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Asymptotic analysis of utility-based hedging strategies for small number of contingent claims

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Abstract

We study the linear approximation of utility-based hedging strategies for small number of contingent claims. We show that this approximation is actually a mean-variance hedging strategy under an appropriate choice of a numéraire and a risk-neutral probability. In contrast to previous studies, we work in the general framework of a semimartingale financial model and a utility function defined on the positive real line.

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1. Introduction

In complete markets, any contingent claim can be replicated by trading. The wealth process of such a hedging strategy follows the price process of the claim, which is uniquely defined by no-arbitrage arguments. In incomplete markets the risk of holding a contingent claim may not be "traded away" and the role of hedging strategy is to provide an optimal trade-off between risk (replication error) and return.

One of the most popular approaches to hedging is to quantify risk as variance, in the spirit of Markowitz, resulting in the so called *mean-variance hedging*. This line of research was initiated

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by Föllmer and Sondermann [5] and continued by Föllmer and Schweizer [4], Schweizer [15] and many other authors. We single out here the paper of Gourieroux, Laurent and Pham [6]: through a change of numéraire they convert the problem of mean-variance hedging under historical measure to the hedging under a *martingale measure*, thus reducing it to the Föllmer–Sondermann case. A nice overview of the literature can be found in Schweizer [16].

Mean-variance hedging is tractable, but it has some economic disadvantages (like penalizing equally shortfalls and earnings). Therefore, more recently, a number of authors studied the concept of *utility-based hedging*, where a portfolio's performance is measured by expected utility. We just mention Duffie et al. [3], which uses direct PDE approach in the study of hedging problem for a non-replicable income stream in the case of power utilities, and Delbaen et al. [2], that relies on duality and martingale methods for the case of exponential utility.

Since explicit computations of utility-based hedging strategies are rarely possible, several authors proposed asymptotic techniques. For example, in the framework of Black and Scholes model with *basis risk* and for power and exponential utilities, Davis [1] and further Monoyios [13,14] approximate hedging strategies with respect to the small parameter $1 - \rho^2$, where ρ is the correlation between traded and non-traded assets. Henderson and Hobson [8] and Henderson [7] derive the first order expansion with respect to the number of contingent claims.

In this paper we generalize the results of [8] and [7] to the case of general semimartingale financial model and general utility function defined on the positive real line. Our main statement is Theorem 2 which shows that the asymptotic hedging strategy is, in fact, the mean-variance hedging strategy (as in [5]), where the role of the pricing measure is played by $Y'(y)$ (the derivative of the dual minimizer) and the role of the numéraire is played by $X'(x)$ (the derivative of the optimal investment strategy). The paper is a companion to our work [12] and relies heavily on ideas and results there.

2. The model

We work in the same model as in [12] and refer to this paper for more details. We have $d + 1$ assets, one bond and d stocks. The price of the bond is constant and the price process of the stocks $S = (S^i)_{1\le i\le d}$ is assumed to be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. Here T is a finite time horizon and $\mathcal{F} = \mathcal{F}_T$.

A portfolio is defined as a pair (x, H), where the constant x represents the initial capital and $H = (H^i)_{1\le i\le d}$ is a predictable S-integrable process. The wealth process $X = (X_t)_{0\le t\le T}$ of the portfolio evolves in time as the stochastic integral of H with respect to S:

$$X_t = x + \int_0^t H_u \mathrm{d}S_u, \quad 0 \le t \le T. \tag{1}$$

We denote by $\mathcal{X}(x)$ the family of wealth processes with non-negative capital at any instant and with initial value equal to x:

$$\mathcal{X}(x) \triangleq \{X \ge 0 : \ X \text{ is defined by (1)}\}. \tag{2}$$

A non-negative wealth process is said to be *maximal* if its terminal value cannot be dominated by that of any other non-negative wealth process with the same initial value. A (signed) wealth process X is said to be *maximal* if it admits a representation of the form

$$X = X' - X'',$$

where both X' and X'' are non-negative maximal wealth processes. A wealth process X is said to be *acceptable* if it admits a representation as above, where both X' and X'' are non-negative wealth processes and, in addition, X'' is maximal.

A probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an *equivalent local martingale measure* if any $X \in \mathcal{X}(1)$ is a local martingale under $\mathbb{Q}$. We denote by $\mathcal{Q}$ the set of equivalent local martingale measures and assume, as usually, that

$$\mathcal{Q} \neq \emptyset. \tag{3}$$

In addition to the set of traded securities we consider a family of N non-traded European contingent claims with payment functions $f = (f_i)_{1\leq i\leq N}$, which are $\mathcal{F}$-measurable random variables, and maturity T. We assume that this family is dominated by the terminal value of some non-negative wealth process X, that is

$$\sum_{i=1}^{N} |f_i| \leq X_T. \tag{4}$$

For $x \in \mathbf{R}$ and $q \in \mathbf{R}^N$ we denote by $\mathcal{X}(x, q)$ the set of *acceptable wealth processes* with initial capital x whose terminal values cover the potential losses from the q contingent claims, that is

$$\mathcal{X}(x, q) \triangleq \{X : X \text{ is acceptable, } X_0 = x \text{ and } X_T + \langle q, f\rangle \geq 0\}.$$

