A model for a large investor trading at market indifference prices. II: continuous-time case.

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Abstract

We develop a continuous-time model for a large investor trading at market indifference prices. In analogy to the construction of stochastic integrals, we investigate the transition from simple to general predictable strategies. A key role is played by a stochastic differential equation for the market makers' utility process. The analysis of this equation relies on conjugacy relations between the stochastic processes

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with values in the spaces of saddle functions associated with the representative agent's utility.

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Contents

1 Introduction

In this paper we extend to continuous time the single-period model for a large trader developed in [1]. We refer to the latter paper for a more detailed introduction to the topic of price impact models.

As usual in mathematical finance we describe a (self-financing) strategy by a predictable process Q of the number of stocks. The role of a "model" is to define a predictable process $X(Q)$ representing the evolution of the cash balance for the strategy Q. Our approach parallels the classical construction of stochastic integrals with respect to semimartingales. The starting point is the specification of the market dynamics for simple strategies, where the trades occur only at a finite number of times. This is accomplished in an inductive manner, building on the results from the single-period case in [1]. The main challenge is then to show that this construction allows for *a consistent passage* to general predictable strategies. For instance, it is an issue to verify that the cash balance process $X(Q)$ is stable with respect to uniform perturbations of the strategy Q.

These stability questions are addressed by deriving and analyzing a non-linear stochastic differential equation for the market makers' indirect utility processes. A key role is played by the fact, that together with the strategy Q, these utilities form a "sufficient statistics" in our model. More precisely, given the strategy Q, at any point in time, the indirect utilities of market makers are in one-to-one correspondence with the cash balance position of the large trader and the weights of the Pareto optimal allocation of wealth among the market makers. The corresponding functional dependences are explicitly given as the gradients of the conjugate saddle functions associated with the description of the economy in terms of a representative market maker.

An outline of this paper is as follows. In Section 2 we collect some standard concepts and notations. In Section 3 we define the model and study the case when the investor trades according to a simple strategy. Section 4 extends to continuous time the duality results from [1] for stochastic processes with values in saddle functions originating from the utility function of the representative market maker. With these tools at hand, we formally define the strategies with general continuous dynamics in Section 5. We conclude with Section 6 by showing that the construction of strategies in Section 5 is consistent with the original idea based on the approximation by simple strategies. In the last two sections we restrict ourselves to a Brownian setting, due to convenience of references to the book [4] by Kunita.

2 Notations

To streamline the presentation we collect below some standard concepts and notations used throughout the paper.

We will work on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P})$ satisfying the usual conditions of right-continuity and completeness. As usual, we identify random variables differing on a set of $\mathbb{P}$-measure zero; $\mathbf{L}^0(\mathbf{R}^d)$ stands for the metric space of such equivalence classes with values in $\mathbf{R}^d$ endowed with the topology of convergence in probability and $\mathbf{L}^p(\mathbf{R}^d)$, $p \geq 1$, denotes the Banach space of p-integrable random variables. For a σ-field $\mathscr{A} \subset \mathscr{F}$ and a set $A \subset \mathbf{R}^d$ denote $\mathbf{L}^0(\mathscr{A}, A)$ and $\mathbf{L}^p(\mathscr{A}, A)$, $p \geq 1$, the respective subsets of $\mathbf{L}^0(\mathbf{R}^d)$ and $\mathbf{L}^p(\mathbf{R}^d)$ consisting of all $\mathscr{A}$-measurable random variables with values in A.

For a set $A \subset \mathbf{R}^d$ a map $\xi : A \to \mathbf{L}^0(\mathbf{R}^e)$ is called a *random field*; ξ is continuous, convex, etc., if its *sample paths* $\xi(\omega) : A \to \mathbf{R}^e$ are continuous, convex, etc., for all $\omega \in \Omega$. A random field η is called a *modification* of ξ if $\xi(x) = \eta(x)$ for every $x \in A$; ξ and η are *indistinguishable* if $\sup_{x\in A}|\xi(x) - \eta(x)| = 0$. A random field $X : A \times [0,T] \to \mathbf{L}^0(\mathbf{R}^e)$ is called an (adapted) *stochastic process* if, for $t \in [0,T]$, $X_t \triangleq X(\cdot, t) : A \to \mathbf{L}^0(\mathscr{F}_t, \mathbf{R}^e)$; X has values in the space of functions which are continuous, convex, etc., if for any $t \in [0,T]$ the random field X_t is continuous, convex, etc..

Let m be a non-negative integer and U be an open subset of $\mathbf{R}^d$. Denote by $\mathbf{C}^m = \mathbf{C}^m(U, \mathbf{R}^e)$ the Fréchet space of m-times continuously differentiable maps $f : U \to \mathbf{R}^e$ with the topology generated by the semi-norms

$$\|f\|_{m,C} \triangleq \sum_{0\leq|a|\leq m} \sup_{x\in C}|\partial^a f(x)|, \tag{2.1}$$

where C is a compact subset of U, $a = (a_1, \ldots, a_d)$ is a multi-index of non-negative integers, $|a| \triangleq \sum_{i=1}^d a_i$, and

$$\partial^a \triangleq \frac{\partial^{|a|}}{\partial x_1^{a_1} \ldots \partial x_d^{a_d}}. \tag{2.2}$$

In particular, for $m = 0$, ∂^0 is the identity operator and $\|f\|_{0,C} \triangleq \sup_{x\in C} |f(x)|$.

For a metric space $\mathbf{X}$ we denote by $\mathbf{D}(\mathbf{X}, [0,T])$ the space of RCLL (right-continuous with left limits) maps of $[0,T]$ to $\mathbf{X}$; the notation $\mathbf{C}(\mathbf{X}, [0,T])$ is used for the space of continuous maps $[0,T] \mapsto \mathbf{X}$.

For a stochastic process X we let $X^*_t \triangleq \sup_{0 \leq s \leq t} |X_s|$. We shall say that $(X^n)_{n \geq 1}$ converges to X in the *ucp* topology (uniform convergence in probability) if $(X^n - X)^*_T$ converges to 0 in probability.

The boundary of a set $A \subset \mathbf{R}^d$ is denoted by ∂A. For vectors x and y in $\mathbf{R}^d$, we denote by $\langle x, y \rangle$ their Euclidean scalar product and by $|x| \triangleq \sqrt{\langle x, x \rangle}$ the norm of x. The relations $x < y$ and $x \leq y$ are understood in the per coordinate sense. The symbol $\mathbf{1} \triangleq (1, \dots, 1)$ is used for the vector in $\mathbf{R}^d$ with components equal to one.

We use the symbols $\wedge$ for min and $\vee$ for max. The positive and negative parts of $x \in \mathbf{R}$ are denoted by $x^+ \triangleq \max(x, 0) = x \vee 0$ and $x^- \triangleq \max(-x, 0)$.

3 Model

3.1 Market makers and the large investor

We consider a financial model where $M \in \{1, 2, \dots\}$ market makers quote prices for a finite number of traded assets. Uncertainty and the flow of information are modeled by a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ satisfying the standard conditions of right-continuity and completeness; the initial σ-field $\mathscr{F}_0$ is trivial, T is a finite maturity, and $\mathscr{F} = \mathscr{F}_T$.

The way the market makers serve the incoming orders crucially depends on their attitude toward risk, which we model in the classical framework of expected utility. Thus, we interpret the probability measure $\mathbb{P}$ as a description of the common beliefs of our market makers (same for all) and denote by $u_m = (u_m(x))_{x \in \mathbf{R}}$ market maker m's utility function for terminal wealth.

Assumption 3.1. Each $u_m = u_m(x)$, $m = 1, \dots, M$, is a strictly concave, strictly increasing, continuously differentiable, and bounded from above function on the real line $\mathbf{R}$ satisfying

$$\lim_{x \to \infty} u_m(x) = 0. \tag{3.1}$$

The normalizing condition (3.1) is added only for notational convenience. Our main results will be derived under the following additional condition on the utility functions, which, in particular, implies their boundedness from above.

