\alpha(L[\mathcal{U}], \mathcal{U} \cap L[\mathcal{U}])$ for some ordinal α . Thus, if $k = i_{0\alpha}^{\mathcal{U} \cap L[\mathcal{U}]}$, k is also an elementary embedding: $L[\mathcal{U}] \rightarrow N$.

If id is the identity function: $\kappa \rightarrow \kappa$, let $\zeta = [\text{id}]$ in $\text{Ult}_1(V, \mathcal{V})$. Then for any $x \subseteq \kappa$, $x \in \mathcal{V}$ iff $\zeta \in i_{01}^{\mathcal{V}}(x)$. In particular,

$$\mathcal{V} \cap L[\mathcal{U}] = \{x \in \mathcal{P}(\kappa) \cap L[\mathcal{U}] : \zeta \in j(x)\} .$$

Since

$$\{x \in \mathcal{P}(\kappa) \cap L[\mathcal{U}] : \zeta \in k(x)\} \in L[\mathcal{U}] ,$$

we need now only show that j and k agree on $\mathcal{P}(\kappa) \cap L[\mathcal{U}]$.

The proof of this last fact is similar to that of Theorem 6.4. Fix γ_μ ($\mu < \delta$) an increasing sequence of ordinals such that $\gamma_0 > \sigma$ and $\delta \geq \kappa^+$, and let θ be a cardinal greater than all the γ_μ . Furthermore, assume that the γ_μ and θ are fixed by j and k . If $x \in \mathcal{P}(\kappa) \cap L[\mathcal{U}]$, we can write

$$\begin{aligned} x &= \{ \xi < \kappa : \langle L_\theta[\mathcal{U}] ; \epsilon, \mathcal{U} \cap L_\theta[\mathcal{U}] \rangle \\ &\models \varphi(\xi, \eta_1, \dots, \eta_m, \kappa, \gamma_{\mu_1}, \dots, \gamma_{\mu_n}) \} \end{aligned}$$

for suitable φ , $\eta_1, \dots, \eta_m < \kappa$, and $\mu_1, \dots, \mu_n < \delta$. Then, if \mathcal{W} is the normal ultrafilter on σ in N , $j(x)$ and $k(x)$ must both equal

$$\begin{aligned} \{ \xi < \sigma : \langle L_\theta[\mathcal{W}] ; \epsilon, \mathcal{W} \rangle \\ \models \varphi(\xi, \eta_1, \dots, \eta_m, \sigma, \gamma_{\mu_1}, \dots, \gamma_{\mu_n}) \} , \end{aligned}$$

proving the theorem.

We now consider the question of how many κ -complete free ultrafilters there are on κ . The following lemma gives some bounds.

7.2. Lemma. *If κ is measurable, there are at least 2^κ and no more than 2^{2^κ} κ -complete free ultrafilters on κ .*

Proof. Each ultrafilter is a subset of $\mathcal{P}(\kappa)$, and there are no more than 2^{2^κ} of these. Now let $\{x_\xi : \xi < 2^\kappa\}$ be a family of almost disjoint subsets of κ . For each ξ , there is a κ -complete free ultrafilter, \mathcal{U}_ξ , such that $x_\xi \in \mathcal{U}_\xi$, and these \mathcal{U}_ξ must be distinct.

The upper bound is possible. For example, as we mentioned in §6, it is consistent that there be even 2^{2^κ} normal ultrafilters on κ .

Another example is when κ is strongly compact. Since κ is inaccessible, an immediate generalization of a theorem of Hausdorff [3] shows that there are subsets A_β of κ for $\beta < 2^\kappa$ such that whenever x, y are disjoint subsets of 2^κ of cardinality $< \kappa$,

$$(\cap \{A_\beta : \beta \in x\}) \cap (\cap \{\kappa - A_\gamma : \gamma \in y\}) \neq 0.$$

Hence, as pointed out by W. Rudin [12] for $\kappa = \omega$, strong compactness of κ implies that for each $X \subseteq 2^\kappa$, there is a κ -complete ultrafilter \mathcal{U}_X on κ such that $A_\beta \in \mathcal{U}_X$ iff $\beta \in X$. Thus, there are 2^{2^κ} κ -complete ultrafilters on κ .

In contrast to the above, we have:

7.3. Theorem. *If $V = L[\mathcal{U}]$, \mathcal{U} a normal ultrafilter on κ , then*

- (i) *There are exactly κ^+ κ -complete free ultrafilters on κ ;*
- (ii) *Every κ -complete free ultrafilter, \mathcal{V} , on κ , is of the form $\{x \subseteq \kappa : \xi \in i_{0\omega}^{\mathcal{U}}(x)\}$ for some $\xi < i_{0\omega}^{\mathcal{U}}(\kappa)$.*

Proof. (i) follows from (ii) by Lemma 7.2.

For (ii), we see, as in the proof of Theorem 7.1, that for some α , $i_{01}^{\mathcal{V}}(\kappa) = i_{0\alpha}^{\mathcal{U}}(\kappa)$, that $i_{01}^{\mathcal{V}}$ and $i_{0\alpha}^{\mathcal{U}}$ agree on $\mathcal{P}(\kappa)$, and that hence for

some $\zeta < i_{0\alpha}^{\mathcal{U}}(\kappa)$, $\mathcal{V} = \{x \subseteq \kappa : \zeta \in i_{0\alpha}^{\mathcal{U}}(x)\}$. Since then also $\mathcal{V} = \{x \subseteq \kappa : \zeta \in i_{0\beta}^{\mathcal{U}}(x)\}$ for any $\beta \geq \alpha$, we are done if we show that α must be finite.

Suppose not. Then $\alpha \geq \omega$. Then $i_{0\omega}^{\mathcal{U}}(\kappa)$ must be inaccessible in $\text{Ult}_{\alpha}(V, \mathcal{U})$, since it is in $\text{Ult}_{\omega}(V, \mathcal{U})$ and both these models have the same subsets of $i_{0\omega}^{\mathcal{U}}(\kappa)$. But $\text{Ult}_{\alpha}(V, \mathcal{U}) = \text{Ult}_1(V, \mathcal{V})$, and hence contains all countable sets of ordinals, so $i_{0\omega}^{\mathcal{U}}(\kappa)$ is cofinal with ω there, a contradiction.

Theorem 7.3. was noticed independently by Jeffrey Paris.

Another description of the κ -complete free ultrafilters on κ arises from considering equivalence classes under permutations. It is convenient to consider base sets other than κ .

7.4. Definition. *If $\mathcal{U} \subseteq \mathcal{P}(I)$ and f is a function from I into J , let $f_*(\mathcal{U}) = \{y \subseteq J : f^{-1}(y) \in \mathcal{U}\}$. If $\mathcal{V} \subseteq \mathcal{P}(J)$, \mathcal{V} and \mathcal{U} are equivalent iff there is a 1-1 function, f , from I onto J such that $\mathcal{V} = f_*(\mathcal{U})$.*

If \mathcal{U} is a κ -complete ultrafilter on some set I of cardinality κ , we shall use the same notation, $\text{Ult}_{\alpha}(V, \mathcal{U})$, $i_{0\alpha}^{\mathcal{U}}$, etc., as for ultrafilters on κ . It is clear that all the basic theorems are essentially the same as for ultrafilters on κ .

7.5. Lemma. *Let κ be measurable, $\bar{I} = \bar{J} = \kappa$, \mathcal{U} a κ -complete free ultrafilter on I , \mathcal{V} a κ -complete free ultrafilter on J . Then \mathcal{U} and \mathcal{V} are equivalent iff $i_{01}^{\mathcal{U}}(\kappa) = i_{01}^{\mathcal{V}}(\kappa)$ and $i_{01}^{\mathcal{U}}$ and $i_{01}^{\mathcal{V}}$ agree on $\mathcal{P}(\kappa)$.*

Proof. The implication from left to right is obvious, so we prove the implication from right to left. We may assume for convenience that $I = J = \kappa$. In general, if $f \in V^{\kappa}$, let $[f]_{\mathcal{U}}$, $[f]_{\mathcal{V}}$ be the equivalence class of f in $\text{Ult}_1(V, \mathcal{U})$, $\text{Ult}_1(V, \mathcal{V})$ respectively.