The set of points (x, q) where $\mathcal{X}(x, q)$ is not empty is a closed convex cone in $\mathbf{R}^{N+1}$. We denote by $\mathcal{K}$ the interior of this cone, that is

$$\mathcal{K} \triangleq \operatorname{int}\left\{(x, q) \in \mathbf{R}^{N+1} : \mathcal{X}(x, q) \neq \emptyset\right\}.$$

In this financial model we consider an economic agent whose preferences over terminal wealth are described by a utility function $U : (0, \infty) \to (-\infty, \infty)$. The function U is assumed to be strictly concave and strictly increasing. In addition, motivated by [11,12], we make the following assumption on U:

Assumption 1. The utility function U is two-times continuously differentiable on $(0, \infty)$ and its relative risk-aversion coefficient

$$A(x) \triangleq -\frac{xU''(x)}{U'(x)}, \quad x > 0, \tag{5}$$

is uniformly bounded away from zero and infinity, that is, there are constants $c_1 > 0$ and $c_2 < \infty$ such that

$$c_1 < A(x) < c_2, \quad x > 0. \tag{6}$$

Assume that the agent has some initial capital x and quantities $q = (q_i)_{1\leq i\leq N}$ of the contingent claims f such that $(x, q) \in \mathcal{K}$. The quantities q of the contingent claims will be held constant up to maturity. The capital x can be freely invested into the stocks and the bond according to some dynamic strategy. Therefore, the maximal expected utility that the agent can achieve by trading in the financial market is given by

$$u(x, q) \triangleq \sup_{X\in\mathcal{X}(x,q)} \mathbb{E}\left[U\left(X_T + \langle q, f\rangle\right)\right], \quad (x, q) \in \mathcal{K}. \tag{7}$$

Under our conditions there is a unique optimizer $X(x, q)$ in (7), see [9, Theorem 2].

Abusing notation, we denote by $u(x) \triangleq u(x,0)$ the value function for the problem of optimal investment with no random endowment, i.e.

$$u(x) \triangleq u(x,0) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0, \tag{8}$$

and by $X(x) \triangleq X(x,0)$ the optimizer in (8). To exclude the trivial case we shall assume that

$$u(x) < \infty \quad \text{for some } x > 0, \tag{9}$$

which together with (4) implies that

$$u(x,q) < \infty \quad \text{for all } (x,q) \in \mathcal{K}. \tag{10}$$

The *dual* problem to (8) is given as follows:

$$v(y) \triangleq \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[V\left(Y_T\right)\right], \quad y > 0. \tag{11}$$

Here V is the convex conjugate to U, that is

$$V(y) \triangleq \sup_{x>0} \left\{U(x) - xy\right\}, \quad y > 0, \tag{12}$$

and $\mathcal{Y}(y)$ is the family of non-negative supermartingales Y such that $Y_0 = y$ and XY is a supermartingale for all $X \in \mathcal{X}(1)$. The unique minimizer in (11) is denoted by $Y(y)$. If $y = u'(x)$ (it is known that under our assumptions the function u in (8) is continuously differentiable) then the process $Y(y)/y$ is called *the state price density* corresponding to initial cash endowment $x > 0$. For such initial position we denote

$$p(x) \triangleq \mathbb{E}\left[\frac{Y_T(u'(x))}{u'(x)} f\right] \tag{13}$$

the vector of marginal utility-based prices for the contingent claims f.

The *certainty equivalent value* $c(x,q)$ of the position $(x,q) \in \mathcal{K}$ is defined as the solution of the equation

$$u(c(x,q)) = u(x,q). \tag{14}$$

In other words, the agent is indifferent between having the position (x,q) and the cash amount $c(x,q)$. Note that for $(x,q) \in \mathcal{K}$ we have $u(x,q) \in (u(0), u(\infty))$. Since $u(x) \triangleq u(x,0)$ is continuous and strictly increasing, Eq. (14) has a unique solution $c(x,q) > 0$.

In this paper we are interested to know how the above agent “hedges” the q contingent claims he cannot trade, starting from position $(x,q) \in \mathcal{K}$. The formal definition of the hedging strategy is as follows.

Definition 1. Fix $(x,q) \in \mathcal{K}$. The wealth process $G(x,q)$ of *the utility-based hedging strategy* is given by

$$G(x,q) \triangleq X(c(x,q)) - X(x,q), \tag{15}$$

where $c(x,q)$ is the certainty equivalent value defined by (14), $X(c(x,q))$ is the solution of (8) for initial wealth $c(x,q)$ and $X(x,q)$ is the solution of (7).

We would like to explain Definition 1. Eq. (15) can be rewritten as

$$X(x,q) = X(c(x,q)) - G(x,q),$$

which means that the optimal investment strategy of the investor with initial position (x, q) is to *hedge* the q contingent claims and invest optimally the certainty equivalent value of his initial position. Recall that the contingent claim $f = (f_i)_{i \le i \le N}$ is *replicable* if for every i there is a maximal wealth process X^i with the terminal value f_i. If f is indeed replicable, it is an easy exercise to show that the utility-based hedging strategy coincides with the replicating strategy:

$$G(x,q) = \langle q, X \rangle, \quad (x,q) \in \mathcal{K}.$$

Definition 1 is therefore a *preference-dependent* generalization of replicating strategy to non-replicable claims. However, if f is non-replicable, it is usually not possible to compute $G(x, q)$ explicitly. The goal of the paper is to study the linear approximation of the hedging strategy in the case of small q.