Assumption 3.2. Each utility function $u_m = u_m(x)$, $m = 1, \dots, M$, is twice continuously differentiable and its absolute risk aversion coefficient is bounded away from zero and infinity, that is, for some $c > 0$,

$$\frac{1}{c} \leq a_m(x) \triangleq -\frac{u_m''(x)}{u_m'(x)} \leq c, \quad x \in \mathbf{R}.$$

The prices quoted by the market makers are also influenced by their initial endowments $\alpha_0 = (\alpha_0^m)_{m=1,\dots,M} \in \mathbf{L}^0(\mathbf{R}^M)$, where α_0^m is an $\mathscr{F}$-measurable random variable describing the terminal wealth of the mth market maker (if the large investor, introduced later, will not trade at all). We assume that the initial allocation α_0 is *Pareto optimal* in the sense of

Definition 3.3. Let $\mathscr{G}$ be a σ-field contained in $\mathscr{F}$. A vector of $\mathscr{F}$-measurable random variables $\alpha = (\alpha^m)_{m=1,\dots,M}$ is called *a Pareto optimal allocation given the information* $\mathscr{G}$ or just a *$\mathscr{G}$-Pareto allocation* if

$$\mathbb{E}[|u_m(\alpha^m)||\mathscr{G}] < \infty, \quad m = 1, \dots, M, \tag{3.2}$$

and there is no other allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with the same total endowment,

$$\sum_{m=1}^{M} \beta^m = \sum_{m=1}^{M} \alpha^m, \tag{3.3}$$

leaving all market makers not worse and at least one of them better off in the sense that

$$\mathbb{E}[u_m(\beta^m)|\mathscr{G}] \geq \mathbb{E}[u_m(\alpha^m)|\mathscr{G}] \text{ for all } m = 1, \dots, M, \tag{3.4}$$

and

$$\mathbb{P}[\mathbb{E}[u_m(\beta^m)|\mathscr{G}] > \mathbb{E}[u_m(\alpha^m)|\mathscr{G}]] > 0 \text{ for some } m \in \{1, \dots, M\}. \tag{3.5}$$

A Pareto optimal allocation given the trivial σ-field $\mathscr{F}_0$ is simply called a *Pareto allocation.*

Finally, we consider an economic agent or investor who is going to trade dynamically the *marketed* contingent claims $\psi = (\psi^j)_{j=1,\dots,J} \in \mathbf{L}^0(\mathbf{R}^J)$, where ψ^j determines the cash payoff of the jth security at the common maturity T. As the result of trading with the investor, up to and including time $t \in [0,T]$, the total endowment of the market makers may change from $\Sigma_0 \triangleq \sum_{m=1}^M \alpha_0^m$ to

$$\Sigma(\xi,\theta) \triangleq \Sigma_0 + \xi + \langle \theta, \psi \rangle = \Sigma_0 + \xi + \sum_{j=1}^J \theta^j \psi^j, \tag{3.6}$$

where $\xi \in \mathbf{L}^0(\mathscr{F}_t, \mathbf{R})$ and $\theta \in \mathbf{L}^0(\mathscr{F}_t, \mathbf{R}^J)$ are, respectively, the cash amount and the number of contingent claims *acquired* by the market makers from the investor. Our model will assume that $\Sigma(\xi,\theta)$ is allocated among the market makers in the form of an $\mathscr{F}_t$-Pareto allocation. For this to be possible we have to impose

Assumption 3.4. For any $x \in \mathbf{R}$ and $q \in \mathbf{R}^J$ there is an allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with total random endowment $\Sigma(x,q)$ defined in (3.6) such that

$$\mathbb{E}[u_m(\beta^m)] > -\infty, \quad m = 1, \dots, M. \tag{3.7}$$

Lemma 3.5. *Under Assumptions 3.1 and 3.4, for any σ-field $\mathscr{G} \subset \mathscr{F}$ and random variables $\xi \in \mathbf{L}^0(\mathscr{G}, \mathbf{R})$ and $\theta \in \mathbf{L}^0(\mathscr{G}, \mathbf{R}^J)$ there is an allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with total endowment $\Sigma(\xi,\theta)$ such that*

$$\mathbb{E}[u_m(\beta^m)|\mathscr{G}] > -\infty, \quad m = 1, \dots, M. \tag{3.8}$$

Proof. Without restricting generality we can assume that ξ and θ are bounded. Then (ξ,θ) can be written as a convex combination of finitely many points $(x_k, q_k) \in \mathbf{R}^{1+J}$, $k = 1, \dots, K$ with $\mathscr{G}$-measurable weights $\lambda^k \geq 0$, $\sum_{k=1}^K \lambda^k = 1$. By Assumption 3.4, for each $k = 1, \dots, K$ there is an allocation β_k with the total endowment $\Sigma(x_k, q_k)$ such that

$$\mathbb{E}[u_m(\beta_k^m)] > -\infty, \quad m = 1, \dots, M.$$

Thus the allocation

$$\beta \triangleq \sum_{k=1}^K \lambda^k \beta_k$$

has the total endowment $\Sigma(\xi,\theta)$ and, by the concavity of the utility functions, satisfies (3.7) and, hence, also (3.8). $\square$

3.2 Simple strategies

An investment strategy of the agent is described by a predictable J-dimensional process $Q = (Q_t)_{0\le t\le T}$, where $Q_t = (Q_t^j)_{j=1,\dots,J}$ is the cumulative number of the contingent claims $\psi = (\psi^j)_{j=1,\dots,J}$ *sold* by the investor through his transactions up to time t. For a strategy to be self-financing we have to complement Q by a corresponding predictable process $X = (X_t)_{0\le t\le T}$ describing the cumulative amount of cash *spent* by the investor. Hereafter, we shall call such an X a *cash balance* process.

Remark 3.6. Our description of a trading strategy follows the standard practice of mathematical finance except for the sign: positive values of Q or X now mean *short* positions for the investor in stocks or cash, and, hence, total *long* positions for the market makers. This convention makes future notations more simple and intuitive.

To facilitate the understanding of the economic assumptions behind our model we consider first the case of a simple strategy Q where trading occurs only at a finite number of times, that is,

$$Q_t = \sum_{n=1}^{N} \theta_n 1_{(\tau_{n-1},\tau_n]}(t), \quad 0 \le t \le T, \tag{3.9}$$

with stopping times $0 = \tau_0 \le \dots \le \tau_N = T$ and random variables $\theta_n \in \mathbf{L}^0(\mathscr{F}_{\tau_{n-1}}, \mathbf{R}^J)$, $n = 1, \dots, N$. It is natural to expect that, for such a strategy Q, the cash balance process X has a similar form:

$$X_t = \sum_{n=1}^{N} \xi_n 1_{(\tau_{n-1},\tau_n]}(t), \quad 0 \le t \le T, \tag{3.10}$$

with $\xi_n \in \mathbf{L}^0(\mathscr{F}_{\tau_{n-1}}, \mathbf{R})$, $n = 1, \dots, N$. In our model, these cash amounts will be determined by (forward) induction along with a sequence of conditionally Pareto optimal allocations $(\alpha_n)_{n=1,\dots,N}$ such that each α_n is an $\mathscr{F}_{\tau_{n-1}}$-Pareto allocation with the total endowment

$$\Sigma(\xi_n, \theta_n) = \Sigma_0 + \xi_n + \langle \theta_n, \psi \rangle .$$

Recall that at time 0, before any trade with the investor has taken place, the market makers have the initial Pareto allocation α_0 and the total endowment Σ_0. After the first transaction of θ_1 securities and ξ_1 in cash, the total random endowment becomes $\Sigma(\xi_1, \theta_1)$. The central assumptions of our model, which will allow us to identify the cash amount ξ_1 uniquely, are that, as a result of the trade,

1. the random endowment $\Sigma(\xi_1, \theta_1)$ is redistributed between the market makers to form a new *Pareto* allocation α_1,
2. the market makers' expected utilities *do not change*:

$$\mathbb{E}[u_m(\alpha_1^m)] = \mathbb{E}[u_m(\alpha_0^m)], \quad m = 1, \dots, M.$$

We postpone the discussion of the economic features of these conditions until Section 3.3.

Proceeding by induction we arrive at the rebalance time τ_n with the economy characterized by an $\mathscr{F}_{\tau_{n-1}}$-Pareto allocation α_n of the random endowment $\Sigma(\xi_n, \theta_n)$. We assume that after exchanging $\theta_{n+1} - \theta_n$ securities and $\xi_{n+1} - \xi_n$ in cash the market makers will hold an $\mathscr{F}_{\tau_n}$-Pareto allocation α_{n+1} of $\Sigma(\xi_{n+1}, \theta_{n+1})$ satisfying the key condition of the preservation of expected utilities:

$$\mathbb{E}[u_m(\alpha_{n+1}^m)|\mathscr{F}_{\tau_n}] = \mathbb{E}[u_m(\alpha_n^m)|\mathscr{F}_{\tau_n}], \quad m = 1, \dots, M. \tag{3.11}$$

The fact that this inductive procedure indeed works is ensured by the following result, established in a single-period framework in [1], Theorem 2.7.

Theorem 3.7. *Under Assumptions 3.1 and 3.4, every sequence of positions* $(\theta_n)_{n=1,\dots,N}$ *of a simple strategy as in* (3.9) *yields a unique sequence of cash balances* $(\xi_n)_{n=1,\dots,N}$ *as in* (3.10) *and a unique sequence of allocations* $(\alpha_n)_{n=1,\dots,N}$ *such that, for each* $n = 1, \dots, N$, α_n *is an* $\mathscr{F}_{\tau_{n-1}}$*-Pareto allocation of* $\Sigma(\xi_n, \theta_n)$ *preserving the market makers' expected utilities in the sense of* (3.11).