Let id be the identity: $\kappa \rightarrow \kappa$. Fix $f, g: \kappa \rightarrow \kappa$ such that $[\text{id}]_{\mathcal{U}} = [f]_{\mathcal{V}}$ and $[\text{id}]_{\mathcal{V}} = [g]_{\mathcal{U}}$.

Since $i_{01}^{\mathcal{U}}$ and $i_{01}^{\mathcal{V}}$ agree on $\mathcal{P}(\kappa)$, we have, for any $x \in \mathcal{P}(\kappa)$,

$\in \mathcal{U}$ iff $[id]_{\mathcal{U}} \in i_{01}^{\mathcal{U}}(x)$ iff $[f]_{\mathcal{V}} \in i_{01}^{\mathcal{V}}(x)$ iff $f^{-1}(x) \in \mathcal{V}$. Thus, $\mathcal{U} = f_*(\mathcal{V})$. Similarly, $\mathcal{V} = g_*(\mathcal{U})$.

Thus, $\mathcal{U} = (f \circ g)_*(\mathcal{U})$. It follows that for some set $a \in \mathcal{U}$, $(f \circ g) \upharpoonright a = id \upharpoonright a$, so g is 1-1 on a . Thus there is a 1-1 function, \tilde{g} , from κ onto κ such that $\{\xi < \kappa : g(\xi) = \tilde{g}(\xi)\} \in \mathcal{U}$. Then $\mathcal{V} = \tilde{g}_*(\mathcal{U})$, so \mathcal{U} and \mathcal{V} are equivalent.

6. Theorem. *Suppose $V = L[\mathcal{U}]$, \mathcal{U} a normal ultrafilter on κ . Let \mathcal{V} be any other κ -complete free ultrafilter on κ . Then for some n , \mathcal{V} is equivalent to the ultrafilter \mathcal{U}_n on κ^n (see Definition 2.4).*

proof. In the proof of Theorem 7.3, we saw that for some n , $i_{01}^{\mathcal{V}}(\kappa) = i_{0n}^{\mathcal{U}}(\kappa)$, and $i_{01}^{\mathcal{V}}$ and $i_{0n}^{\mathcal{U}}$ agree on $\mathcal{P}(\kappa)$. But $i_{0n}^{\mathcal{U}} = i_{01}^{\mathcal{U}_n}$, so the theorem follows by Lemma 7.5.

§8. Model theory in $L[\mathcal{U}]$

In this section we give two examples to show that model theory is rather pathological in the universe constructed from a normal ultrafilter on a measurable cardinal. The first involves Hanf numbers, the second, Rowbottom cardinals.

8.1. Definition

- (i) *If κ and λ are regular infinite cardinals, $\mathcal{L}_{\kappa\lambda}$ is the infinitary language consisting of finitary function and predicate symbols, with $< \kappa$ conjunctions and disjunctions, and $< \lambda$ strings of quantifiers;*
- (ii) *If \mathcal{L} is any language, the Hanf number of \mathcal{L} , $H(\mathcal{L})$, is the least cardinal, α , such that whenever a sentence, φ , of \mathcal{L} has a model of cardinality $\geq \alpha$, φ has models of arbitrarily large cardinality.*

For more on infinitary languages, see Karp [4].

We remind the reader of some well-known elementary facts about Hanf numbers.

8.2. Theorem. *Let $\alpha = H(\mathcal{L}_{\kappa\lambda})$.*

- (i) *α is a limit cardinal, and if $\varphi \in \mathcal{L}_{\kappa\lambda}$ has models of arbitrarily large cardinality below α , φ has models of arbitrarily large cardinality;*
- (ii) *$\alpha = \beth_\gamma$ for some ordinal γ ;*
- (iii) *If $\lambda > \omega$, $\alpha = \beth_\alpha$;*
- (iv) *$\kappa \leq \text{cf}(\alpha) \leq (\mathcal{L}_{\kappa\lambda})^\omega$.*

There are reasonable bounds known for $H(\mathcal{L}_{\kappa\omega})$, given by

8.3. Theorem

- (i) (Lopez-Escobar [8]) $H(\mathcal{L}_{\omega_1\omega}) = \beth_{\omega_1}$;
- (ii) (ibid.) $H(\mathcal{L}_{\kappa^+, \omega}) < \beth_{(2^\kappa)^+}$;
- (iii) $H(\mathcal{L}_{\kappa^+, \omega}) > \beth_{\kappa^+}$ when $\text{cf}(\kappa) > \omega$;
- (iv) (Helling) *If GCH and $\text{cf}(\kappa) = \omega$, then $H(\mathcal{L}_{\kappa^+, \omega}) = \beth_{\kappa^+}$.*

In § 15 of [7], we showed that for $\kappa = \omega_1$, results (ii) and (iii) are best possible.

As soon as λ becomes bigger than ω , bounds for $H(\mathcal{L}_{\kappa\lambda})$ can no longer be stated in terms of elementary cardinal arithmetic (i.e., sums, products, and exponentiation). Thus,

8.4. Theorem (Silver [15]). *Let \mathcal{L} be the language consisting of those sentences of $\mathcal{L}_{\omega_1\omega_1}$ which are conjunctions of sentences of $\mathcal{L}_{\omega_1\omega}$ and purely universal sentences of $\mathcal{L}_{\omega_1\omega_1}$.*

- (i) *If $\kappa \rightarrow (\omega_1)^{<\omega}$, $\kappa > H(\mathcal{L})$;*
- (ii) *For each $\gamma < \omega_1$, $H(\mathcal{L})$ is greater than the first cardinal κ such that $\kappa \rightarrow (\gamma)^{<\omega}$ (if it exists).*

One might hope to generalize 8.4 (i) and get a bound on $H(\mathcal{L}_{\omega_1\omega_1})$ in terms of partition properties of the type $\kappa \rightarrow (\sigma)^{<\omega}$. In this section (Theorem 8.8) we show that this is impossible, since it is consistent to assume that $H(\mathcal{L}_{\omega_1\omega_1})$ is greater than the first measurable cardinal.

8.5. Definition. For the rest of this section:

- (i) Assume κ is a measurable cardinal, \mathcal{U} a normal ultrafilter on κ . $i_{\alpha\beta}$ is the embedding: $\text{Ult}_\alpha(L[\mathcal{U}], \mathcal{U} \cap L[\mathcal{U}]) \rightarrow \text{Ult}_\beta(L[\mathcal{U}], \mathcal{U} \cap L[\mathcal{U}])$ of Definition 2.8;
- (ii) $a = \{i_{0n}(\kappa) : n \in \omega\}$;
- (iii) φ_n is the conjunction of the first n axioms of ZFC (in some fixed enumeration). $\psi(v_0)$ asserts that v_0 is a measurable cardinal and that the universe is constructed from a normal ultrafilter on v_0 .

8.6. Lemma. *There is some fixed m such that whenever M is a transitive (set) model for φ_m satisfying $\psi(\sigma)$ (where $\sigma \in M$) and $\sigma \geq i_{0\omega}(\kappa)$, then $a \notin M$.*

Proof. First assume M is a model for all of ZFC.

Let M satisfy that it is constructed from the normal ultrafilter, \mathcal{V} , on σ , where $\sigma \geq i_{0\omega}(\kappa)$.

We see, as in the proof of Theorem 5.8, that if γ is a regular cardinal greater than σ^+ and \mathcal{F} the closed unbounded filter on γ , then $\text{Ult}_\gamma(M, \mathcal{V}) = L_\delta[\mathcal{F}]$ for some δ .

Thus, if $a \in M$, also $a \in L_\delta[\mathcal{F}] \subseteq L[\mathcal{F}]$.

But also by Theorem 5.8, $L[\mathcal{F}] = \text{Ult}_\gamma(L[\mathcal{U}], \mathcal{U} \cap L[\mathcal{U}]) \subset \subset \text{Ult}_\omega(L[\mathcal{U}], \mathcal{U} \cap L[\mathcal{U}])$, and a cannot be in $\text{Ult}_\omega(L[\mathcal{U}], \mathcal{U} \cap L[\mathcal{U}])$, since $i_{0\omega}(\kappa)$ is inaccessible there. This is a contradiction.

By examining the above, we see that we really only needed that M is a model for some φ_m .

For the rest of the section, fix m to be as in Lemma 8.6.