3. Asymptotic analysis

The main object of the paper is specified in the following

Definition 2. Let $x > 0$. An N-dimensional semimartingale $H(x)$ is called a (wealth process of) *marginal hedging strategy* for the contingent claims f if each component $H^i(x)$ is a wealth process (that is, a stochastic integral w.r.t. S) and

1. the terminal value $H_T(x)$ satisfies
$$\lim_{\|q\| \to 0} \frac{|G_T(x,q) - \langle q, H_T(x) \rangle|}{\|q\|} = 0, \tag{16}$$
where the above limit is in $\mathbb{P}$-probability.
2. for $y = u'(x)$ the product $H(x)Y(y)$ is a martingale, where $Y(y)$ is the solution to (11).

The marginal hedging strategy $H(x)$ represents (up to a sign) the *marginal* action the investor needs to take in order to compensate the risk coming from adding to his portfolio a small number of contingent claims. It is easy to see, that $H(x)$ is defined uniquely by Definition 2. Following [12] we specify below precise mathematical conditions for the existence of marginal hedging strategy and also describe some methods of its computation.

Since $X(x)Y(y)$ is a uniformly integrable martingale, we can define a probability measure $\mathbb{R}(x)$ by

$$\frac{d\mathbb{R}(x)}{d\mathbb{P}} = \frac{X_T(x)Y_T(y)}{xy}, \quad y = u'(x). \tag{17}$$

Note that as $X(x)$ and $Y(y)$ are strictly positive processes, $\mathbb{R}(x)$ is equivalent to $\mathbb{P}$.

Denote by $S^{X(x)}$ the price process of the traded securities discounted by $X(x)/x$, that is,

$$S^{X(x)} = \left(\frac{x}{X(x)}, \frac{xS}{X(x)} \right). \tag{18}$$

Let $\mathbf{H}_0^2(\mathbb{R}(x))$ be the space of square integrable martingales under $\mathbb{R}(x)$ with initial value 0 and

$$\mathcal{M}^2(x) = \left\{ M \in \mathbf{H}_0^2(\mathbb{R}(x)) : M_t = \int_0^t H_u dS_u^{X(x)}, 0 \le t \le T \right\}.$$

Note that if $M \in \mathcal{M}^2(x)$, then $\frac{X(x)}{x}M$ is a wealth process (under the original *numéraire*). We also denote by

$$g_i(x) = x\frac{f_i}{X_T(x)}, \quad 1 \le i \le N, \tag{19}$$

the payoffs of the European options discounted by $X_T(x)/x$. The computation of the marginal hedging strategy is based on the solutions of the following auxiliary optimization problems:

$$c_i(x) = \inf_{M\in\mathcal{M}^2(x)} \mathbb{E}_{\mathbb{R}(x)}[A(X_T(x))(p_i(x) + M_T - g_i(x))^2], \quad 1 \le i \le N, \tag{20}$$

where the function A and the vector $p(x)$ were defined in (5) and (13).

To state the result we require two technical assumptions from [12].

Assumption 2. The process $S^{X(x)}$ defined in (18) is sigma-bounded, that is, there is a strictly positive predictable (one-dimensional) process h such that the stochastic integral $\int h\mathrm{d}S^{X(x)}$ is well defined and is locally bounded.

Assumption 3. There are a constant $c > 0$ and a process $M \in \mathcal{M}^2(x)$, such that

$$\sum_{i=1}^{N} |g_i(x)| \le c + M_T. \tag{21}$$

Theorem 1. *Assume (3) and (9) and also that Assumptions 1–3 hold true. Then the marginal hedging strategy $H(x)$ exists and is given by*

$$H^i(x) = \frac{X(x)}{x}(p_i(x) + M^i(x)), \quad 1 \le i \le N, \tag{22}$$

where $p(x)$ is defined by (13) and $M^i(x)$ are the solutions of (20).

The proof of Theorem 1 as well as the proofs of Theorems 2 and 3 below will be given in Section 5. Our next goal is to characterize $H(x)$ in terms of the solution of a mean-variance hedging problem. We denote by $X'(x)$ and $Y'(y)$, $y = u'(x)$, the derivatives to $X(x)$ and $Y(y)$ in the sense that $X'(x)Y(y)$ and $Y'(y)X(x)$ are martingales and

$$X'_T(x) = \lim_{\epsilon\to 0}\left(\frac{X_T(x+\epsilon) - X_T(x)}{\epsilon}\right), \tag{23}$$

$$Y'_T(y) = \lim_{\epsilon\to 0}\left(\frac{Y_T(y+\epsilon) - Y_T(y)}{\epsilon}\right), \tag{24}$$

where the convergences take place in probability. Under conditions of Theorem 1 (see [11, Theorem 1]) we have that $X'(x)$ and $Y'(y)$ are well-defined semimartingales with initial value 1. Hereafter we assume that $X'(x)$ is a strictly positive wealth process, that is,

$$X'_T(x) > 0. \tag{25}$$

(A simple example when this condition is violated can be found in [11].) In this case the product $X'(x)Y'(y)$ is a strictly positive martingale with initial value 1, so we can define an equivalent probability measure $\widetilde{\mathbb{R}}(x)$ such that