Proof. Follows from Lemma 3.5, Lemma 3.8 below, and a standard induction argument. □

Lemma 3.8. *Let Assumption 3.1 hold and consider a* σ*-field* $\mathscr{G} \subset \mathscr{F}$ *and random variables* $\gamma \in \mathbf{L}^0(\mathscr{G}, (-\infty, 0)^M)$ *and* $\Sigma \in \mathbf{L}^0(\mathbf{R})$*. Suppose there is an allocation* $\beta \in \mathbf{L}^0(\mathbf{R}^m)$ *which has the total endowment* Σ *and satisfies the integrability condition* (3.8)*.*

Then there are unique $\xi \in \mathbf{L}^0(\mathscr{G}, \mathbf{R})$ *and a* $\mathscr{G}$*-Pareto allocation* α *with the total endowment* $\Sigma + \xi$ *such that*

$$\mathbb{E}[u_m(\alpha^m)|\mathscr{G}] = \gamma^m, \quad m = 1, \dots, M.$$

Proof. The uniqueness of such ξ and α is a direct consequence of the definition of the $\mathscr{G}$-Pareto optimality and the strict concavity and monotonicity of the utility functions. To verify the existence we shall use a conditional version of the argument from the proof of Theorem 2.7 in [1].

To facilitate references we assume hereafter that $\gamma \in \mathbf{L}^1(\mathscr{G}, (-\infty, 0)^M)$. This extra condition does not restrict any generality as, if necessary, we can replace the reference probability measure $\mathbb{P}$ with the equivalent measure $\mathbb{Q}$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathrm{const}\frac{1}{1+|\gamma|}.$$

Note that because γ is $\mathscr{G}$-measurable this change of measure does not affect $\mathscr{G}$-Pareto optimality.

For $\eta \in \mathbf{L}^0(\mathscr{G}, \mathbf{R})$ denote by $\mathscr{B}(\eta)$ the family of allocations $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with total endowments less than or equal to $\Sigma + \eta$ such that

$$\mathbb{E}[u_m(\beta^m)|\mathscr{G}] \geq \gamma^m, \quad m = 1, \dots, M.$$

From the properties of the utility functions in Assumption 3.1 and the existence of an allocation β having the total endowment Σ and satisfying (3.8) we deduce that the set

$$\mathscr{H} \triangleq \{\eta \in \mathbf{L}^0(\mathscr{G}, \mathbf{R}) : \ \mathscr{B}(\eta) \neq \emptyset\}$$

is not empty. If $\eta \in \mathscr{H}$, then $\mathscr{B}(\eta)$ is convex (even with respect to $\mathscr{G}$-measurable weights) by the concavity of the utility functions. Now observe that if $(\eta_i)_{i=1,2}$ belong to $\mathscr{H}$ so does $\eta_1 \wedge \eta_2$. It follows that there is a decreasing sequence $(\eta_n)_{n\geq 1}$ in $\mathscr{H}$ such that its limit ξ is less than or equal to any element of $\mathscr{H}$. Let $\beta_n \in \mathscr{B}(\eta_n)$, $n \geq 1$.

From the properties of utility functions in Assumption 3.1 we deduce the existence of $c > 0$ such that, for $m = 1, \dots, M$,

$$x^- \leq c(-u_m(x)), \quad x < 0.$$

Hence, for $m = 1, \dots, M$,

$$\mathbb{E}[(\beta_n^m)^-)] \leq c\mathbb{E}[(-u_m(\beta^m))] \leq -c\mathbb{E}[\gamma^m] < \infty, \quad n \geq 1,$$

implying that the sequence $((\beta_n)^-)_{n\geq 1}$ is bounded in $\mathbf{L}^1(\mathbf{R}^M)$. Since

$$\sum_{m=1}^M \beta_n^m \leq \Sigma + \eta_n \leq \Sigma + \eta_1, \quad n \geq 1,$$

we deduce that the family of all possible convex combinations of $(\beta_n)_{n\geq 1}$ is bounded in $\mathbf{L}^0(\mathbf{R}^M)$.

By Lemma A1.1 in Delbaen and Schachermayer [3] we can then choose convex combinations ζ_n of $(\beta_k)_{k\geq n}$, $n \geq 1$, converging almost surely to a random variable $\alpha \in \mathbf{L}^0(\mathbf{R}^M)$. It is clear that

$$\sum_{m=1}^{M} \alpha^m \leq \Sigma + \xi. \tag{3.12}$$

Since the utility functions are bounded above and, by the convexity of $\mathscr{B}(\eta_n)$, $\zeta_n \in \mathscr{B}(\eta_n)$, an application of Fatou's lemma yields:

$$\mathbb{E}[u_m(\alpha^m)|\mathscr{G}] \geq \limsup_{n\to\infty} \mathbb{E}[u_m(\zeta_n^m)|\mathscr{G}] \geq \gamma^m, \quad m = 1, \dots, M. \tag{3.13}$$

It follows that $\alpha \in \mathscr{B}(\xi)$. The minimality property of ξ then immediately implies that in (3.12) and (3.13) we have, in fact, equalities and that α is a $\mathscr{G}$-Pareto allocation. □

In Section 5 we shall prove a more constructive version of Theorem 3.7, namely, Theorem 5.1, where the cash balances ξ_n and the Pareto allocations α_n will be given as explicit functions of their predecessors and of the new position θ_n.

The main goal of this paper is to extend the definition of the cash balance processes X from simple to general predictable strategies Q. This task has a number of similarities with the construction of a stochastic integral with respect to a semimartingale. In particular, we are interested in the following questions:

Question 3.9. For simple strategies $(Q^n)_{n\geq 1}$ that converge to another simple strategy Q in ucp, i.e., such that

$$(Q^n - Q)^*_T \triangleq \sup_{0\leq t\leq T} |Q^n_t - Q_t| \to 0, \tag{3.14}$$

do the corresponding cash balance processes converge in ucp as well:

$$(X^n - X)^*_T \to 0?$$

Question 3.10. For any sequence of simple strategies $(Q^n)_{n\geq 1}$ converging in ucp to a predictable process Q, does the sequence $(X^n)_{n\geq 1}$ of their cash balance processes converge to a predictable process X in ucp?

Naturally, when we have an affirmative answer to Question 3.10, the process X will define the cash balance process for the strategy Q. Note that a predictable process Q can be approximated by simple processes as in (3.14) if and only if it has LCRL (left-continuous with right limits) trajectories.

The construction of cash balance processes X and processes of Pareto allocations for general strategies Q will be accomplished in Section 5 and the answers to the Questions 3.9 and 3.10 will be given in Section 6 after a (rather long and technical) study of stochastic processes associated with Pareto optimal allocations in Section 4.

3.3 Economic considerations

This section contains a few remarks concerning the economic features of our model; see also the introduction to our companion paper [1].

Remark 3.11. The above model for the interaction between a large trader and the market makers is essentially based on two economic assumptions:

1. After every trade the market makers can redistribute additional securities and cash to form a conditionally Pareto optimal allocation.

2. As a result of a trade, the indirect utilities of the market makers do not change.

Concerning the first condition, we remark that only in some special cases (for example, if the utility functions for all market makers are exponential) the market makers can achieve the required Pareto allocations by *static* trading of the marketed contingent claims ψ among themselves. In general, a larger set of contingent claims containing non-linear functions of the initial endowment α_0 and contingent claims ψ is needed. An interesting problem for future research is to determine conditions under which the market makers can accommodate any strategy of the large investor by *dynamic* trading in the stocks ψ only.

The second condition can be viewed as a consequence of the following two assumptions:

2a. At any time $t \in [0, T]$ the market makers quote *marginal prices*

$$S_t = \mathbb{E}_{\mathbb{Q}}[\psi|\mathscr{F}_t], \tag{3.15}$$

for *infinitesimally small* quantities of contingent claims ψ, where $\mathbb{Q}$ is the *pricing measure* of the current $\mathscr{F}_t$-Pareto allocation α:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{u'_m(\alpha^m)}{\mathbb{E}[u'_m(\alpha^m)|\mathscr{F}_t]}, \quad m = 1, \dots, M. \tag{3.16}$$

It is well-known that the existence of such a measure follows from the first-order condition of Pareto optimality; see (4.5) in Theorem 4.1 below. Note that the trading of small quantities q of the stocks at these prices will not change the conditional expected utilities for market makers to the first order:

$$\mathbb{E}[u_m(\alpha^m + \langle q, \psi - S_t \rangle)|\mathscr{F}_t] = \mathbb{E}[u_m(\alpha^m)|\mathscr{F}_t] + o(|q|), \quad |q| \to 0.$$

2b. The market makers have no *a priori* knowledge about the subsequent trading strategy of the economic agent. They base their decisions entirely on the past without speculating about the future positions the large investor may take. As a consequence, two trading strategies with identical history but different future lead to the same current Pareto allocation.

To see how conditions 2a and 2b imply assumption 2, note that the investor can split any large transaction into a number of very small orders. Each of these orders will be filled at marginal prices leaving the conditional expected utilities for the market makers unchanged.

Remark 3.12. We emphasize the fact that for the simple strategy Q in Theorem 3.7 the allocation α_n is Pareto optimal only with respect to the information $\mathscr{F}_{\tau_{n-1}}$ (and is, thus, not necessarily $\mathscr{F}_0$-)Pareto optimal!). This may lead to the situation where for two simple strategies Q and R the terminal gain (viewed from the collective position of the market makers) of the former strictly dominates the one of the latter:

$$\langle Q_T, \psi \rangle + X_T(Q) > \langle R_T, \psi \rangle + X_T(R). \tag{3.17}$$

To construct an example, take any simple strategy Q resulting in the terminal allocation $\alpha(Q)$ which is *not* (unconditionally) Pareto optimal. Consider the model with the additional traded asset:

$$\psi_{J+1} \triangleq \langle Q_T, \psi \rangle + X_T(Q),$$

and take R to be its *buy and hold* strategy:

$$R^j = 0, \quad j \leq J, \quad R^{J+1} = 1.$$

As all transactions in R take place at initial time, the terminal allocation $\alpha(R)$ is ($\mathscr{F}_0$-)Pareto optimal and the cash balance process $X(R)$ is constant. Both terminal allocations $\alpha(Q)$ and $\alpha(R)$ yield same expected utilities to the market makers, while their total random endowments differ by $X(R)$. Since $\alpha(Q)$ is not Pareto optimal, it follows that $X(R) < 0$, implying (3.17).

