Satisfiability and the Giant Component in Online Variants of the Classical Random Models

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- Duplications within k-sets won't change any of our results so we ignore them.

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A component of size $\Omega(n)$ is called a giant component .

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Solving $f' = 2f^2$ and $f(0) = 1$ gives $f(x) \;=\; \frac{1}{1-2x}$, which blows up at $x = \frac{1}{2}$.

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- Online Algorithms - Pairs appear sequentially.
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  - This is called an Achlioptas Process, named after Dimitris Achlioptas who first posed the question of online avoidance of a giant component.

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- A bounded size algorithm is a size algorithm that makes no distinction between components larger than some fixed constant m.
- A bounded first-edge algorithm is a bounded size algorithm that chooses between e_{i+1} and f_{i+1} only by looking at the sizes of the components in $G_A(i)$ connected by e_i.

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- The critical value c_A is the given by the blow-up point in the differential equation for the susceptibility.

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Theorem 2: If $c > 0.385$ then **whp** this algorithm will create a graph with a component of size $\Omega(n)$.

A Difference Between Online and Offline

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$c = 1$ is called the threshold density for $k = 2$.

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Satisfiability Threshold Conjecture

For each $k > 2$ there exists a threshold density c_k such that:

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- Friedgut used Fourier Analysis in his proof of this theorem.

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Theorem: (Achlioptas, Naor, Peres, 2003)
For $k \geq 1$, there exist constants α and β such that

- If $c < 2^k \ln 2 - \alpha k$ then $F_k(cn)$ is satisfiable **whp**.
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Theorem: (Dubois, Boufkhad, Mandler, 2000)
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(either c is fixed or $c \to \infty$)

Easy

There is an online algorithm which accepts an expected $(1 - \frac{1}{2^k})cn$ clauses.

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So, if $k = 2$ then accept $\{{\color{green}\bullet}, {\color{green}\bullet}\}$, $\{{\color{green}\bullet}, {\color{red}\bullet}\}$, $\{{\color{red}\bullet}, {\color{green}\bullet}\}$, and reject $\{{\color{red}\bullet}, {\color{red}\bullet}\}$. This accepts an expected $\frac{3}{4}cn$ clauses.

Online-Lazy

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Given: $(k = 2)$	Accept?	Set to:
{●(yellow), ●(yellow)} , {●(yellow), ●(red)} , {●(red), ●(yellow)}	Yes	{●(green), ●(yellow)}, {●(green), ●(red)} , {●(red), ●(green)}
{●(red), ●(red)}	No	
{●(red), ●(green)}, {●(green), ●(red)}, {●(green), ●(yellow)}, {●(yellow), ●(green)}, {●(green), ●(green)}	Yes	

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Given: ($k = 2$)	**Accept?**	**Set to:**
{🟡, 🟡} ,{🟡, 🔴} , {🔴, 🟡}	Yes	{🟢, 🟡}, {🟢, 🔴} , {🔴, 🟢}
{🔴, 🔴}	No	
{🔴, 🟢},{🟢, 🔴}, {🟢, 🟡}, {🟡, 🟢}, {🟢, 🟢}	Yes	

This accepts an expected $\frac{3}{4}cn + \frac{3}{8}n$ clauses as $c \to \infty$.

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This accepts an expected $(1-\frac{1}{2^k})cn + a_k n$ clauses as $c \to \infty$.

k	1	**2**	3	4	5	10
a_k	0.5	**0.375**	$0.2842\ldots$	$0.2209\ldots$	$0.1765\ldots$	$0.0809\ldots$

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Theorem: (K,05)
Any online algorithm accepts less than $(1 - \frac{1}{2^k})cn + \ln 2n$ clauses with high probability.

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Corollary: The naive algorithm is asymptotically optimal.

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Total number of clauses accepted by naive algorithm:
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Need to show $B_{cn} \leq n \ln 2 + o(n)$ with high probability.

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This means $Y_{cn} \leq Y_0 + o(n)$ is true **whp**.