$$\frac{d\widetilde{\mathbb{R}}(x)}{d\mathbb{P}} = X'_T(x)Y'_T(y). \tag{26}$$

We choose the wealth process $X'(x)$ as a numéraire and denote by $S^{X'(x)}$ the price process of the traded securities discounted by $X'(x)$, that is,

$$S^{X'(x)} = \left(\frac{1}{X'(x)}, \frac{S}{X'(x)}\right). \tag{27}$$

Let $\mathbf{H}_0^2(\widetilde{\mathbb{R}}(x))$ be the space of square integrable martingales under $\widetilde{\mathbb{R}}(x)$ with initial value 0 and

$$\widetilde{\mathcal{M}}^2(x) = \left\{M \in \mathbf{H}_0^2(\widetilde{\mathbb{R}}(x)) : M_t = \int_0^t H_u dS_u^{X'(x)}, 0 \le t \le T\right\}.$$

We denote by $\widetilde{g}(x)$ the payoffs of the contingent claims discounted by $X'(x)$, that is,

$$\widetilde{g}_i(x) = \frac{f_i}{X'_T(x)}, \quad 1 \le i \le N, \tag{28}$$

and define the following N-dimensional martingale under $\widetilde{\mathbb{R}}(x)$:

$$\widetilde{P}_t(x) = \mathbb{E}_{\widetilde{\mathbb{R}}(x)}[\widetilde{g}(x) \mid \mathcal{F}_t], \quad 0 \le t \le T. \tag{29}$$

In Lemma 1 we shall show that $\widetilde{P}(x)$ is a square integrable martingale under $\widetilde{\mathbb{R}}(x)$. Hence, it admits the following Kunita–Watanabe decomposition:

$$\widetilde{P}(x) = \widetilde{p}(x) + \widetilde{M}(x) + \widetilde{N}(x), \tag{30}$$

where

$$\widetilde{p}(x) = \mathbb{E}_{\widetilde{\mathbb{R}}(x)}[\widetilde{g}(x)] = \mathbb{E}[Y'_T(y)f], \tag{31}$$

$\widetilde{M}^i(x)$ belongs to $\widetilde{\mathcal{M}}^2(x)$ and $\widetilde{N}^i(x)$ is an element of $\mathbf{H}_0^2(\widetilde{\mathbb{R}}(x))$ orthogonal to $\widetilde{\mathcal{M}}^2(x)$.

Theorem 2. *Assume conditions of Theorem 1 and that $X'(x)$ is a strictly positive wealth process. Then the marginal hedging strategy $H(x)$ admits the representation:*

$$H^i(x) = X'(x)(p_i(x) + \widetilde{M}^i(x)), \quad 1 \le i \le N, \tag{32}$$

where $p(x)$ is defined by (13) and $\widetilde{M}^i(x)$ are given by the Kunita–Watanabe decomposition (30).

Theorems 1 and 2 provide characterizations of the marginal hedging strategy in terms of the numéraires $X(x)$ and $X'(x)$ and the corresponding risk-neutral probabilities $\mathbb{R}(x)$ and $\widetilde{\mathbb{R}}(x)$. In our final Theorem 3 we give more explicit description of $H(x)$ under the original numéraire (bank account) and the risk-neutral probability measure $\mathbb{Q}(y)$ defined by

$$\frac{d\mathbb{Q}(y)}{d\mathbb{P}} \triangleq \frac{Y_T(y)}{y}, \quad y = u'(x). \tag{33}$$

Of course, for $\mathbb{Q}(y)$ to be a probability measure we need the following

Assumption 4. $Y(y)$ is a uniformly integrable martingale, i.e. $\mathbb{E}[Y_T(y)] = y$.

Recall from [12] that a semimartingale $R(x)$ is called a *risk-tolerance wealth process* if it is a maximal positive wealth process and

$$R_T(x) = -\frac{U'(X_T(x))}{U''(X_T(x))}. \tag{34}$$

Assumption 5. The risk-tolerance wealth process $R(x)$ exists.

It was shown in [12, Theorem 4] that the existence of $R(x)$ is equivalent to the fact that

$$Y'(y) = \frac{Y(y)}{y}, \tag{35}$$

and that in this case

$$X'(x) = \frac{R(x)}{R_0(x)}. \tag{36}$$

Moreover, Theorem 9 in [12] states that Assumption 5 is a necessary and sufficient condition for some "desirable" qualitative properties of marginal utility-based prices to hold true when $q \approx 0$. Hence, one can argue that Assumption 5 should be valid for any "practical" incomplete financial model. Note that, in particular, this assumption holds true if $U = U(x)$ is a power utility function, that is,

$$U(x) = \frac{x^p - 1}{p}, \quad \text{for some } p < 1,$$

in which case,

$$X'(x) = \frac{R(x)}{R_0(x)} = X(1).$$

To state the result we also have to impose the following

Assumption 6. The stock price process S is continuous.

We would like to point out that Assumption 6 is stronger than Assumption 2 and, as simple examples show, is needed for the validity of Theorem 3 below.