Of course, the existence of Q and R satisfying (3.17) does not lead to an arbitrage opportunity for the large trader. In fact, as we shall see in Lemma 5.7 below, even after the extension to a continuous-time framework our model will not contain arbitrage strategies. Moreover, contrary to the standard, small agent, model of mathematical finance, to prevent arbitrage, we will not need to impose on a general strategy Q any "admissibility" requirements.

Remark 3.13. As we discussed in [1] the above framework is, of course, not the only possible "equilibrium-based" model for a large economic agent. For example, if we replace condition 2b with the more conventional *complete information* assumption:

2b'. Before any trading takes place the market makers possess full knowledge of the large investor's future strategy.

then the market makers' reaction to a strategy Q will be very different. At time 0 the market makers will immediately change the initial Pareto allocation α_0 to a terminal Pareto allocation α_1 (*even if no transaction takes place at $t = 0$!*). In the spirit of the Arrow-Debreu theory of general equilibrium the new Pareto optimal allocation α_1 is characterized by the budget equations:

$$\mathbb{E}_{\mathbb{Q}}[\alpha_0^m] = \mathbb{E}_{\mathbb{Q}}[\alpha_1^m], \quad m = 1, \dots, M,$$

and the clearing condition:

$$\sum_{m=1}^{M} \alpha_1^m = \sum_{m=1}^{M} \alpha_0^m + \int_0^T Q_t \, dS_t,$$

where $\mathbb{Q}$ and S are the pricing measure and the price process of the Pareto optimal allocation α_1; see (3.16) and (3.15).

Note that under this alternative market mechanism the market makers' expected utilities will *increase* and, hence, this model is "more friendly" to them than our model. It is easy to see that for a constant (buy and hold) strategy Q our model leads to a higher terminal gain for the large investor and, therefore, is more favorable to him. For a general strategy Q the relative advantages of these models from the point of view of our trader are not so clear cut, due to the phenomena discussed in Remark 3.12.

4 Stochastic processes of Pareto allocations

The passage from the discrete-time portfolio dynamics of Section 3.2 to general continuous-time dynamics in Section 5 will rely on the study of stochastic processes associated with Pareto allocations presented in this section.

4.1 Parameterization of Pareto allocations

We begin by recalling the results and notations from Section 4 in [1] concerning the classical parameterization of Pareto allocations. As usual in the theory of such allocations, a key role is played by *the utility function of the representative market maker* given by

$$r(v, x) \triangleq \sup_{x^1+\cdots+x^M=x} \sum_{m=1}^{M} v^m u_m(x^m), \quad v \in (0, \infty)^M, x \in \mathbf{R}. \tag{4.1}$$

We shall rely on the properties of this function stated in Section 4.1 of [1], Theorems 4.1 and 4.2. Following [1], we denote

$$\mathbf{A} \triangleq (0, \infty)^M \times \mathbf{R} \times \mathbf{R}^J,$$

the parameter set of Pareto allocations in our economy. An element $a \in \mathbf{A}$ will often be represented as $a = (v, x, q)$, where $v \in (0, \infty)^M$, $x \in \mathbf{R}$, and $q \in \mathbf{R}^J$ will stand for, respectively, weights, a cash amount, and a number of stocks owned collectively by the market makers.

According to Theorem 4.3 in [1], for $a = (v, x, q) \in \mathbf{A}$, the random vector $\pi(a) \in \mathbf{L}^0(\mathbf{R}^M)$ defined by

$$v^m u'_m(\pi^m(a)) = \frac{\partial r}{\partial x}(v, \Sigma(x, q)), \quad m = 1, \dots, M, \tag{4.2}$$

forms a Pareto allocation and, conversely, for $(x,q) \in \mathbf{R} \times \mathbf{R}^J$, any Pareto allocation of the total endowment $\Sigma(x,q)$ is given by (4.2) for some $v \in (0,\infty)^M$. Moreover, $\pi(v_1,x,q) = \pi(v_2,x,q)$ if and only if $v_1 = cv_2$ for some constant $c > 0$ and, therefore, (4.2) defines a one-to-one correspondence between the Pareto allocations with total endowment $\Sigma(x,q)$ and the set

$$\mathbf{S}^M \triangleq \{w \in (0,1)^M : \sum_{m=1}^{M} w^m = 1\}, \tag{4.3}$$

the interior of the simplex in $\mathbf{R}^M$. Following [1], we denote by

$$\pi : \mathbf{A} \to \mathbf{L}^0(\mathbf{R}^M),$$

the random field of Pareto allocations given by (4.2). Clearly, the sample paths of this random field are continuous. Note that, by the properties of the function $r = r(v,x)$, see Theorem 4.1 in [1], the Pareto allocation $\pi(a)$ can be equivalently defined by

$$u_m(\pi^m(a)) = \frac{\partial r}{\partial v^m}(v, \Sigma(x,q)). \tag{4.4}$$

In Corollary 4.2 below we provide the description of the conditional Pareto allocations in our economy, which is analogous to (4.2). The proof of this corollary relies on the following general and well-known fact, which is a conditional version of Theorem 4.3 in [1].

Theorem 4.1. *Consider the family of market makers with utility functions $(u_m)_{m=1,\dots,M}$ satisfying Assumption 3.1. Let $\mathscr{G} \subset \mathscr{F}$ be a σ-field and $\alpha \in \mathbf{L}^0(\mathbf{R}^M)$. Then the following statements are equivalent:*

1. *The allocation α is $\mathscr{G}$-Pareto optimal.*
2. *The integrability condition (3.2) holds and there is $\lambda \in \mathbf{L}^0(\mathscr{G}, \mathbf{S}^M)$ such that*

$$\lambda^m u_m'(\alpha^m) = \frac{\partial r}{\partial x}(\lambda, \sum_{k=1}^{M} \alpha^k), \quad m = 1, \dots, M, \tag{4.5}$$

where the function $r = r(v,x)$ is defined in (4.1).

Moreover, such a random variable λ is defined uniquely in $\mathbf{L}^0(\mathscr{G}, \mathbf{S}^M)$.

Proof. 1 $\implies$ 2: It is enough to show that

$$\frac{u'_m(\alpha^m)}{u'_1(\alpha^1)} \in \mathbf{L}^0(\mathscr{G}, (0,\infty)), \quad m = 1, \dots, M. \tag{4.6}$$

Indeed, in this case we can define

$$\lambda^m = \frac{1/u'_m(\alpha^m)}{\sum_{k=1}^M 1/u'_k(\alpha^k)}, \quad m = 1, \dots, M,$$

and (4.5) follows from the properties of $r = r(v,x)$, see Theorem 4.1 in [1]. Clearly, any $\lambda \in \mathbf{L}^0(\mathscr{G}, \mathbf{S}^M)$ obeying (4.5) also satisfies the equality above and, hence, is defined uniquely.

Suppose (4.6) fails to hold for some index m, for example, for $m = 2$. Then we can find a random variable ξ such that

$$|\xi| \leq 1, \quad (u'_1(\alpha^1 - 1) + u'_2(\alpha^2 - 1))|\xi| \in \mathbf{L}^1(\mathbf{R}),$$

and the set

$$A \triangleq \{\omega \in \Omega : \ \mathbb{E}[u'_1(\alpha^1)\xi|\mathscr{G}](\omega) < 0 < \mathbb{E}[u'_2(\alpha^2)\xi|\mathscr{G}](\omega)\}$$

has a positive probability.

From the continuity of the first derivatives of the utility functions we deduce the existence of $0 < \varepsilon < 1$ such that the set

$$B \triangleq \{\omega \in \Omega : \ \mathbb{E}[u'_1(\alpha^1 - \varepsilon\xi)\xi|\mathscr{G}](\omega) < 0 < \mathbb{E}[u'_2(\alpha^2 + \varepsilon\xi)\xi|\mathscr{G}](\omega)\}$$

also has positive probability. Denoting $\eta \triangleq \varepsilon\xi 1_B$ and observing that, by the concavity of utility functions,

$$\begin{aligned} u_1(\alpha^1) &\leq u_1(\alpha^1 - \eta) + u'_1(\alpha^1 - \eta)\eta, \\ u_2(\alpha^2) &\leq u_2(\alpha^2 + \eta) - u'_2(\alpha^2 + \eta)\eta, \end{aligned}$$

we obtain that the allocation

$$\beta^1 = \alpha^1 - \eta, \quad \beta^2 = \alpha^2 + \eta, \quad \beta^m = \alpha^m, \quad m = 3, \dots, M,$$

satisfies (3.3), (3.4), and (3.5), thus, contradicting the $\mathscr{G}$-Pareto optimality of α.