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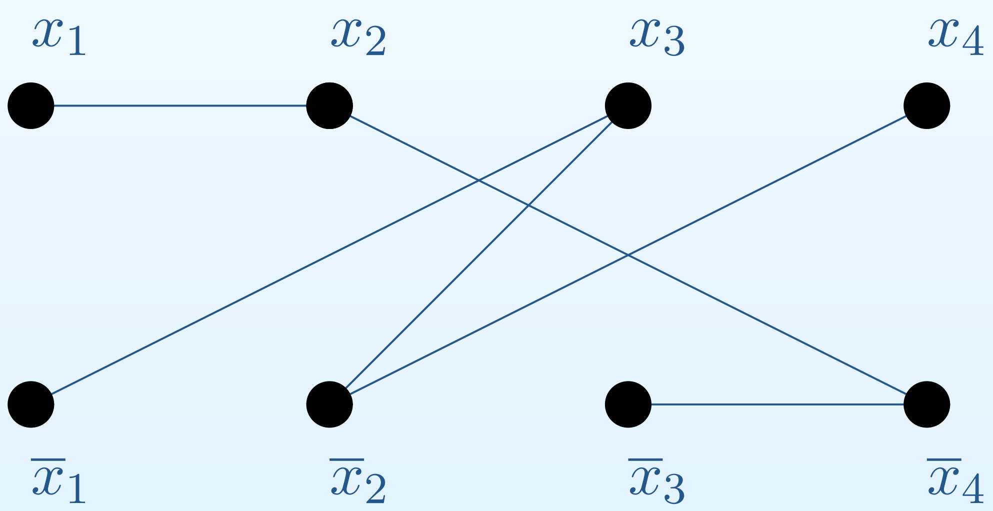

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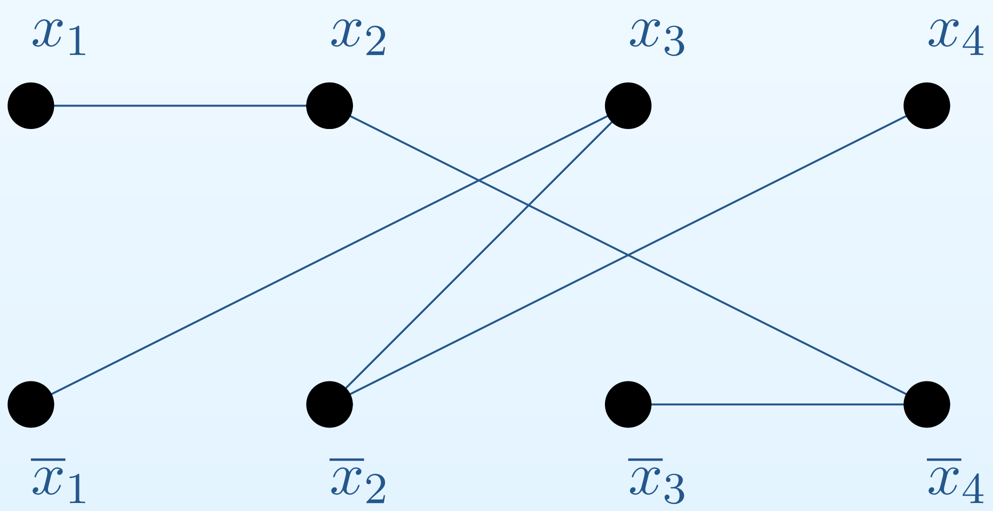

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Question: Is there a correlation?

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Notation: For any graph G, $\Delta(G)$ is the maximum degree and $d_i(G)$ is the number of vertices of degree i ($i \geq 0$).

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Conjecture: If G has more than $(1+\epsilon)n$ edges then there exists ϕ such that if $\Delta(G) = o(n^{\phi})$ then $S(G)$ is not satisfiable **whp**.

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- Solve the conjecture!

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Are there any questions?