Consider now the $\mathbb{Q}(y)$-martingale $P(x)$ (the marginal utility-based price process)

$$P_t(x) = \mathbb{E}_{\mathbb{Q}(y)}[f \mid \mathcal{F}_t], \quad 0 \le t \le T, \tag{37}$$

and let

$$P^i(x) = p_i(x) + \int K^i \mathrm{d}S + L^i, \quad 1 \le i \le N, \tag{38}$$

be its Kunita–Watanabe decomposition, where $L = (L^i)_{1 \le i \le N}$ is a local martingale under $\mathbb{Q}(y)$ orthogonal to S such that $L_0 = 0$ and we used the fact that $P_0(x) = p(x)$.

Theorem 3. *Let the conditions of Theorem 1 and Assumptions 4–6 hold true. Then the marginal hedging strategy satisfies the equation*

$$H^i_t(x) = p_i(x) + \int_0^t K^i_u \mathrm{d}S_u + \int_0^t (H^i_u(x) - P^i_{u-}(x)) \frac{\mathrm{d}R_u(x)}{R_u(x)}, \tag{39}$$

where K^i is defined by (38), $P(x)$ is given by (37) and $R(x)$ is the risk-tolerance wealth process.

The message of Theorem 3 is that the marginal hedging is performed the following way: start with $p(x)$ cash and buy at any moment the quantities of stocks S one would buy to hedge quadratically the payoff f under the martingale measure $\mathbb{Q}(y)$. The missing dollar amount up to the marginal price $P_{t-}(x)$ is invested in the money market. Since perfect replication may not be possible, this strategy is not self-financing. The mismatch $(H_t(x) - P_{t-}(x))$ should be financed by investing in (borrowing from) the risk-tolerance wealth process.

4. An example

To illustrate the general theory we consider now a concrete example, where the marginal hedging strategy $H(x)$ is evaluated in the framework of financial model with *basis risk*. The utility-based hedging in such a model has been studied in [1,13,14,8,7], among others, for power and exponential utilities. The same example (with general utility function U) has been used in our paper [12] to illustrate the computation of sensitivities for utility-based prices.

Let $W = (W_t)_{0\le t\le T}$ and $B = (B_t)_{0\le t\le T}$ be two independent Brownian motions on a filtered probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t)_{0\le t\le T}, \mathcal{F})$, where the filtration is generated by W and B. The evolution of the *non-traded* asset Q is given by

$$\mathrm{d}Q_t = Q_t\left(\nu\mathrm{d}t + \eta\left(\rho\mathrm{d}W_t + \sqrt{1-\rho^2}\mathrm{d}B_t\right)\right) \tag{40}$$

and the *traded* asset S evolves according to

$$\mathrm{d}S_t = S_t(\mu\mathrm{d}t + \sigma\mathrm{d}W_t). \tag{41}$$

Here $\nu \in \mathbf{R}$, $\mu \in \mathbf{R}$, $\eta > 0$, $\sigma > 0$ and $0 < \rho < 1$ are some constants. The money market pays zero interest rate.

Consider an economic agent starting with initial wealth $x > 0$, that can trade only in S. As before, we assume that the agent has a utility function U satisfying Assumption 1. The agent is pricing a contingent claim with payoff $f = h(Q_T)$, where $h = h(x)$ is a bounded function. Of course, this covers the case of a European put written on Q. Note that even in the case when U is a power utility function, an explicit formula for "true" utility-based hedging strategy $G(x, q)$ from Definition 1 is unknown.

It has been shown in [12], Section 7, that the martingale measures $\mathbb{Q}(y)$ defined in (33) do not depend on y, that is, there is a martingale measure $\widehat{\mathbb{Q}}$ such that

$$\mathbb{Q}(y) = \widehat{\mathbb{Q}}, \quad y > 0, \tag{42}$$

and

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{\mu}{\sigma}W_T - \frac{\mu^2}{2\sigma^2}T\right) = \left(\frac{S_T}{S_0}\right)^{-\mu/\sigma^2} \mathrm{e}^{\frac{\mu}{2}(\frac{\mu}{\sigma^2}-1)T}. \tag{43}$$

The terminal wealth of the optimal investment strategy $X(x)$ is given by

$$U'(X_T(x)) = y\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \quad \text{or, equivalently, } X_T(x) = -V'\left(y\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right), \tag{44}$$

where V is defined in (12) and the constant $y = u'(x)$ and can be found from the formula

$$x = \mathbb{E}_{\widehat{\mathbb{Q}}}[X_T(x)] = \mathbb{E}_{\widehat{\mathbb{Q}}}\left[-V'\left(y\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right].$$

The above expressions allow us to calculate the terminal wealth of risk-tolerance wealth process:

$$R_T(x) = A(X_T(x)) = -\frac{U'(X_T(x))}{U''(X_T(x))} = y\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}V''\left(y\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right),$$

where the last equality follows from (44) and the fact that V is the convex conjugate to U. Using (42) we deduce that R is the wealth process of the replication strategy for the European option with the payoff $\phi(S_T/S_0)$, where

$$\phi(z) = yz^{-\mu/\sigma^2}\mathrm{e}^{\frac{\mu}{2}(\frac{\mu}{\sigma^2}-1)T}V''(yz^{-\mu/\sigma^2}\mathrm{e}^{\frac{\mu}{2}(\frac{\mu}{\sigma^2}-1)T}), \quad z > 0,$$

and, hence, it can computed using Black and Scholes type formula for standard European options.