$2 \implies 1$: For any allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with the same total endowment as α we have

$$\sum_{m=1}^{M} \lambda^m u_m(\beta^m) \leq r(\lambda, \sum_{m=1}^{M} \alpha^m) = \sum_{m=1}^{M} \lambda^m u_m(\alpha^m), \tag{4.7}$$

where the last equality follows from (4.5) and the properties of the function $r = r(v, x)$, see Theorem 4.1 in [1]. Granted integrability as in (3.2), this clearly implies the $\mathscr{G}$-Pareto optimality of α. $\square$

From Theorem 4.1 and the definition of the random field $\pi = \pi(a)$ in (4.2) we obtain

Corollary 4.2. *Let Assumptions 3.1 and 3.4 hold and consider a σ-field $\mathscr{G} \subset \mathscr{F}$ and random variables $\xi \in \mathbf{L}^0(\mathscr{G}, \mathbf{R})$ and $\theta \in \mathbf{L}^0(\mathscr{G}, \mathbf{R}^J)$.*

Then for any $\lambda \in \mathbf{L}^0(\mathscr{G}, (0,\infty)^M)$ the random vector $\pi(\lambda, \xi, \theta)$ forms a $\mathscr{G}$-Pareto allocation. Conversely, any $\mathscr{G}$-Pareto allocation of the total endowment $\Sigma(\xi, \theta)$ is given by $\pi(\lambda, \xi, \theta)$ for some $\lambda \in \mathbf{L}^0(\mathscr{G}, (0,\infty)^M)$.

Proof. The only delicate point is to show that the allocation

$$\alpha^m \triangleq \pi^m(\lambda, \xi, \theta), \quad m = 1, \dots, M,$$

satisfies the integrability condition (3.2). Lemma 3.5 implies the existence of an allocation β of $\Sigma(\xi, \theta)$ satisfying (3.8). The result now follows from inequality (4.7) which holds true by the properties of the function $r = r(v, x)$. $\square$

4.2 Stochastic process of indirect utilities

A key role in the construction of the general investment strategies will be played by the stochastic process $F: \mathbf{A} \times [0, T] \to \mathbf{L}^0(-\infty, 0)$ given by

$$F(a, t) \triangleq \mathbb{E}[r(v, \Sigma(x, q))|\mathscr{F}_t], \quad a = (v, x, q) \in \mathbf{A}, \ t \in [0, T], \tag{4.8}$$

where $r = r(v, x)$ is the utility function of the representative market maker defined in (4.1). The value $F(v, x, q, t)$ defines the *indirect utility* of the representative market maker at time t with the weights v and the endowment $\Sigma(x, q)$. The main results of this section, Theorems 4.3 and 4.4 below, describe the structure of the sample paths of this random field.

Fix a constant $c > 0$, which will be used in (F7)–(F9) below. Following Section 3 in [1], for a function

$$f: \mathbf{A} \to (-\infty, 0) \tag{4.9}$$

we define the following conditions:

(F1) The function f is continuously differentiable on $\mathbf{A}$.

(F2) For any $(x, q) \in \mathbf{R} \times \mathbf{R}^J$, the function $f(\cdot, x, q)$ is positively homogeneous:

$$f(cv, x, q) = cf(v, x, q), \text{ for all } c > 0 \text{ and } v \in (0, \infty)^M, \tag{4.10}$$

and strictly decreasing on $(0, \infty)^M$. Moreover, if $M > 1$ then $f(\cdot, x, q)$ is strictly convex on the set $\mathbf{S}^M$ defined in (4.3) (the interior of the simplex in $\mathbf{R}^M$) and for any sequence $(w_n)_{n \geq 1}$ in $\mathbf{S}^M$ converging to a boundary point of $\mathbf{S}^M$

$$\lim_{n \to \infty} f(w_n, x, q) = 0. \tag{4.11}$$

(F3) For any $v \in (0, \infty)^M$, the function $f(v, \cdot, \cdot)$ is concave on $\mathbf{R} \times \mathbf{R}^J$.

(F4) For any $(v, q) \in (0, \infty)^M \times \mathbf{R}^J$, the function $f(v, \cdot, q)$ is strictly concave and strictly increasing on $\mathbf{R}$ and

$$\lim_{x \to \infty} f(v, x, q) = 0. \tag{4.12}$$

(F5) The function f is twice continuously differentiable on $\mathbf{A}$ and, for any $a = (v, x, q) \in \mathbf{A}$,

$$\frac{\partial^2 f}{\partial x^2}(a) < 0,$$

and the matrix $A(f)(a) = (A^{lm}(f)(a))_{l,m=1,\ldots,M}$ given by

$$A^{lm}(f)(a) \triangleq \frac{v^l v^m}{\frac{\partial f}{\partial x}} \left(\frac{\partial^2 f}{\partial v^l \partial v^m} - \frac{1}{\frac{\partial^2 f}{\partial x^2}} \frac{\partial^2 f}{\partial v^l \partial x} \frac{\partial^2 f}{\partial v^m \partial x} \right)(a), \tag{4.13}$$

has full rank.

(F6) If $M > 1$ then for any $(x, q) \in \mathbf{R} \times \mathbf{R}^J$ and any sequence $(w_n)_{n\geq 1}$ in $\mathbf{S}^M$ converging to a boundary point of $\mathbf{S}^M$

$$\lim_{n\to\infty} \sum_{m=1}^{M} \frac{\partial f}{\partial v^m}(w_n, x, q) = -\infty.$$

(F7) For any $a = (v, x, q) \in \mathbf{A}$ and $m = 1, \dots, M$,

$$\frac{1}{c}\frac{\partial f}{\partial x}(a) \leq -v^m \frac{\partial f}{\partial v^m}(a) \leq c\frac{\partial f}{\partial x}(a).$$

(F8) For any $a \in \mathbf{A}$ and $z \in \mathbf{R}^M$,

$$\frac{1}{c}\langle z, z\rangle \leq \langle z, A(f)(a)z\rangle \leq c\langle z, z\rangle,$$

where the matrix $A(f)(a)$ is defined in (4.13).

(F9) For any $a = (v, x, q) \in \mathbf{A}$ and $m = 1, \dots, M$,

$$-\frac{1}{c}\frac{\partial^2 f}{\partial x^2}(a) \leq v^m \frac{\partial^2 f}{\partial v^m \partial x}(a) \leq -c\frac{\partial^2 f}{\partial x^2}(a).$$

Following [1] we define the families of functions:

$$\begin{aligned}
\mathbf{F}^1 &\triangleq \{f \text{ as in (4.9)} : \text{ (F1)–(F4) hold}\},\\
\mathbf{F}^2 &\triangleq \{f \in \mathbf{F}^1 : \text{ (F5) holds}\}.
\end{aligned}$$

We also denote

$$\widetilde{\mathbf{F}}^1 \triangleq \{f \in \mathbf{F}^1 : \text{ (F6) holds}\},$$

and, for a constant $c > 0$,

$$\widetilde{\mathbf{F}}^2(c) \triangleq \{f \in \mathbf{F}^2 : \text{ (F6)–(F9) hold for given } c\}.$$

Note that in the case of a single market maker, when $M = 1$, the condition (F6) and the second part of (F4) hold trivially and, in particular, $\mathbf{F}^1 = \widetilde{\mathbf{F}}^1$.

Hereafter, we shall view $\mathbf{F}^i$, $i = 1, 2$, as topological subspaces of the corresponding Fréchet spaces $\mathbf{C}^i(\mathbf{A})$ of i-times continuously differentiable functions with the semi-norms $\|\cdot\|_{i,C}$ defined in (2.1). A similar convention will also be used for $\widetilde{\mathbf{F}}^1$ and $\widetilde{\mathbf{F}}^2(c)$. Note that, as the elements of $\mathbf{F}^1$ are *saddle* functions, the topology on $\mathbf{F}^1$ is equivalent to the topology of pointwise convergence, see Theorems 35.4 and 35.10 in the book [5] by Rockafellar.

We now state the main results of this section. In a one-period setting they were established in [1], Theorems 4.7 and 4.13. Recall the notation $\mathbf{D}(\mathbf{X}, [0,T])$ from Section 2 for the space of RCLL maps of $[0,T]$ into a metric space $\mathbf{X}$.

Theorem 4.3. *Under Assumptions 3.1 and 3.4 the stochastic process $F = F(a,t)$ defined in* (4.8) *has a modification with sample paths in $\mathbf{D}(\widetilde{\mathbf{F}}^1, [0,T])$.*

Moreover, for any compact set $C \subset \mathbf{A}$

$$\mathbb{E}[\|F(\cdot,T)\|_{1,C}] < \infty, \tag{4.14}$$

and, for $a = (v,x,q) \in \mathbf{A}$, $t \in [0,T]$, and $i = 1,\dots,M+1+J$,

$$\frac{\partial F}{\partial a^i}(a,t) = \mathbb{E}[\frac{\partial F}{\partial a^i}(a,T)|\mathscr{F}_t]. \tag{4.15}$$

For later use, we note the following expressions for the first derivatives of $F = F(a,t)$ with respect to v:

$$\frac{\partial F}{\partial v^m}(a,t) = \mathbb{E}[u_m(\pi^m(a))|\mathscr{F}_t], \quad m = 1,\dots,M, \tag{4.16}$$

which follow from (4.15) and (4.4).