By the standard Löwenheim-Skolem argument,

8.7. Lemma. *If $V = L[\mathcal{U}]$, there is a transitive model, M , of cardinality κ , for φ_m , such that $\kappa \in M$, M satisfies $\psi(\kappa)$, and $\forall x \subset M [\bar{x} = \omega \rightarrow x \in M]$.*

8.8. Theorem. *If $V = L[\mathcal{U}]$, where \mathcal{U} is a normal ultrafilter on the measurable cardinal κ , then the Hanf number of $\mathcal{L}_{\omega_1\omega_1}$ is greater than κ .*

Proof. Let χ be the sentence of $\mathcal{L}_{\omega_1\omega_1}$ in $=, \epsilon$, constant symbol s , and unary function symbol f , which is the conjunction of:

- (i) φ_m ;
- (ii) $\psi(s)$;
- (iii) $\neg \exists v_0 v_1 v_2 \dots [\dots v_n \epsilon \dots \epsilon v_2 \epsilon v_1 \epsilon v_0]$;
- (iv) $\forall v_2 v_3 \dots \exists v_0 \forall v_1 [v_1 \epsilon v_0 \leftrightarrow [v_1 = v_2 \vee v_1 = v_3 \vee \dots]]$;
- (v) $\forall v_0 \exists v_1 [v_1 \epsilon s \wedge f(v_1) = v_0]$.

Thus, models of χ are isomorphic to transitive models, M , of φ_m , satisfying $\psi(\sigma)$ for some ordinal $\sigma \in M$, and such that $\forall x \subseteq M [\bar{x} = \omega \rightarrow x \in M]$ (iv) and $\bar{M} = \sigma$ (v).

By Lemma 8.7, χ has models of cardinality κ . By Lemma 8.6 and the fact that $i_{0\omega}(\kappa) < \kappa^{++}$, χ has no models of cardinality $\geq \kappa^{++}$.

We remark that, by usual ultrapower methods, any sentence χ of $\mathcal{L}_{\kappa\kappa}$ with a model of cardinality κ has one of cardinality 2^κ ($= \kappa^+$ in $L[\mathcal{U}]$).

Once we have that $H(\mathcal{L}_{\omega_1\omega_1}) > \kappa$, Lemma 8.2 shows that it is larger than \beth_{κ^+} , $\beth_{\beth_{\kappa^+}}$, etc., so it is doubtful that any relation could be found between measurable cardinals and $H(\mathcal{L}_{\omega_1\omega_1})$.

Conceivably, some partition properties stronger than $\kappa \rightarrow \kappa^{(<\omega)}$, perhaps involving infinite sequences, could be used to investigate $H(\mathcal{L}_{\omega_1\omega_1})$, but so far the only bound known is the trivial one that $H(\mathcal{L}_{\omega_1\omega_1})$ is less than the first strongly compact cardinal.

Another unusual phenomenon in $L[\mathcal{U}]$ is the behavior of Rowbottom cardinals (see [9], D4007). Prikry [11] has shown that the limit of ω measurable cardinals is a Rowbottom cardinal, and the question of whether the limit of ω Rowbottom cardinals is a Rowbottom cardinal has remained open. We shall show that in $L[\mathcal{U}]$,

this is not the case; in fact, in $L[\mathcal{U}]$, all Rowbottom cardinals are Ramsey cardinals (and hence regular). We shall actually show that in $L[\mathcal{U}]$, a property somewhat weaker than Rowbottom implies Ramsey. Thus,

8.9. Definition

- (i) A Jónsson model is a finitary relational system with no proper elementary subsystems of the same power;
- (ii) λ is a Jónsson cardinal iff there are no Jónsson models of power λ .

We now prove a preliminary lemma on the structure of $L[\mathcal{U}]$.

8.10. Definition. If $x \in L[\mathcal{U}]$, $\text{od}(x)$ is the least α such that $x \in L_{\alpha+1}[\mathcal{U}]$.

8.11. Lemma. Suppose M is a transitive set model for ZFML, with measurable cardinal ρ and normal ultrafilter \mathcal{V} , where $\rho < \kappa$, and $\text{Ult}_\alpha(M, \mathcal{V})$ is well-founded for all α . Let $x \in \mathcal{P}(\rho) \cap M$. Then $x \in L[\mathcal{U}]$, and for any $y \in \mathcal{P}(\rho) \cap L[\mathcal{U}]$ such that $\text{od}(y) \leq \text{od}(x)$, $y \in \mathcal{P}(\rho) \cap M$.

Proof. $\text{Ult}_\kappa(M, \mathcal{V}) = L_\gamma[\mathcal{U}]$ for some γ , and $\mathcal{P}(\rho) \cap M = \mathcal{P}(\rho) \cap L_\gamma[\mathcal{U}]$. Thus $x \in L_\gamma[\mathcal{U}]$. $\text{od}(y) < \gamma$, so $y \in L_\gamma[\mathcal{U}]$ and hence $y \in M$.

Note that $\text{Ult}_\alpha(M, \mathcal{V})$ will be well-founded for all α whenever $\omega_1 \subseteq M$ (by Theorem 3.5).

8.12. Theorem. If $V = L[\mathcal{U}]$, where \mathcal{U} is a normal ultrafilter on the measurable cardinal κ , and λ is a Jónsson cardinal, then λ is a Ramsey cardinal.

Proof. Standard arguments show $\lambda \leq \kappa$, and κ is a Ramsey cardinal, so we may assume $\lambda < \kappa$. Since ω_n is never a Jónsson cardinal for $n < \omega$, $\lambda \geq \omega_\omega$.

Let $P_n: [\lambda]^n \rightarrow 2$, $P = \langle P_n : n < \omega \rangle$. We shall show how to get a homogeneous set for P of cardinality λ .

Special case. Suppose that for some bounded subset x of λ , $\text{od}(P) \leq \text{od}(x)$. Say $x \subseteq \delta$, where $\omega_1 \leq \delta < \lambda$. By standard Löwenheim-Skolem and collapsing arguments, there is a transitive model M for ZFML with measurable cardinal ρ and normal ultrafilter \mathcal{V} such that $\delta < \rho$, $x \in M$, and $\bar{M} = \bar{\delta}$. Note that $i_{0\lambda}^{\mathcal{V}}(\rho) = \lambda$. By Lemma 8.11, $P \in \text{Ult}_\lambda(M, \mathcal{V})$. Since λ is measurable in $\text{Ult}_\lambda(M, \mathcal{V})$, there is a homogeneous set for P of cardinality λ in $\text{Ult}_\lambda(M, \mathcal{V})$.

General case. Now let M be a transitive model for ZFML with measurable cardinal ρ and normal ultrafilter \mathcal{V} such that $\lambda < \rho$, $P \in M$, and $\bar{M} = \lambda$. Let F be a function from λ onto M , and consider the relational system $\langle M; \epsilon, F, \{P\} \rangle$. Since λ is a Jónsson cardinal, there is a proper subset A of M containing P such that $\bar{A} = \lambda$ and $\langle A; \epsilon, F \upharpoonright A \rangle \prec \langle M; \epsilon, F \rangle$. Then $(A \cap \lambda)^{\bar{}} = \lambda$, but $A \cap \lambda \neq \lambda$. If T is the transitive model isomorphic to $\langle A; \epsilon \rangle$, and j is the elementary embedding: $T \rightarrow M$, then the first ordinal, δ , moved by j is less than λ . Also, $j(\lambda) = \lambda$, and $j(\tilde{P}) = P$ for some $\tilde{P} \in T$. Now T cannot contain all subsets of δ , since otherwise $\{x \in \mathcal{P}(\delta) : \delta \in j(x)\}$ would be a normal ultrafilter on δ ; but δ is not measurable by Theorem 5.11. Let $x \in \mathcal{P}(\delta) - T$. Then $\text{od}(x) > \text{od}(\tilde{P})$ by Lemma 8.11, so, by the Special case, there is a set $H \subseteq \lambda$ of cardinality λ homogeneous for \tilde{P} . Then $\{j(\xi) : \xi \in H\}$ is homogeneous for P .

§9. On GCH at a measurable cardinal

It is still unknown whether GCH can fail at a measurable cardinal. The results of this section indicate that this question may be very difficult, since we show that, arguing in $\text{ZFC} + \exists \kappa [\kappa \text{ measurable} \wedge 2^\kappa > \kappa^+]$, one can prove the consistency of the theory $\text{ZFC} + \exists \kappa [\kappa \text{ measurable}]$.