To facilitate future computations we introduce the process

$$\widetilde{Q}_t = \mathrm{e}^{-\kappa t}Q_t, \quad 0 \le t \le T,$$

where

$$\kappa = \nu - \frac{\mu}{\sigma}\rho\eta.$$

The benefit of this transformation is due to the fact that $\widetilde{Q}$ is a martingale (along with S and B) under $\widehat{\mathbb{Q}}$ and, hence, can be viewed as a trading asset in an artificial complete financial model with martingale measure $\widehat{\mathbb{Q}}$. In such a complete model the contingent claim $f = h(\mathrm{e}^{\kappa T}\widetilde{Q}_T)$ could be replicated by a bounded wealth process

$$P_t = \mathbb{E}_{\widehat{\mathbb{Q}}}[h(\mathrm{e}^{\kappa T}\widetilde{Q}_T) \mid \mathcal{F}_t] = g(\widetilde{Q}_t, t) = g(\widetilde{Q}_0, 0) + \int_0^t g_x(\widetilde{Q}_s, s)\mathrm{d}\widetilde{Q}_s. \tag{45}$$

In the above expression, we denoted by $g(z, t)$ the price of the claim $f = h(\mathrm{e}^{\kappa T}\widetilde{Q}_T)$ at time t if $\widetilde{Q}_t = z$.

Going back to our incomplete model we note that P is the (independent on initial wealth x) marginal utility-based price process. From Theorem 3 we deduce that the marginal utility-based hedging strategy allocates money between 3 financial assets: bank account, stock S and, finally, the risk-tolerance wealth process R. Due to self-financing condition the complete description of such a strategy is given by its initial wealth $H_0(x)$, and the amounts of money β_t and γ_t invested at time t in, respectively, bank account and the stock S. From (45) and the decomposition

$$\frac{\mathrm{d}\widetilde{Q}_t}{\widetilde{Q}_t} = \frac{\eta\rho}{\sigma}\frac{\mathrm{d}S_t}{S_t} + \eta\sqrt{1-\rho^2}\mathrm{d}B_t, \tag{46}$$

we deduce that

$$H_0(x) = g(\widetilde{Q}_0, 0),$$
$$\gamma_t = \frac{\eta\rho}{\sigma}g_x(\widetilde{Q}_t, t)\widetilde{Q}_t,$$
$$\beta_t = P_t - \gamma_t = g(\widetilde{Q}_t, t) - \frac{\eta\rho}{\sigma}g_x(\widetilde{Q}_t, t)\widetilde{Q}_t.$$

It is interesting to note that the initial capital of the marginal hedging strategy as well as the wealth allocations between bank account and the stock do not depend on the subjective "parameters" of economic agent: utility function U and initial wealth x. The only way these "parameters" show up in the computation of the marginal hedging strategy is through the risk-tolerance wealth process $R(x)$.

5. Proofs

Proof of Theorem 1. Let $g_0(x) = 1$ and consider the optimization problems

$$a_i(x) = \inf_{N \in \mathcal{M}^2(x)} \mathbb{E}_{\mathbb{R}(x)}[A(X_T(x))(g_i(x) + N_T)^2], \quad 0 \le i \le N. \tag{47}$$

For $0 \le i \le N$ we denote by $N^i(x)$ the solution to (47). From [12, Theorem 1] we know that

$$X'(x) = \frac{X(x)}{x}(1 + N^0(x)) \tag{48}$$

($X'(x)$ is defined by (23) and the martingale property of $X'(x)Y(y)$). Also, denoting

$$Z^i(x) \triangleq \frac{X(x)}{x} N^i(x), \quad 1 \le i \le N, \tag{49}$$

we have by the same [12, Theorem 1] that

$$\lim_{|\Delta x| + |q| \to 0} \left(\frac{X_T(x + \Delta x, q) - X_T(x) - X'_T(x)\Delta x - \langle Z_T(x), q \rangle}{|\Delta x| + |q|} \right) = 0, \tag{50}$$

and the process $Z(x)Y(y)$ is a uniformly integrable martingale. Taking into account the dual (Lagrange multiplier) characterization of the minimizers in the quadratic optimization problems (47), we conclude that the operator

$$g \to N,$$

mapping a random variable g into the corresponding minimizer in (47) (where $g_i(x)$ is replaced by the generic g) is linear. Therefore the minimizer $M(x)$ in (20) can be written as

$$M^i(x) = p_i(x)N^0(x) - N^i(x).$$

Using (48) and (49), we obtain

$$\frac{X(x)}{x}(p_i(x) + M^i(x)) = p_i(x)X'(x) - Z^i(x).$$

We know from [12, Theorem 10] that

$$\left.\frac{\partial c(x, q)}{\partial q^i}\right|_{q=0} = p_i(x), \quad 1 \le i \le N,$$

so we can use Definition 2, relation (50) and a simple chain rule argument to finish the proof. □

For the proof of Theorem 2 we denote by $\mathcal{N}^2(y)$ the orthogonal complement of $\mathcal{M}^2(x)$ in $\mathbf{H}_0^2(\mathbb{R}(x))$, $y = u'(x)$, and by $\widetilde{\mathcal{N}}^2(y)$ the orthogonal complement of $\widetilde{\mathcal{M}}^2(x)$ in $\mathbf{H}_0^2(\widetilde{\mathbb{R}}(x))$.