Theorem 4.4. *Under Assumptions 3.1, 3.2, and 3.4 the stochastic process $F = F(a,t)$ defined in* (4.8) *has a modification with sample paths in $\mathbf{D}(\widetilde{\mathbf{F}}^2(c), [0,T])$ with the constant $c > 0$ from Assumption 3.2.*

Moreover, for any compact set $C \subset \mathbf{A}$

$$\mathbb{E}[\|F(\cdot,T)\|_{2,C}] < \infty, \tag{4.17}$$

and, for $a = (v,x,q) \in \mathbf{A}$, $t \in [0,T]$, and $i,j = 1,\dots,M+1+J$,

$$\frac{\partial^2 F}{\partial a^i \partial a^j}(a,t) = \mathbb{E}[\frac{\partial^2 F}{\partial a^i \partial a^j}(a,T)|\mathscr{F}_t]. \tag{4.18}$$

The rest of the section is devoted to the proofs of these two theorems. We start with the following basic result on the existence of smooth modifications for parametric families of martingales.

Lemma 4.5. *Let m be a non-negative integer, U be an open set in $\mathbf{R}^d$, and $\xi : U \to \mathbf{L}^0$ be a random field with sample paths in $\mathbf{C}^m = \mathbf{C}^m(U)$ such that for any compact set $C \subset U$*

$$\mathbb{E}[\|\xi\|_{m,C}] < \infty.$$

Then the stochastic process

$$M_t(x) \triangleq \mathbb{E}[\xi(x)|\mathscr{F}_t], \quad 0 \le t \le T,\, x \in U,$$

has a modification with sample paths in $\mathbf{D}(\mathbf{C}^m, [0,T])$ and, for any multi-index $a = (a_1, \dots, a_d)$ of non-negative integers with $|a| \triangleq \sum_{i=1}^d a_i \le m$,

$$\partial^a M_t(x) = \mathbb{E}[\partial^a \xi(x)|\mathscr{F}_t], \quad 0 \le t \le T,\, x \in U,$$

where ∂^a is the differential operator of the order a with respect to x defined in (2.2).

Proof. By induction, it is sufficient to consider the cases $m = 0, 1$.

Assume first that $m = 0$. It is well-known that, for any $x \in U$, the martingale $M(x)$ has a modification in $\mathbf{D}(\mathbf{R}, [0,T])$. Fix a compact set $C \subset U$ and let $(x_i)_{i \ge 1}$ be a dense countable subset of C. Standard arguments show that the stochastic process $M : C \times [0,T] \to \mathbf{R}$ has a modification in $\mathbf{D}(\mathbf{C}, [0,T])$ if

$$\lim_{a\to\infty} \mathbb{P}[\sup_{x_i} (M(x_i))^*_T \ge a] = 0, \tag{4.19}$$

$$\lim_{\delta\to 0} \mathbb{P}[\sup_{|x_i - x_j| \le \delta} (M(x_i) - M(x_j))^*_T \ge \varepsilon] = 0, \text{ for any } \varepsilon > 0, \tag{4.20}$$

where we recall the notation $X^*_t \triangleq \sup_{0 \le s \le t} |X_s|$.

From the conditions on $\xi = \xi(x)$ we deduce that the martingales:

$$\begin{aligned} X_t &\triangleq \mathbb{E}[\sup_{x\in C} |\xi(x)||\mathscr{F}_t], \\ Y_t(\delta) &\triangleq \mathbb{E}[\sup_{|x_i - x_j| \le \delta} |\xi(x_i) - \xi(x_j)||\mathscr{F}_t], \end{aligned}$$

are well-defined and

$$\lim_{\delta\to 0} \mathbb{E}[Y_T(\delta)] = 0. \tag{4.21}$$

Since, clearly,

$$\sup_{x_i} |M_t(x_i)| \le X_t,$$
$$\sup_{|x_i - x_j| \le \delta} |M_t(x_i) - M_t(x_j)| \le Y_t(\delta),$$

we deduce from Doob's inequality:

$$\mathbb{P}[\sup_{x_i} (M(x_i))^*_T \ge a] \le \mathbb{P}[X^*_T \ge a] \le \frac{1}{a} \mathbb{E}[X_T],$$
$$\mathbb{P}[\sup_{|x_i - x_j| \le \delta} (M(x_i) - M(x_j))^*_T \ge \varepsilon] \le \mathbb{P}[(Y(\delta))^*_T \ge \varepsilon] \le \frac{1}{\varepsilon} \mathbb{E}[Y_T(\delta)],$$

which, jointly with (4.21), implies (4.19) and (4.20). This concludes the proof for the case $m = 0$.

Assume now that $m = 1$ and define the stochastic process

$$D_t(x) \triangleq \mathbb{E}[\nabla \xi(x) | \mathscr{F}_t] : \ U \times [0, T] \to \mathbf{L}^0(\mathbf{R}^d),$$

where $\nabla \triangleq (\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d})$ is the gradient operator. From the case $m = 0$ we obtain that the stochastic processes $M = M_t(x)$ and $D = D_t(x)$ have modifications in $\mathbf{D}(\mathbf{C}, [0, T])$, which we shall use. For $M = M_t(x)$ to have a modification in $\mathbf{D}(\mathbf{C}^1, [0, T])$ with the derivatives given by $D = D_t(x)$ we have to show that

$$\lim_{\delta \to 0} \mathbb{P}[\sup_{x \in C, |x - y| \le \delta} \frac{1}{\delta} (N(x, y))^*_T \ge \varepsilon] = 0, \text{ for any } \varepsilon > 0, \tag{4.22}$$

where

$$N(x, y) \triangleq M(y) - M(x) - \langle D(x), y - x \rangle .$$

We follow the same path as in the proof of the previous case. Our assumptions on $\xi = \xi(x)$ imply that, for sufficiently small $\delta > 0$, the martingale

$$Z_t(\delta) \triangleq \mathbb{E}[\sup_{x \in C, |x - y| \le \delta} \frac{1}{\delta} |\xi(y) - \xi(x) - \langle \nabla \xi(x), y - x \rangle | \, | \mathscr{F}_t],$$

is well-defined and

$$\lim_{\delta \to 0} \mathbb{E}[Z_T(\delta)] = 0. \tag{4.23}$$

Since

$$\sup_{x\in C,|x-y|\leq\delta}\frac{1}{\delta}N_t(x,y)\leq Z_t(\delta),$$

we have, by Doob's inequality,

$$\mathbb{P}[\sup_{x\in C,|x-y|\leq\delta}\frac{1}{\delta}(N(x,y))^*_T\geq\varepsilon]\leq\mathbb{P}[(Z(\delta))^*_T\geq\varepsilon]\leq\frac{1}{\varepsilon}\mathbb{E}[Z_T(\delta)],$$

and (4.22) follows from (4.23). $\square$

Lemma 4.6. *Let U be an open set in $\mathbf{R}^m$ (and, in addition, be a convex set for conditions (C3) and (C7) and a cone for (C4)). Let furthermore V be an open set in $\mathbf{R}^l$, and $\xi=\xi(x,y):\ U\times V\to\mathbf{L}^0$ be a random field with continuous sample paths such that for any compact set $C\subset U\times V$*

$$\mathbb{E}[\sup_{(x,y)\in C}|\xi(x,y)|]<\infty.$$

Then the stochastic process

$$M_t(x,y)\triangleq\mathbb{E}[\xi(x,y)|\mathscr{F}_t],\quad 0\leq t\leq T,\ x\in U,\ y\in V,$$

has a modification with sample paths in $\mathbf{D}(\mathbf{C}(U\times V),[0,T])$. Moreover, if the sample paths of ξ belong to $\widetilde{\mathbf{C}}$, then there is a modification of M with sample paths in $\mathbf{D}(\widetilde{\mathbf{C}},[0,T])$, where $\widetilde{\mathbf{C}}=\widetilde{\mathbf{C}}(U\times V)$ is any one of the following subspaces of $\mathbf{C}=\mathbf{C}(U\times V)$:

(C1) $\widetilde{\mathbf{C}}$ *consists of all non-negative functions;*

(C2) $\widetilde{\mathbf{C}}$ *consists of all functions $f=f(x,y)$ which are non-decreasing with respect to x;*

(C3) $\widetilde{\mathbf{C}}$ *consists of all functions $f=f(x,y)$ which are convex with respect to x;*

(C4) $\widetilde{\mathbf{C}}$ *consists of all functions $f=f(x,y)$ which are positively homogeneous with respect to x:*

$$f(cx,y)=cf(x,y),\quad c>0.$$

(C5) $\widetilde{\mathbf{C}}$ *consists of all strictly positive functions;*

(C6) $\widetilde{\mathbf{C}}$ *consists of all functions* $f = f(x, y)$ *which are strictly increasing with respect to* x*:*

$$f(x_1, y) < f(x_2, y), \quad x_1 \leq x_2, \ x_1 \neq x_2;$$

(C7) $\widetilde{\mathbf{C}}$ *consists of all functions* $f = f(x, y)$ *which are strictly convex with respect to* x*:*

$$\frac{1}{2}(f(x_1, y) + f(x_2, y)) > f(\frac{1}{2}(x_1 + x_2), y), \quad x_1 \neq x_2.$$