9.1. Definition. For the rest of this section, κ will be a measurable

cardinal, \mathcal{U} a normal ultrafilter on κ , and $M = L[\mathcal{U}]$. For each α , let $M_\alpha = \text{Ult}_\alpha(M, \mathcal{U} \cap L[\mathcal{U}])$. For $\alpha \leq \beta$, let $i_{\alpha\beta}$ be the usual embedding: $M_\alpha \rightarrow M_\beta$.

9.2. Definition. For the rest of this section, let A, B, C, D, E abbreviate the following propositions:

- A. $\kappa^+ > \kappa^{+(M)}$.
- B. All uncountable cardinals are inaccessible in M .
- C. There is a ρ -model for some ordinal $\rho < \kappa$ (see Definition 6.2).
- D. For some κ -complete ultrafilter, \mathcal{V} , on κ , $i_{01}^{\mathcal{V}}(\kappa) \geq i_{0\omega}(\kappa)$.
- E. Solovay's 0^\dagger exists (see Matthias [9], D2040).

We shall show (Theorems 9.4, 9.5) that A, B, C, D, E are equivalent, and that they follow from $2^\kappa > \kappa^+$. Thus, using e.g. B, $2^\kappa > \kappa^+$ implies the existence of a set model for $\text{ZFC} + \exists \kappa [\kappa \text{ measurable}]$.

The following lemma is well known.

9.3. Lemma. Let \mathcal{V} be any κ -complete free ultrafilter on κ , $i_{01}^{\mathcal{V}}$ the embedding: $V \rightarrow \text{Ult}_1(V, \mathcal{V})$. Then $i_{01}^{\mathcal{V}}(\kappa) > 2^\kappa$.

Proof. Let $N = \text{Ult}_1(V, \mathcal{V})$. Since $i_{01}^{\mathcal{V}}(\kappa)$ is inaccessible in N and $\mathcal{P}(\kappa) \subseteq N$, we have $i_{01}^{\mathcal{V}}(\kappa) > 2^{\kappa(N)} \geq 2^\kappa$.

9.4. Theorem. The propositions A, B, C, D, E are all equivalent.

Proof. Clearly E implies B and B implies A.

To see that A implies D, note that $i_{0\omega}(\kappa) < \kappa^{++(M)} \leq \kappa^+$, whereas $i_{01}^{\mathcal{V}}(\kappa) > \kappa^+$ for any κ -complete free \mathcal{V} by Lemma 9.3.

We now assume D, and shall conclude C. Let $N = \text{Ult}_1(V, \mathcal{V})$. Note that $i_{01}^{\mathcal{V}}(\kappa) > i_{0\omega}(\kappa)$; equality cannot hold, since $\text{cf}(i_{0\omega}(\kappa)) = \omega$, and N contains all countable sets of ordinals, but $i_{01}^{\mathcal{V}}(\kappa)$ is inaccessible in N . Now let $\mathcal{F} = \{x \subseteq i_{0\omega}(\kappa) : \exists m \forall n > m [i_{0n}(\kappa) \in x]\}$. $\mathcal{F} \cap N \in N$, since $\{i_{0n}(\kappa) : n < \omega\} \in N$. Furthermore, by Lemma 4.5, $i_{0\omega}(\mathcal{U} \cap M) = \mathcal{F} \cap \text{Ult}_\omega(M, \mathcal{U} \cap M)$, so \mathcal{F} is

a strong filter on $i_{0\omega}(\kappa)$ (see Definition 5.4). Hence, the sentence

$$\exists \rho < i_{01}^{\mathcal{V}}(\kappa) \exists \mathcal{G} [\mathcal{G} \text{ is a strong filter on } \rho]$$

is true in N (take $\rho = i_{0\omega}(\kappa)$, $\mathcal{G} = \mathcal{F} \cap N$). By elementarity of $i_{01}^{\mathcal{V}}$,

$$\exists \rho < \kappa \exists \mathcal{G} [\mathcal{G} \text{ is a strong filter on } \rho]$$

is true in V , so C holds.

We now derive E from C. Let P be the ρ -model for some $\rho < \kappa$. Let \mathcal{U} be the normal ultrafilter on ρ in P , \mathcal{V} any κ -complete free ultrafilter on κ . For any α , $i_{0\alpha}^{\mathcal{V}}(\rho) = \rho$ and $i_{0\alpha}^{\mathcal{V}}(\mathcal{U}) = \mathcal{U}$, so $i_{0\alpha}^{\mathcal{V}}$ takes P into P . It follows from Lemma 3.3 that $\{i_{0\alpha}^{\mathcal{V}}(\kappa) : \kappa \in \text{ORD}\}$ is a class of indiscernibles for $\langle P; \epsilon, \xi \rangle_{\xi < \rho}$. Hence, the class $K = \{\lambda : \lambda \text{ regular and } \lambda > 2^\kappa\}$ is a class of indiscernibles for $\langle P; \epsilon, \xi \rangle_{\xi < \rho}$, since $i_{0\lambda}^{\mathcal{V}}(\kappa) = \lambda$ for $\lambda \in K$. Since $\text{Ult}_\kappa(P, \mathcal{U})$ is the κ -model M and $i_{0\kappa}^{\mathcal{V}}(\lambda) = \lambda$ for $\lambda \in K$, K is also a class of indiscernibles for $\langle M; \epsilon, \xi \rangle_{\xi < \kappa}$. We can now, as usual, pick $I \in \mathcal{U}$ such that I is a set of indiscernibles for $\langle M; \epsilon, \lambda_n \rangle_{n < \omega}$ (where λ_n is the n th element of K) to show that 0^\dagger exists.

9.5. Theorem. *If $2^\kappa > \kappa^+$, then the propositions A – E hold.*

Proof. Let \mathcal{V} be any κ -complete free ultrafilter on κ . By Lemma 9.3,

$$i_{0\omega}(\kappa) < \kappa^{++(M)} \leq 2^\kappa < i_{01}^{\mathcal{V}}(\kappa),$$

so D holds.

Another question that might be asked about κ is whether every κ -complete filter on κ can be extended to a κ -complete ultrafilter. We shall show that this statement would also imply A – E.

Consider the set $\mathcal{U}_\omega \subset \kappa^\omega$ (see Definition 2.7). By the method of proof of Theorem 3.6, any intersection of $< \kappa$ elements of \mathcal{U}_ω

is non-empty, so \mathcal{U}_ω generates a κ -complete filter, \mathcal{F} , on κ^ω , which is a set of cardinality κ . Suppose \mathcal{F} could be extended to a κ -complete ultrafilter, \mathcal{V} . Then the inclusion $\text{Fn}_\omega(\kappa) \subset \text{Fn}_1(\kappa^\omega)$ defines an elementary embedding $e: \text{Ult}_\omega(V, \mathcal{U}) \rightarrow \text{Ult}_1(V, \mathcal{V})$. In particular, $e(i_{0\omega}^{\mathcal{U}}(\kappa)) = i_{01}^{\mathcal{V}}(\kappa)$, so $i_{0\omega}^{\mathcal{U}}(\kappa) \leq i_{01}^{\mathcal{V}}(\kappa)$, so D holds. Hence we have shown:

9.6. Theorem. *If \mathcal{U}_ω can be extended to a κ -complete ultrafilter on κ^ω , then propositions A–E hold.*

Actually, using methods of §10, one can derive from the hypothesis of this theorem the existence of an inner model with two measurable cardinals, but we omit the proof here.

Theorem 9.6 implies:

9.7. Corollary. *If $V = L[\mathcal{U}]$, not every κ -complete filter on κ can be extended to a κ -complete ultrafilter.*

§10. Strongly compact cardinals

κ is called strongly compact iff for every λ , every κ -complete filter on λ can be extended to a κ -complete ultrafilter.

All strongly compact cardinals are measurable. Are all measurable cardinals strongly compact?

Vopěnka and Hrbáček [19] showed that one could not prove this in set theory, since if there is a strongly compact cardinal, the universe is not constructible from any set, so that in an $L[\mathcal{U}]$, there is a measurable cardinal but no strongly compact cardinals.