Lemma 1. *Assume the hypotheses of Theorem 2 and let $y = u'(x)$. Then:*

1. *For a random vector h, we have*

$$\frac{xh}{X_T(x)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{R}(x)) \quad \textit{if and only if} \quad \frac{h}{X'_T(x)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \widetilde{\mathbb{R}}(x)),$$

$$\frac{yh}{Y_T(y)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{R}(x)) \quad \textit{if and only if} \quad \frac{h}{Y'_T(y)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \widetilde{\mathbb{R}}(x)).$$

2. *For a semimartingale Z and a fixed number a we have*

$$\frac{xZ}{X(x)} \in a + \mathcal{M}^2(x) \quad \textit{if and only if} \quad \frac{Z}{X'(x)} \in a + \widetilde{\mathcal{M}}^2(x).$$

3. *For a semimartingale W and a fixed number b we have*

$$\frac{yW}{Y(y)} \in b + \mathcal{N}^2(y) \quad \textit{if and only if} \quad \frac{W}{Y'(y)} \in b + \widetilde{\mathcal{N}}^2(y).$$

Proof. From [11, Theorem 1] we know that the function u is two-times differentiable at x, and

$$U''(X_T(x))X'_T(x) = u''(x)Y'_T(y), \qquad y = u'(x). \tag{51}$$

Relation (51) together with $U'(X_T(x)) = Y_T(y)$ imply

$$A(X_T(x))\frac{xX'_T(x)}{X_T(x)} = -\frac{xu''(x)}{u'(x)}\frac{yY'_T(y)}{Y_T(y)}.$$

By Assumption 1 we have that $c_1 \leq A(X_T(x)) \leq c_2$ and by [11, Theorem 1] we know that

$$0 < c_1 \leq a(x) \triangleq -\frac{xu''(x)}{u'(x)} \leq c_2 < \infty.$$

Since $\frac{xX'(x)}{X(x)}$ and $\frac{yY'(y)}{Y(y)}$ are uniformly integrable martingales under $\mathbb{R}(x)$ we conclude that

$$\frac{c_1}{c_2}\frac{xX'(x)}{X(x)} \leq \frac{yY'(y)}{Y(y)} \leq \frac{c_2}{c_1}\frac{xX'(x)}{X(x)}. \tag{52}$$

Note that $\frac{xh}{X_T(x)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{R}(x))$ if and only if

$$\mathbb{E}\left[\|h\|^2\frac{Y_T(y)}{X_T(x)}\right] < \infty,$$

and, similarly, $\frac{h}{X'_T(x)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \widetilde{\mathbb{R}}(x))$ if and only if

$$\mathbb{E}\left[\|h\|^2\frac{Y'_T(y)}{X'_T(x)}\right] < \infty.$$

Taking into account relation (52) (at time T) we complete the proof of the first assertion of item 1. The proof of the second statement of this item follows along the same line of arguments and is omitted here.

If $\frac{xZ}{X(x)} \in a + \mathcal{M}^2(x)$, then $\frac{Z}{X'(x)}$ is a wealth process starting at a, under the numéraire $X'(x)$, and, according to item 1, we also know that $\frac{Z_T}{X'_T(x)} \in \mathbf{L}^2(\Omega, \mathcal{F}, \widetilde{\mathbb{R}}(x))$. Furthermore, since

$$\frac{d\widetilde{\mathbb{R}}(x)}{d\mathbb{R}(x)} = \frac{xyX'_T(x)Y'_T(y)}{X_T(x)Y_T(y)},$$

and

$$\frac{Z}{X'(x)}\frac{X'(x)Y'(y)}{X(x)Y(y)} = \frac{Z}{X(x)}\frac{Y'(y)}{Y(y)}$$

is a uniformly integrable martingale under $\mathbb{R}(x)$ (because $\frac{yY'(y)}{Y(y)} \in 1 + \mathcal{N}^2(y)$, see [11, Theorem 1]), we conclude that $\frac{Z}{X'(x)}$ is a uniformly integrable martingale under $\widetilde{\mathbb{R}}(x)$. It follows that

$$\frac{Z}{X'(x)} \in a + \widetilde{\mathcal{M}}^2(x). \tag{53}$$

Assume now that (53) holds true. Then $\frac{xZ}{X(x)}$ is a wealth process starting at a, under the numéraire $X(x)/x$, that is, it is a stochastic integral with respect to $S^{X(x)}$. Using the fact that any bounded stochastic integral of $S^{X(x)}$ is a martingale under $\mathbb{R}(x)$ we deduce from Assumption 2 that $S^{X(x)}$ and, hence, also $\frac{xZ}{X(x)}$, are *sigma-martingales* under $\mathbb{R}(x)$, that is, they can be represented as a stochastic integrals with respect to martingales under $\mathbb{R}(x)$. We refer to [10], Page 214, for definitions and properties of sigma-martingales. The process $\frac{xZ}{X(x)}$ being a sigma-martingale under $\mathbb{R}(x)$ is square integrable martingale under $\mathbb{R}(x)$ if and only

$$\sup_{0\le\tau\le T} \mathbb{E}_{\mathbb{R}(x)}\left[\frac{Z_\tau^2}{X_\tau^2(x)}\right] < \infty, \tag{54}$$

where the supremum above is taken with respect to all stopping times τ. In view of relation (52) this amounts to

$$\sup_{0\le\tau\le T} \mathbb{E}_{\widetilde{\mathbb{R}}(x)}\left[\frac{Z_\tau^2}{(X'_\tau(x))^2}\right] < \infty,$$

which is true because of assumption (53). The proof of item 2 of the lemma is complete.