If, in addition, the random field ξ *is such that for any compact set* $D \subset V$

$$\mathbb{E}[\sup_{(x,y)\in U\times D} \xi(x, y)] < \infty, \tag{4.24}$$

then the assertion also holds for the following subspaces:

(C8) $\widetilde{\mathbf{C}}$ *consists of all non-negative functions* $f = f(x, y)$ *such that for any increasing sequence* $(C_n)_{n\geq 1}$ *of compact sets in* U *with* $\cup_{n\geq 1} C_n = U$ *and for any compact set* $D \subset V$

$$\lim_{n\to\infty} \sup_{x\in U/C_n} \sup_{y\in D} f(x, y) = 0;$$

(C9) $\widetilde{\mathbf{C}}$ *consists of all functions* $f = f(x, y)$ *such that for any increasing sequence* $(C_n)_{n\geq 1}$ *of compact sets in* U *with* $\cup_{n\geq 1} C_n = U$ *and for any compact set* $D \subset V$

$$\lim_{n\to\infty} \sup_{x\in U/C_n} \sup_{y\in D} f(x, y) = -\infty.$$

Proof. The existence of a modification for M with sample paths in $\mathbf{D}(\mathbf{C}, [0, T])$ has been proved in Lemma 4.5. Hereafter we shall use this modification.

The assertions of items (C1)–(C4) are straightforward, since for any $t \in [0, T]$ these conditions are obviously satisfied for the random field $M_t : U \to \mathbf{L}^0$ and the sample paths of M belong to $\mathbf{D}(\mathbf{C}, [0, T])$.

To verify (C5) recall the well-known fact that if N is a martingale on $[0, T]$ such that $N_T > 0$, then $\inf_{t\in[0,T]} N_t > 0$. For any compact set $C \subset U \times V$ we have by (C5) that $\inf_{(x,y)\in C} \xi(x, y) > 0$ and, hence,

$$\inf_{t\in[0,T]} \inf_{(x,y)\in C} M_t(x, y) \geq \inf_{t\in[0,T]} \mathbb{E}[\inf_{(x,y)\in C} \xi(x, y)|\mathscr{F}_t] > 0,$$

implying (C5). Observe that this argument clearly extends to the case, when U is an F_σ-set, that is, a countable union of closed sets.

The cases (C6) and (C7) follow from (C5) by re-parameterization. For example, to obtain (C6) define the set $\widetilde{U} \subset \mathbf{R}^{2m}$ and the random fields $\eta : \widetilde{U} \times V \to \mathbf{L}^0$ and $N : \widetilde{U} \times V \times [0,T] \to \mathbf{L}^0$ as

$$\begin{aligned} \widetilde{U} &\triangleq \{(x_1, x_2) : x_i \in U,\ x_1 \leq x_2,\ x_1 \neq x_2\}, \\ \eta(x_1, x_2, y) &\triangleq \xi(x_2, y) - \xi(x_1, y), \\ N_t(x_1, x_2, y) &\triangleq M_t(x_2, y) - M_t(x_1, y). \end{aligned}$$

While the set $\widetilde{U}$ is not open, it is an F_σ-set. An application of (C5) to η and N then yields (C6) for ξ and M.

For the proof of (C8) recall that by Doob's inequality, if $(N^n)_{n\geq 1}$ is a sequence of martingales such that $N^n_T \to 0$ in $\mathbf{L}^1$, then $(N^n)^*_T \triangleq \sup_{0\leq t\leq T}|N^n_t| \to 0$ in $\mathbf{L}^0$. Accounting for (4.24) we deduce that, for the compact sets $(C_n)_{n\geq 1}$ and D as in (C8),

$$\lim_{n\to\infty} \mathbb{E}[\sup_{x\in U/C_n} \sup_{y\in D} \xi(x,y)] = 0.$$

The validity of (C8) for the sample paths of M follows now from

$$\sup_{x\in U/C_n} \sup_{y\in D} (M(x,y))^*_T \leq \sup_{0\leq t\leq T} \mathbb{E}[\sup_{x\in U/C_n} \sup_{y\in D} \xi(x,y) | \mathscr{F}_t],$$

where we used the fact that in (C8) $\xi \geq 0$.

Finally, (C9) follows from (C8) if we observe that a function $f = f(x,y)$ satisfies (C9) if and only if for any positive integer n the function

$$g_n(x,y) \triangleq \max(f(x,y) + n, 0), \quad (x,y) \in U \times V,$$

satisfies (C8). $\square$

The following result links the condition (F6) used in the definition of the space $\widetilde{\mathbf{F}}^1$ with the condition (C9) in Lemma 4.6.

Lemma 4.7. *Let $M > 1$. A function $f \in \mathbf{F}^1$ satisfies* (F6) *(that is, belongs to $\widetilde{\mathbf{F}}^1$) if and only if for any increasing sequence $(C_n)_{n\geq 1}$ of compact sets in $\mathbf{S}^M$ with $\cup_{n\geq 1} C_n = \mathbf{S}^M$ and for any compact set $D \subset \mathbf{R}^{1+J}$*

$$\lim_{n\to\infty} \sup_{w\in \mathbf{S}^M/C_n} \sup_{(x,q)\in D} \sum_{m=1}^M \frac{\partial f}{\partial v^m}(w,x,q) = -\infty. \tag{4.25}$$

Proof. The "if" statement is straightforward. Hereafter we shall focus on the opposite implication.

To verify (4.25) we have to show that for $f \in \widetilde{\mathbf{F}}^1$ and any $a_n = (w_n, x_n, q_n) \in \mathbf{S}^M \times \mathbf{R} \times \mathbf{R}^J$, $n \geq 1$, converging to $(w, x, q) \in \partial \mathbf{S}^M \times \mathbf{R} \times \mathbf{R}^J$ we have

$$\lim_{n\to\infty} \sum_{m=1}^{M} \frac{\partial f}{\partial v^m}(a_n) = \lim_{n\to\infty} \left\langle \frac{\partial f}{\partial v}(a_n), \mathbf{1} \right\rangle = -\infty, \tag{4.26}$$

where $\mathbf{1} \triangleq (1, \dots, 1)$.

Let $\varepsilon > 0$. Accounting for the convexity and the positive homogeneity of the functions $f(\cdot, x_n, q_n)$, $n \geq 1$, on $(0, \infty)^M$ we deduce

$$\begin{aligned} \lim_{n\to\infty} \left\langle \frac{\partial f}{\partial v}(a_n), \mathbf{1} \right\rangle &\leq \lim_{n\to\infty} \left\langle \frac{\partial f}{\partial v}(w_n + \varepsilon\mathbf{1}, x_n, q_n), \mathbf{1} \right\rangle \\ &= \left\langle \frac{\partial f}{\partial v}(w + \varepsilon\mathbf{1}, x, q), \mathbf{1} \right\rangle = \left\langle \frac{\partial f}{\partial v}(w(\varepsilon), x, q), \mathbf{1} \right\rangle, \end{aligned}$$

where $w(\varepsilon) \triangleq \frac{w+\varepsilon\mathbf{1}}{1+\varepsilon M}$ belongs to $\mathbf{S}^M$. By (F6), the passage to the limit when $\varepsilon \to 0$ yields (4.26). $\square$

After these preparations we are ready to proceed with the proof of Theorem 4.3.

Proof of Theorem 4.3. The inequality (4.14) and the fact that the sample paths of $F(\cdot, T)$ belong to $\widetilde{\mathbf{F}}^1$ have been established in [1], Theorem 4.7, which is a single-period version of Theorem 4.3. Lemma 4.5 then implies that the stochastic process F defined in (4.8) has its sample paths in $\mathbf{D}(\mathbf{C}^1(\mathbf{A}), [0, T])$ and that the equality (4.15) holds.

To verify that the sample paths of F belong to $\mathbf{D}(\mathbf{F}^1, [0, T])$ it is sufficient to match the properties (F1)–(F4) in the description of $\mathbf{F}^1$ with the properties (C1)–(C8) in Lemma 4.6. For the most part these correspondences are straightforward with the links between (4.11) in (F2) or (4.12) in (F4) and their respective versions of (C8) holding due to the equivalence of the pointwise and the uniform on compact sets convergences for a sequence of convex or saddle functions.

Note that in order to use (C8) in Lemma 4.6 we still have to verify the integrability condition (4.24). The adaption of this integrability condition to (4.11) in (F2) has the form:

$$\mathbb{E}[\inf_{w\in\mathbf{S}^M} \inf_{(x,q)\in D} F(w, x, q, T)] > -\infty,$$

for any compact set $D \subset \mathbf{R}^{1+J}$. This inequality holds due to (4.14) and since the sample paths of $F(\cdot, T)$ are decreasing with respect to v (hence, $F(w, x, q, T) > F(\mathbf{1}, x, q, T)$, $w \in \mathbf{S}^M$, where $\mathbf{1} \triangleq (1, \dots, 1)$).