One might still hope to prove that $\text{Con}(\text{ZFC} + \exists \kappa [\kappa \text{ measurable}])$ implies $\text{Con}(\text{ZFC} + \exists \kappa [\kappa \text{ strongly compact}])$. However, this statement is also not provable in set theory, since we shall show, in $\text{ZFC} + \exists \kappa [\kappa \text{ strongly compact}]$, the existence of submodels of V with many measurable cardinals.

For the rest of this section, we let κ be a fixed strongly compact cardinal.

We begin with some remarks on the method of Vopěnka and Hrbáček.

Whenever ρ is a cardinal $\geq \kappa$, there is a κ -complete ultrafilter, \mathcal{U} , on ρ^+ , such that $\forall x \subseteq \rho^+ [\bar{x} \leq \rho \rightarrow x \notin \mathcal{U}]$. Vopěnka and Hrbáček realized that this \mathcal{U} could be used to get an extension of Scott's result with measurable cardinals. The following definition is due to them (with different notation).

10.1. Definition (Vopěnka-Hrbáček). *For any cardinal $\rho \geq \kappa$ and any κ -complete free ultrafilter, \mathcal{U} , on ρ^+ satisfying $\forall x \subseteq \rho^+ [\bar{x} \leq \rho \rightarrow x \notin \mathcal{U}]$, we construct models M_1, M_2 , and embeddings $i_1: V \rightarrow M_1, i_2: V \rightarrow M_2$, and $k: M_1 \rightarrow M_2$, as follows:*

- (i) $M_2 = \text{Ult}_1(V, \mathcal{U}) = \{[f] : f \in \text{Fn}_1(\rho^+)\}$ of Definition 2.8(ii).
Note that this definition makes sense even though \mathcal{U} is not ρ^+ -complete. j_2 is the embedding i_{01} of 2.8 (iv), so $j_2(x) = [f]$, where $\forall \xi < \rho^+ [f(\xi) = x]$;
- (ii) $\text{Fn}_1^-(\rho^+) = \{f \in \text{Fn}_1(\rho^+) : (\text{range}(f))^- \leq \rho\}$. For $f \in \text{Fn}_1^-(\rho^+)$, $[f]^- = \{g \in \text{Fn}_1^-(\rho^+) : g \approx f \wedge \forall h \in \text{Fn}_1^-(\rho^+) [h \approx f \rightarrow \text{rank}(h) \geq \text{rank}(g)]\}$. $M_1 = \{[f]^- : f \in \text{Fn}_1^-(\rho^+)\}$. $j_1(x) = [f]^-$, where $\forall \xi < \rho^+ [f(\xi) = x]$;
- (iii) ϵ relations are defined as in Definition 2.8 on M_1 and M_2 , but again we always identify M_1 and M_2 with the transitive classes to which they are isomorphic;
- (iv) For $f \in \text{Fn}_1^-(\rho^+)$, $k([f]^-) = [f]$;
- (v) If K is a class, set $j_1(K) = \{[f]^- : f \in \text{Fn}_1^-(\rho^+) \wedge \text{range}(f) \subseteq K\}$;
 $j_2(K) = \{[f] : f \in \text{Fn}_1(\rho^+) \wedge \text{range}(f) \subseteq K\}$.

Actually, the definition of M_1 is a special case of Keisler's notion of a limit ultrapower. In the notation of Keisler [5] p. 389, $M_1 = V^{\rho^+}_{\mathcal{U}} \upharpoonright G$, where G is the filter on $\rho^+ \times \rho^+$ generated by those equivalence relations on ρ^+ with no more than ρ equivalence classes.

10.2. **Lemma.** With the notation of Definition 8.1:

- (i) j_1, j_2 , and k are elementary embeddings and $j_2 = k \circ j_1$;
- (ii) If $\xi < j_1(\rho^+)$, $k(\xi) = \xi$;
- (iii) $j_1(\rho^+) < j_2(\rho^+)$;
- (iv) $j_1(\rho^+) = \sup\{j_1(\xi) : \xi < \rho^+\}$;
- (v) If each $y_\xi \in M_2$ for $\xi < \kappa$, then $\langle y_\xi : \xi < \kappa \rangle \in M_2$;
- (vi) For any x_ξ ($\xi < \kappa$), $\langle j_1(x_\xi) : \xi < \kappa \rangle \in M_1$;
- (vii) If each $y_\xi \in M_1$ for $\xi < \alpha$, where α is an ordinal $\leq \kappa$ such that $(\rho^\alpha)^{\bar{}} = \rho$, then $\langle y_\xi : \xi < \alpha \rangle \in M_1$.

Proof. (i)–(vi) are standard.

For (vii), let $y_\xi = [f_\xi]^-$. Define $g(\eta) = \langle f_\xi(\eta) : \xi < \alpha \rangle$ for $\eta < \rho^+$. $g \in \text{Fn}_1^-(\rho^+)$ since $(\text{range}(g))^{\bar{}} \leq (\rho^\alpha)^{\bar{}} = \rho$. Then $\langle y_\xi : \xi < \alpha \rangle = ([g]^-) \upharpoonright \alpha$.

M_1 cannot equal M_2 since $j_1(\rho) = j_2(\rho)$, $j_1(\rho^+) < j_2(\rho^+)$, and $j_l(\rho^+)$ is the successor of $j_l(\rho)$ in M_l ($l = 1, 2$). Vopěnka and Hrbáček concluded from this that $V \neq L[a]$ for any $a \subseteq \rho$, and, since ρ can be made arbitrarily large, $V \neq L[a]$ for any a .

The restriction, $(\rho^\alpha)^{\bar{}} = \rho$ in Lemma 10.2 (vii) cannot in general be eliminated. For example, suppose $(\rho^\omega)^{\bar{}} = 2^\rho$ (e.g. take $\rho = \beth_{\kappa+\omega}$). Let t be a 1–1 function from $\mathcal{P}(\rho)$ onto the set of functions ρ^ω . It follows from $j_1(\rho^+) < j_2(\rho^+)$ that there is an $a \in \mathcal{P}(j_1(\rho)) - M_1$, and hence $(j_2(t))(a)$ is an ω -sequence of ordinals absent from M_1 . Also, if $(\rho^\omega)^{\bar{}} = \rho^+$, we could take t to be a 1–1 function from ρ^+ onto ρ^ω ; then $(j_2(t))(\rho^+) \notin M_1$. We do not know what happens when $\rho^+ < (\rho^\omega)^{\bar{}} < 2^\rho$, or even whether this situation is possible.

The elementary embedding, k , naturally suggests defining $\mathcal{W} = \{x \in \mathcal{P}(j_1(\rho^+)) \cap M_1 : j_1(\rho^+) \in k(x)\}$. \mathcal{W} is not quite a normal M_1 -ultrafilter. \mathcal{W} satisfies (i)–(iv) of Definition 1.1, along with the criterion for normality in Definition 4.1. But $j_1(\rho^+)$ is a successor in M_1 and hence not weakly compact, so that \mathcal{W} cannot satisfy 1.1 (v). But note that by Lemma 10.2 (vii), \mathcal{W} is closed under arbi-

trary countable intersections whenever $\rho^\omega = \rho$. This, along with Theorem 3.6 allows us to conclude, using Lemma 4.6,

10.3. Lemma. *With the notation of Definition 10.1, suppose K is a transitive model of ZFC containing all the ordinals, such that $j_1(K) = j_2(K)$, and suppose $\rho^\omega = \rho$. Then $\mathcal{W} = \{x \in \mathcal{P}(j_1(\rho^+)) \cap j_1(K) : j_1(\rho^+) \in k(x)\}$ is a normal $j_1(K)$ -ultrafilter on $j_1(\rho^+)$ such that all $\text{Ult}_\alpha(j_1(K), \mathcal{W})$ are well-founded.*

Whenever $j_1(K) = j_2(K)$, it is known that we can conclude, using the embedding, k , that $j_1(\rho^+)$ is inaccessible and Π_m^n indescribable in $j_1(K)$ for all n and m , and hence the same holds for ρ^+ in K . In particular, we have this situation if $K = L[\mathcal{V}]$, where \mathcal{V} is a normal ultrafilter on κ , so that ρ^+ is Π_m^n indescribable in $L[\mathcal{V}]$ for all $n, m < \omega$ and $\rho \geq \kappa$. This fact was noted also by Reinhardt.