Choose now two arbitrary semimartingales Z and W. We observe that the process

$$\frac{xZ}{X(x)}\frac{yW}{Y(y)}$$

is a uniformly integrable martingale under $\mathbb{R}(x)$ if and only if

$$\frac{Z}{X'(x)}\frac{W}{Y'(y)}$$

is a uniformly integrable martingale under $\widetilde{\mathbb{R}}(x)$. The above observation, applied for fixed W and any Z satisfying conditions of item 2, together with the assertions of the first and second items, finishes the proof of item 3. $\square$

Proof of Theorem 2. From Lemma 1 (item 1) and Assumption 3 we have $\widetilde{g}(x) \in \mathbf{L}^2(\Omega, \mathcal{F}, \widetilde{\mathbb{R}}(x))$. This implies that the process $(\widetilde{P}_t(x))_{0\le t\le T}$ defined in (29) is a square integrable martingale under $\widetilde{\mathbb{R}}(x)$ and, hence, admits the unique Kunita–Watanabe decomposition (30).

A standard argument in constraint optimization applied to problem (20) leads to

$$A(X_T(x))(p_i(x) + M_T^i(x) - g_i(x)) = L_T^i,$$

where $L^i \in b_i + \mathcal{N}^2(y)$ for some real number b_i. Using (5) and (51) we obtain

$$\frac{f_i}{X'_T(x)} = \frac{X_T(x)}{xX'_T(x)}(p_i(x) + M_T^i(x)) + \frac{y}{xu''(x)}\frac{Y_T(y)L_T^i}{yY'_T(y)}.$$

According to Lemma 1 we have

$$\frac{X(x)}{xX'(x)}(p_i(x)+M^i(x)) \in p_i(x)+\widetilde{\mathcal{M}}^2(x)$$

and

$$\frac{y}{xu''(x)}\frac{Y(y)L^i}{yY'(y)} \in \frac{y}{xu''(x)}b_i+\widetilde{\mathcal{N}}^2(y).$$

Since the Kunita–Watanabe decomposition (30) is unique, we obtain

$$\widetilde{M}^i(x)=\frac{X(x)}{xX'(x)}(p_i(x)+M^i(x))-p_i(x).$$

Taking into account Theorem 1 we finally conclude that

$$H^i(x)=X'(x)(p_i(x)+\widetilde{M}^i(x)). \quad \square$$

Proof of Theorem 3. We remind the reader that under the assumptions of Theorem 3 we have

$$Y'(y)=\frac{Y(y)}{y}, \qquad X'(x)=\frac{R(x)}{R_0(x)}.$$

Consider decomposition (30). Since $\widetilde{p}(x)=p(x)$ and $P(x)=R(x)\widetilde{P}(x)/R_0(x)$, we know from Theorem 2 that

$$P(x)=H(x)+\frac{R(x)}{R_0(x)}\widetilde{N}(x). \tag{55}$$

Under the measure $\widetilde{\mathbb{R}}(x)$, the process $\widetilde{N}(x)$ is a martingale orthogonal to the continuous local martingale

$$S^{X'(x)}=\left(\frac{1}{X'(x)},\frac{S}{X'(x)}\right)=\left(\frac{R_0(x)}{R(x)},\frac{SR_0(x)}{R(x)}\right).$$

This implies that $\widetilde{N}(x)$ and S are orthogonal local martingales under $\mathbb{Q}(y)$. The process $R(x)$ is a stochastic integral with respect to S, which is continuous, so $[\widetilde{N}(x),R(x)]=0$. We can now apply the Itô formula to the product $\widetilde{N}(x)R(x)$ in (55) to obtain

$$P_t(x)=H_t(x)+\int_0^t\frac{\widetilde{N}_{u-}(x)}{R_0(x)}\mathrm{d}R_u(x)+\int_0^t\frac{R_u(x)}{R_0(x)}\mathrm{d}\widetilde{N}_u(x).$$

Using again the fact that $\widetilde{N}(x)$ and S are orthogonal local martingales under $\mathbb{Q}(y)$, we can identify the terms in the Kunita–Watanabe decomposition (38) as

$$p(x)+\int_0^t K_u\mathrm{d}S_u=H_t(x)+\int_0^t\frac{\widetilde{N}_{u-}(x)}{R_0(x)}\mathrm{d}R_u(x). \tag{56}$$

Using (55) we have

$$\widetilde{N}(x)=\frac{R_0(x)(P(x)-H(x))}{R(x)}.$$

Hence, (56) can be rewritten as

$$H_t(x)=p(x)+\int_0^t K_u\mathrm{d}S_u-\int_0^t(P_{u-}(x)-H_u(x))\frac{\mathrm{d}R_u(x)}{R_u(x)},$$

which ends the proof. $\square$

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