It will be convenient to actually get a K -ultrafilter on some ordinal. Conceivably, \mathcal{W} may not be in M_1 , so it is not clear that ρ^+ has a normal K -ultrafilter. However,

10.4. Lemma. *Continuing the notation and assumptions of Lemma 10.3, suppose in addition that $\text{Ult}_\omega(j_1(K), \mathcal{W}) = j_1(\tau)$. Then there is a normal K -ultrafilter, \mathcal{W}' , on some ordinal, σ , such that $\rho^+ < \sigma < 2_5^\rho$, and such that all $\text{Ult}_\alpha(K, \mathcal{W}')$ are well-founded ($2_1^\rho = 2^\rho$, and $2_{n+1}^\rho = 2^{2_n^\rho}$).*

Proof. Let $i_{0\omega}$ be $i_{0\omega}^{\mathcal{W}} : j_1(K) \rightarrow \text{Ult}_\omega(j_1(K), \mathcal{W}) = j_1(K)$. $\mathcal{W}^{(\omega)}$ (see Definition 2.10) is a normal $j_1(K)$ -ultrafilter on $\tau = i_{0\omega}(j_1(\rho^+))$, and for all $x \in \mathcal{P}(\tau) \cap j_1(K)$, $x \in \mathcal{W}^{(\omega)}$ iff $\exists m \forall n > m [i_{0n}(j_1(\rho^+)) \in x]$. Hence, $\mathcal{W}^{(\omega)} \in M_1$, since it is definable from the countable set, $\{i_{0n}(j_1(\rho^+)) : n < \omega\}$. Furthermore, all $\text{Ult}_\alpha(j_1(K), \mathcal{W}^{(\omega)})$ are well-founded by Theorem 2.11 and Lemma 10.3.

$j_1(\tau) \geq \tau$. Thus, M_1 satisfies that there is a normal $j_1(K)$ -ultrafilter, \mathcal{W}' , on some ordinal σ , such that $j_1(\rho^+) < \sigma \leq j_1(\tau)$ and such that all $\text{Ult}_\alpha(j_1(K), \mathcal{W}')$ are well-founded.

Since j_1 is elementary, we shall have the desired conclusion once we know that $\tau < 2_5^\rho$. Now $j_1(\rho^+) < (2^{\rho^+})^+ \leq 2_3^\rho$, so $\tau = i_{0\omega}(j_1(\rho^+)) < (2^{j_1(\rho^+)})^+ \leq 2_5^\rho$.

We were not very careful about getting the best bound for σ , but this will not matter.

We now launch into the main body of the proof. The plan is as follows: We shall fix a sequence of π limit cardinals, $\vec{\lambda}$, and attempt to prove that the sequence of cardinal filters on $\vec{\lambda}$ is strong (see Definitions 5.4, 5.5). That this be true does not seem too surprising in view of Solovay's Theorem 5.6. However, now we do not know that we have $\pi + 1$ measurable cardinals, but only one strongly compact cardinal. Nevertheless, the desired result will eventually be obtained by Lemma 10.4 and iterated ultrapowers.

First, an exercise in cardinal arithmetic to justify the next definition.

10.5. Lemma. *Let α be any cardinal.*

- (i) *There are arbitrarily large cardinals, β , such that $\beta^\alpha = \beta$;*
- (ii) *If $\beta^\alpha = \beta$ then $(\beta^+)^{\alpha} = \beta^+$.*

10.6. Definition. For the rest of this section fix $\pi < \kappa$. Also, fix cardinals $\rho_\mu, \gamma_{\mu n}, \lambda_\mu$ for $\mu < \pi, n < \omega$, such that:

- (i) $(\gamma_{\mu 0}) \exp(2_6^{\rho_\mu}) = \gamma_{\mu 0}$;
- (ii) $\gamma_{\mu, n+1} = (\gamma_{\mu n})^+$;
- (iii) $\lambda_\mu = \sup\{\gamma_{\mu n} : n < \omega\}$;
- (iv) $\rho_\mu > \sup\{\lambda_\nu : \nu < \mu\}$; $\rho_0 > \kappa$;
- (v) $(\rho_\mu)^\omega = \rho_\mu$.

10.7. Definition

- (i) For $\mu < \pi$, let \mathcal{F}_μ be the cardinal filter on λ_μ ;
- (ii) Let $a = \{\gamma_{\mu n} : \mu < \pi \wedge 1 \leq n < \omega\}$;
- (iii) $L[a]$ is the universe constructed from a under the usual definition of construction from a set of ordinals.

Note that $\vec{\mathcal{F}}$ is definable from a , so that $L[\vec{\mathcal{F}}] \subseteq L[a]$. In our effort to prove that $\vec{\mathcal{F}}$ is strong, we shall use normal $L[a]$ -ultrafilters on ordinals, σ_μ , situated between ρ_μ and $\gamma_{\mu,0}$.

10.8. Lemma. *For each $\mu < \pi$, there is a σ_μ and \mathcal{W}_μ such that $(\rho_\mu)^+ < \sigma_\mu < 2_5^{\rho_\mu}$, \mathcal{W}_μ is a normal $L[a]$ -ultrafilter on σ_μ , and all $\text{Ult}_\alpha(L[a], \mathcal{W}_\mu)$ are well-founded.*

Proof. We wish to apply Lemma 10.4 with $L[a]$ as K and ρ_μ as ρ .

To check that $j_1(L[a]) = j_2(L[a])$, we need only show that $j_1(a) = j_2(a)$, and, since a is of length $\omega \cdot \pi < \kappa$, we need only show that $j_1(\gamma_{\nu n}) = j_2(\gamma_{\nu n})$ for $\nu < \pi$, $1 \leq n < \omega$. This is clear from Lemma 10.2 (ii) for $\nu < \mu$ since then $\gamma_{\nu n} < (\rho_\mu)^+$. Now for $\nu \geq \mu$, $j_l(\gamma_{\nu n}) = \sup \{j_l(\xi) : \xi < \gamma_{\nu n}\}$ ($l = 1, 2$), since $\gamma_{\nu n}$ is regular and $> (\rho_\mu)^+$. By Definition 10.6 (ii) and Lemma 10.5 (ii), each $j_l(\xi) < < ((\gamma_{\nu, n-1})^{(\rho_\mu)^+})^+ = \gamma_{\nu n}$. Hence, $j_l(\gamma_{\nu n}) = \gamma_{\nu n}$ ($l = 1, 2$).

We similarly check that $\text{Ult}_\omega(j_1(L[a]), \mathcal{W}) = j_1(L[a])$.

Hence, let σ_μ be the σ and \mathcal{W}_μ the \mathcal{W}' of Lemma 10.4.

For the rest of this section, σ_μ and \mathcal{W}_μ will be as in Lemma 10.8. Also, $i_{\alpha\beta}^\mu$ will be the embedding from $\text{Ult}_\alpha(L[a], \mathcal{W}_\mu)$ into $\text{Ult}_\beta(L[a], \mathcal{W}_\mu)$ defined by \mathcal{W}_μ .

10.9. Lemma. *Let $\alpha < \lambda_\mu$, $\nu < \pi$.*

- (i) $i_{0\alpha}^\mu(\lambda_\nu) = \lambda_\nu$; $i_{0\alpha}^\mu(\vec{\lambda}) = \vec{\lambda}$;
- (ii) For $x \in \mathcal{P}(\lambda_\nu) \cap L[a] \cap i_{0\alpha}^\mu(L[a])$, $x \in \mathcal{F}_\nu$ iff $x \in i_{0\alpha}^\mu(\mathcal{F}_\nu \cap L[a])$;
- (iii) $i_{0\alpha}^\mu(L[\vec{\mathcal{F}}]) = L[\vec{\mathcal{F}}]$;
- (iv) $i_{0\alpha}^\mu(\vec{\mathcal{F}} \cap L[\vec{\mathcal{F}}]) = \vec{\mathcal{F}} \cap L[\vec{\mathcal{F}}]$.

Proof. (iii) and (iv) follow from (i) and (ii).

$i_{0\alpha}^\mu(\gamma_{\nu n}) = \gamma_{\nu n}$ when $\alpha < \gamma_{\mu n}$ by Theorem 3.9 (iii), using Definition 10.6 (i) and Lemma 10.8. Hence (i) and (ii).

10.10. Lemma. $\vec{\mathcal{F}} \cap L[\vec{\mathcal{F}}]$ is, in $L[\vec{\mathcal{F}}]$, a sequence of normal ultrafilters on $\vec{\lambda}$.

Proof. We show first that they are ultrafilters, next that they are $\vec{\lambda}$ -complete, and finally that they are normal.

If $\mathcal{F}_\mu \cap L[\vec{\mathcal{F}}]$ is not an ultrafilter, let b be the first (in order of construction from $\vec{\mathcal{F}}$) of the subsets, x , of λ_μ in $L[\vec{\mathcal{F}}]$ such that $x \notin \mathcal{F}_\mu$ and $(\lambda_\mu - x) \notin \mathcal{F}_\mu$. Then by Lemma 10.9, $i_{0\alpha}^\mu(b) = b$ for all $\alpha < \lambda_\mu$. Also, $i_{0,\gamma_{\mu n}}^\mu(\sigma_\mu) = \gamma_{\mu n}$ for $1 \leq n < \omega$. Thus, $\gamma_{\mu n} \in b$ iff $\sigma_\mu \in b$ for $1 \leq n < \omega$, so $b \in \mathcal{F}_\mu$ or $(\lambda_\mu - b) \in \mathcal{F}_\mu$, a contradiction.

Similarly, if $\mathcal{F}_\mu \cap L[\vec{\mathcal{F}}]$ is not λ_μ -complete in $L[\vec{\mathcal{F}}]$, let $\langle b_\xi : \xi < \delta \rangle \in L[\vec{\mathcal{F}}]$ be the first counter-example to λ_μ -completeness. Thus, each $b_\xi \in \mathcal{F}_\mu$ but $\bigcap \{b_\xi : \xi < \delta\} \notin \mathcal{F}_\mu$, and $\delta < \gamma_{\mu m}$ for some m . $i_{0\alpha}^\mu(\langle b_\xi : \xi < \delta \rangle) = \langle b_\xi : \xi < \delta \rangle$ for each $\alpha < \lambda_\mu$. Also, for $n \geq m$, $i_{\gamma_{\mu m}\gamma_{\mu n}}(\xi) = \xi$ for $\xi < \delta$ and $i_{\gamma_{\mu m}\gamma_{\mu n}}(\gamma_{\mu m}) = \gamma_{\mu n}$, so $\gamma_{\mu n} \in b_\xi$ iff $\gamma_{\mu m} \in b_\xi$. Thus, $\{\gamma_{\mu n} : n \geq m\} \subseteq b_\xi$ for each ξ , so $\bigcap \{b_\xi : \xi < \delta\} \in \mathcal{F}_\mu$, a contradiction.

Similarly, suppose $\langle b_\xi : \xi < \lambda_\mu \rangle \in L[\vec{\mathcal{F}}]$ is the first counter-example to normality of $\mathcal{F}_\mu \cap L[\vec{\mathcal{F}}]$. Then each $b \in \mathcal{F}_\mu$ and $b_\xi = \bigcap \{b_\eta : \eta < \xi\}$ for limit ξ . As before, $\{\gamma_{\mu n} : n \geq m\} \subseteq b_\xi$ whenever $\xi < \gamma_{\mu m}$, so each $\gamma_{\mu m} \in b_{\gamma_{\mu m}}$, so $\{\xi : \xi \in b_\xi\} \in \mathcal{F}_\mu$, a contradiction.

By somewhat more careful computation of cardinal bounds, we could have put somewhat less stringent conditions on $\vec{\lambda}$ than those in Definition 10.6. This does not seem worthwhile, however, in view of Solovay's Theorem 5.6, which implies (after we have gone through Lemma 10.10 with one sequence of length $\pi + 1$) that we in fact could have been very free in choosing $\vec{\lambda}$.

Even without Solovay's theorem we have:

10.11. Theorem. *If there is a strongly compact cardinal, κ , then for any ordinal π , there is a transitive model M of ZFC with π measurable cardinals. M may be taken to be either a class containing all the ordinals, or a set.*

Proof. For $\pi < \kappa$, take $M = L[\vec{\mathcal{F}}]$ and use Lemma 10.10. Or, if we want a set, apply Lemma 10.10 to a sequence of length $\pi + 1$, and take $M = L[\vec{\mathcal{F}}] \cap R(\lambda_\pi)$.

For an arbitrary π , let \mathcal{V} be a κ -complete free ultrafilter on κ . For a suitable $\text{Ult}_\alpha(\mathcal{V}, \mathcal{V})$, $i_{0\alpha}(\kappa) > \pi$, and apply the above within $\text{Ult}_\alpha(\mathcal{V}, \mathcal{V})$.

§11. Saturated ideals

Solovay [17] has shown that if a cardinal, κ , has a λ -saturated κ -complete nontrivial ideal, where $\lambda < \kappa$, then κ is measurable in some sub-model of the universe. In this section we shall, by methods similar to those of §10, extend this result to κ^+ -saturated ideals.

11.1. Definition. For the rest of this section, κ is an uncountable cardinal and \mathcal{I} is a normal, κ -complete, κ^+ -saturated, non-trivial ideal on κ .

κ must be regular. Also, assuming \mathcal{I} to be normal is no loss in generality, since Solovay [17] shows that the existence of any κ -complete, κ^+ -saturated, non-trivial ideal on κ implies the existence of a normal one.

We shall eventually show (Theorem 11.12) that $\mathcal{I} \cap \mathcal{L}[\mathcal{I}]$ is, in $\mathcal{L}[\mathcal{I}]$, a prime ideal, so that κ is measurable in $\mathcal{L}[\mathcal{I}]$.

We first describe some ideas due to Solovay [17]. The material through Lemma 11.5 is taken from there, with slightly changed notation.

11.2. Definition

- (i) \mathcal{B} is the Boolean algebra, $\mathcal{P}(\kappa)/\mathcal{I}$;
- (ii) If $x \in \mathcal{P}(\kappa)$, $[x]$ is the equivalence class of x in \mathcal{B} .

\mathcal{B} has the κ^+ -chain condition. Also, by a theorem of Tarski, \mathcal{B} is complete. We use the standard notation regarding the \mathcal{B} -valued universe, $V^{(\mathcal{B})}$ (see Scott-Solovay [14]). It is convenient to extend the \forall notation to proper classes. Thus,

11.3. Definition. If C is a proper class, \check{C} is the \mathfrak{B} -valued sub-class of $V^{(\mathfrak{B})}$ defined by

$$\llbracket u \in \check{C} \rrbracket = V \{ \llbracket \check{p} = u \rrbracket : p \in C \}.$$

11.4. Definition. \mathcal{U} is the object in $V^{(\mathfrak{B})}$ such that $\llbracket \mathcal{U} \subseteq \mathcal{P}(\check{\kappa}) \cap \check{V} \rrbracket = 1$, and, for each $x \in \mathcal{P}(\kappa)$, $\llbracket \check{x} \in \mathcal{U} \rrbracket = [x]$.

It is easy to see that, with value 1, \mathcal{U} satisfies (i)–(iv) of Definition 1.1 (with $M = \check{V}$), even though it may not satisfy (v). Thus, within $V^{(\mathfrak{B})}$, one may, as in Definition 2.8, form the ultrapowers $\text{Ult}_1(\check{V}, \mathcal{U})$ and the embedding, $i_{01}^{\mathcal{U}} : \check{V} \rightarrow \text{Ult}_1(\check{V}, \mathcal{U})$. Of course, since condition (v) is lacking, it is not clear how to iterate ultrapowers by \mathcal{U} .

11.5. Lemma. $\llbracket \text{Ult}_1(\check{V}, \mathcal{U}) \text{ is well-founded} \rrbracket = 1$.

Now, before showing that κ is measurable in $L[\mathcal{S}]$, we first prove that some larger λ is measurable in an inner model. The following is analogous to Definitions 10.6 and 10.7.

11.6. Definition. For the rest of this section,

- (i) Let γ_n ($n < \omega$) be an increasing sequence of cardinals such that $\text{cf}(\gamma_n) > \kappa$ and $\gamma_n = \beth_{\gamma_n}$;
- (ii) $\lambda = \sup \{ \gamma_n : n < \omega \}$;
- (iii) \mathcal{F} is the filter on λ defined by $x \in \mathcal{F}$ iff $\exists m \forall n > m [\gamma_n \in x]$;
- (iv) $a = \{ \gamma_n : n < \omega \}$;
- (v) \mathcal{W} is the object in $V^{(\mathfrak{B})}$ such that $\llbracket \mathcal{W} = \mathcal{U} \cap L[\check{a}] \rrbracket = 1$.

Analogously to Lemma 10.8, we have

11.7. Lemma. $\llbracket \mathcal{W} \text{ is a normal } L[\check{a}] \text{-ultrafilter on } \kappa \rrbracket = 1$.

Proof. By normality of \mathcal{S} , $\llbracket \check{x} \in \mathcal{W} \rrbracket = \llbracket \check{\kappa} \in i_{01}^{\mathcal{U}}(\check{x}) \rrbracket$ for any

$x \in \mathcal{P}(\kappa) \cap L[a]$. The lemma then follows from the fact that $\llbracket i_{01}^u(\check{a}) = \check{a} \rrbracket = 1$.

Although the fact that $\text{Ult}_1(L[\check{a}], \mathcal{W})$ is, with value 1, well-founded follows from Lemma 11.5, it is not immediately obvious that all $\text{Ult}_\alpha(L[\check{a}], \mathcal{W})$ are well-founded, so we cannot proceed immediately as in Lemma 10.10. What we shall do instead is to first prove (Lemma 11.10) that \mathcal{W} is 2-valued. The proof is like the uniqueness proof for ρ -models in §6.

11.8. Definition

- (i) $\delta_{0\xi} = \beth_{\kappa+\omega \cdot (\xi+1)} \quad (\xi < \gamma_0)$;
- (ii) $\delta_{n+1, \xi} = \beth_{\gamma_n+\omega \cdot (\xi+1)} \quad (\xi < \gamma_{n+1})$;
- (iii) $\delta_{\omega\xi} = \beth_{\lambda+\omega \cdot (\xi+1)} \quad (\xi \in \text{ORD})$;
- (iv) $\Delta = \{ \delta_{n\xi} : n < \omega \wedge \xi < \gamma_n \} \cup \{ \delta_{\omega\xi} : \xi \in \text{ORD} \}$.

Note that $\llbracket i_{01}^u(\check{\delta}) = \check{\delta} \rrbracket = 1$ for each $\delta \in \Delta$.

11.9. Lemma. *Let $x \in \mathcal{P}(\kappa) \cap L[a]$. Then there is a finite subset, F , of $\kappa \cup \Delta$ and a formula, φ , of set theory such that*

$$x = \{ \xi < \kappa : L[a] \models \varphi(\xi, F, a) \}$$

Proof. Let A be the class of elements of $L[a]$ which are first order definable in $L[a]$ from a and some finite subset of $\Delta \cup \kappa$. Then $A \prec L[a]$. Furthermore, A is isomorphic to $L[a]$ since each γ_n is γ_n is the γ_n th ordinal in A . Let j be the isomorphism: $A \rightarrow L[a]$, and let $y \in A$ be such that $x = j(y)$. Then for some φ, F ,

$$y = \{ \xi : L[a] \models \varphi(\xi, F, a) \},$$

and, since j is the identity on κ ,

$$x = \{ \xi < \kappa : L[a] \models \varphi(\xi, F, a) \}.$$

11.10. **Lemma.** *If $x \in \mathcal{P}(\kappa) \cap L[a]$, $\llbracket \check{x} \in \mathcal{W} \rrbracket$ is either 0 or 1.*

Proof. Since $\llbracket \check{x} \in \mathcal{W} \rrbracket = \llbracket \check{\kappa} \in i_{01}^{\mathcal{Q}}(\check{x}) \rrbracket$, it follows, if we write x as in Lemma 11.9, that $\llbracket \check{x} \in \mathcal{W} \rrbracket = 1$ if $L[a] \models \varphi(\kappa, F, a)$, 0 otherwise.

From now on, we identify \mathcal{W} with $\{x \in \mathcal{P}(\kappa) \cap L[a] : \llbracket \check{x} \in \mathcal{W} \rrbracket = 1\}$. Thus, $\mathcal{W} = \{x \in \mathcal{P}(\kappa) \cap L[a] : \kappa - x \in \mathcal{G}\}$. We may now forget about $V^{(\mathcal{Q})}$. \mathcal{W} is a normal $L[a]$ -ultrafilter on κ , and, since \mathcal{G} is κ -complete, arbitrary countable intersections of elements of \mathcal{W} are non-empty, so that all $\text{Ult}_{\alpha}(L[a], \mathcal{W})$ are well-founded. Thus, we may prove, as we did Lemma 10.10,

11.11. **Lemma.** *$\mathcal{F} \cap L[\mathcal{F}]$ is, in $L[\mathcal{F}]$, a normal ultrafilter on λ .*

It follows immediately by Theorem 6.9 that there is a κ -model M , with \mathcal{W} the normal ultrafilter on κ in M . Since \mathcal{W} is the dual filter to $\mathcal{G} \cap M$ in M , $M = L[\mathcal{G}]$ and $\mathcal{G} \cap L[\mathcal{G}]$ is, in $L[\mathcal{G}]$, a normal prime ideal.

We remark finally that if z is any bounded subset of κ , the above would through exactly the same for $L[\mathcal{G}, z]$, since z would be fixed by any elementary embeddings we considered. Thus,

11.12. **Theorem.** *If \mathcal{G} is a normal, κ -complete, κ^+ -saturated, non-trivial ideal on κ , and z is a bounded subset of κ , then $\mathcal{G} \cap L[\mathcal{G}, z]$ is, in $L[\mathcal{G}, z]$, a normal prime ideal on κ .*

There are many open questions concerning κ -saturated and κ^+ -saturated ideals. We can show (by methods of §10 in [7]) that it is consistent that $\kappa = 2^{\aleph_0}$ and carries a κ -saturated ideal, but no λ -saturated ideals for $\lambda < \kappa$. However, it is not known whether κ can be strongly inaccessible and carry a κ -saturated ideal without being measurable.

Even less is known about κ^+ -saturated ideals. For example, it is not known whether ω_1 can have an ω_2 -saturated ideal, or even

whether the ideal on ω_1 dual to the closed unbounded filter can be ω_2 -saturated. Some indication of the difficulty of these problems is given by

11.13. Theorem. *If $\kappa = \lambda^+$ and κ has a κ^+ -saturated κ -complete non-trivial ideal, \mathcal{I} , then Solovay's 0^\dagger (see [9], D2040) exists.*

Proof. As before, we assume \mathcal{I} is normal.

$i_{01}^{\mathcal{U}}(\kappa)$ is, with value 1, greater than κ , but still the successor cardinal to λ in $\text{Ult}_1(\check{V}, \mathcal{U})$. Since \mathcal{B} has the κ^+ chain condition, κ^+ is a cardinal in $\text{Ult}_1(\check{V}, \mathcal{U})$, so $\llbracket i_{01}^{\mathcal{U}}(\check{\kappa}) = (\kappa^+)^{\check{\vee}} \rrbracket = 1$. Thus, there is a κ^+ -model. Since $i_{01}^{\mathcal{W}}(\kappa) > \kappa^{+(M)}$ (where $i_{01}^{\mathcal{W}}: M \rightarrow \text{Ult}_1(M, \mathcal{W})$), Theorem 6.7 implies that $\kappa^{+(M)} < \kappa^+$.

Since \mathcal{W} has cardinality κ , $\llbracket \check{\mathcal{W}} \in \text{Ult}_1(\check{V}, \mathcal{U}) \rrbracket = 1$. By elementarity of $i_{01}^{\mathcal{U}}$, there is a ρ -model, N , for some ρ such that $\rho^{+(N)} < \kappa$. If z is a subset of $\rho^{+(N)}$ which codes the normal ultrafilter on ρ in N , we have, by Theorem 11.12, that

$$L[z, \mathcal{I}] \models [\kappa \text{ is measurable and there is a } \rho\text{-model}] ,$$

so 0^\dagger exists by Theorem 9.4.

In fact, we can show by a somewhat more complicated argument that, under the hypothesis of this theorem, there is an inner model with 2 measurable cardinals.

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