To perform a similar verification for the convergence (4.12) in (F4) we restrict the domain of x to $[0, \infty)$. The analog of (4.24) then has the form:

$$\mathbb{E}[\inf_{x \geq 0} \inf_{(v,q) \in D} F(v, x, q, T)] > -\infty,$$

for any compact set $D \subset (0, \infty)^M \times \mathbf{R}^J$, and follows from (4.14) and the monotonicity of F with respect to x.

Finally, the connection between (C9) and (F6) has been established in Lemma 4.7. The adaptation of (4.24) to this case holds trivially as $\frac{\partial F}{\partial v} < 0$. □

For a two-times continuously differentiable $f = f(a) : \mathbf{A} \to \mathbf{R}$ recall the notation $A(f)$ for the matrix defined in (4.13). For the proof of Theorem 4.4 we have to verify (F8) for the matrices $A(F_t)$, $t \in [0, T]$.

Towards the end of this (sub)section we shall work under Assumptions 3.1, 3.2, and 3.4. According to [1], Theorem 4.13, these conditions imply (4.17). From Lemma 4.5 we then deduce the existence of a modification for $F = F(a, t)$ with sample paths in $\mathbf{D}(\mathbf{C}^2(\mathbf{A}), [0, T])$. In the future we shall use this modification.

Following Section 4.3 in [1], for $a \in \mathbf{A}$, define the probability measure $\mathbb{R}(a)$ with

$$\frac{d\mathbb{R}(a)}{d\mathbb{P}} \triangleq \frac{\partial^2 F}{\partial x^2}(a, T) / \frac{\partial^2 F}{\partial x^2}(a, 0), \tag{4.27}$$

the stochastic process

$$R_t(a) \triangleq -\frac{\partial F}{\partial x}(a, t) / \frac{\partial^2 F}{\partial x^2}(a, t), \quad 0 \leq t \leq T, \tag{4.28}$$

and the random variable $\tau(a) \in \mathbf{L}^0(\mathbf{R}^M)$:

$$\tau^m(a) \triangleq t_m(\pi^m(a)), \quad m = 1, \dots, M, \tag{4.29}$$

where $t_m = t_m(x)$ is the absolute risk-aversion of $u_m = u_m(x)$:

$$t_m(x) \triangleq -\frac{u'_m(x)}{u''_m(x)} = \frac{1}{a_m(x)}, \quad x \in \mathbf{R}.$$

Lemma 4.8. *Under Assumptions 3.1, 3.2, and 3.4, for $t \in [0,T]$ and $a \in \mathbf{A}$, the matrix $A(F_t)(a)$ is given by, for $l,m = 1,\dots,M$,*

$$\begin{aligned} A^{lm}(F_t)(a) = \frac{1}{R_t(a)} \mathbb{E}_{\mathbb{R}(a)}[\tau^l(a)(\delta_{lm} \sum_{k=1}^{M} \tau^k(a) - \tau^m(a))|\mathscr{F}_t] \\ + \frac{1}{R_t(a)} \mathbb{E}_{\mathbb{R}(a)}[\tau^l(a)|\mathscr{F}_t] \mathbb{E}_{\mathbb{R}(a)}[\tau^m(a)|\mathscr{F}_t], \end{aligned}$$

where the probability measure $\mathbb{R}(a)$, the stochastic process $R(a)$, and the random variable $\tau(a)$ are defined in (4.27), (4.28), *and* (4.29), *respectively.*

Moreover, for any $z \in \mathbf{R}^n$,

$$\frac{1}{c}|z|^2 \le \langle z, A(F_t)(a)z \rangle \le c|z|^2,$$

where the constant $c > 0$ is given in Assumption 3.2.

Proof. The proof is, essentially, a word-by-word reproduction of the proofs of Lemmas 4.14 and 4.15 from [1] obtained for the case $t = 0$. All we have to do is to replace $\mathbb{E}[\cdot]$ and R_0 there with $\mathbb{E}[\cdot|\mathscr{F}_t]$ and R_t. □

Proof of Theorem 4.4. The inequality (4.17) and the fact that the sample paths of $F(\cdot,T)$ belong to $\widetilde{\mathbf{F}}^2(c)$ have been established in [1], Theorem 4.13. From Lemma 4.5 we then deduce that $F = F(a,t)$ has sample paths in $\mathbf{D}(\mathbf{C}^2(\mathbf{A}),[0,T])$ and that (4.18) holds. The rest of the proof is an easy consequence of Lemma 4.6 if we account for the properties of the sample paths for $F(\cdot,T)$ and use Lemma 4.8. □

4.3 Stochastic process of cash balances

For ease of notation denote

$$\mathbf{B} \triangleq (-\infty,0)^M \times (0,\infty) \times \mathbf{R}^J.$$

We shall often decompose $b \in \mathbf{B}$ as $b = (u,y,q)$, where $u \in (-\infty,0)^M$, $y \in (0,\infty)$, and $q \in \mathbf{R}^J$. In our economy, u will denote the *indirect utilities* of the market makers, y will stand for the *marginal utility* of the representative market maker, and q will define the collective quantities of the contingent claims ψ accumulated by the market makers as the result of trading with the large investor.

In addition to the random field $F = F(a,t)$ of (4.8) studied in the previous section an important role in the analysis of investment strategies for the large investor is played by the stochastic process

$$G = G(b,t) : \mathbf{B} \times [0,T] \to \mathbf{L}^0(\mathbf{R}),$$

which is conjugate to $F = F(a,t)$ in the sense that, for $b = (u,y,q) \in \mathbf{B}$ and $t \in [0,T]$,

$$G(b,t) \triangleq \sup_{v \in (0,\infty)^M} \inf_{x \in \mathbf{R}} [\langle v, u \rangle + xy - F(v,x,q,t)]. \tag{4.30}$$

In Theorem 4.9 we shall show that such a stochastic process is well-defined and we shall describe the properties of its sample paths. As pointed out in Remark 4.15 below, the random variable $G(u,1,q,t)$ represents the *total cash amount* that the market makers have accumulated up to time t as the result of trading with the large investor, given that their indirect utilities are at the level $u \in (-\infty,0)^M$ and that they acquired in total $q \in \mathbf{R}^J$ of the contingent claims ψ.

To describe the sample paths of $G = G(b,t)$ we shall use the spaces of saddle functions from [1], Section 3.2. Fix a constant $c > 0$, used below in (G7)–(G9), and for a function

$$g : \mathbf{B} \to \mathbf{R} \tag{4.31}$$

define the following conditions:

(G1) The function g is continuously differentiable on $\mathbf{B}$.

(G2) For any $(y,q) \in (0,\infty) \times \mathbf{R}^J$, the function $g(\cdot,y,q)$ is strictly increasing and strictly convex on $(-\infty,0)^M$. Moreover,

(a) If $(u_n)_{n\geq 1}$ is a sequence in $(-\infty,0)^M$ converging to 0, then

$$\lim_{n\to\infty} g(u_n,y,q) = \infty.$$

(b) If $(u_n)_{n\geq 1}$ is a sequence in $(-\infty,0)^M$ converging to a boundary point of $(-\infty,0)^M$, then

$$\lim_{n\to\infty} |\frac{\partial g}{\partial u}(u_n,y,q)| = \infty.$$

(c) If $(u_n)_{n\geq 1}$ is a sequence in $(-\infty,0)^M$ such that

$$\limsup_{n\to\infty} u_n^m < 0 \text{ for all } m = 1,\dots,M$$

and

$$\lim_{n\to\infty} u_n^{m_0} = -\infty \text{ for some } m_0 \in \{1,\dots,M\},$$

then

$$\lim_{n\to\infty} g(u_n,y,q) = -\infty.$$

(G3) For any $y \in (0,\infty)$, the function $g(\cdot,y,\cdot)$ is convex on $(-\infty,0)^M \times \mathbf{R}^J$.

(G4) For any $(u,q) \in (-\infty,0)^M \times \mathbf{R}^J$, the function $g(u,\cdot,q)$ is positively homogeneous, that is,

$$g(u,y,q) = yg(u,1,q), \quad y > 0. \tag{4.32}$$

(G5) The function g is twice continuously differentiable on $\mathbf{B}$ and, for any $b = (u,y,q) \in \mathbf{B}$, the matrix $B(g)(b) = (B^{lm}(g)(b))_{l,m=1,\dots,M}$ given by

$$B^{lm}(g)(b) \triangleq \frac{y}{\frac{\partial g}{\partial u^l}\frac{\partial g}{\partial u^m}} \frac{\partial^2 g}{\partial u^l \partial u^m}(b) \tag{4.33}$$

has full rank.

(G6) For any $(y,q) \in (0,\infty) \times \mathbf{R}^J$ and any sequence $(u_n)_{n\geq 1}$ in $(-\infty,0)^M$ converging to a boundary point of $(-\infty,0)^M$

$$\lim_{n\to\infty} g(u_n,y,q) = \infty.$$

(G7) For any $(u,q) \in (-\infty,0)^M \times \mathbf{R}^J$ and $m = 1,\dots,M$,

$$\frac{1}{c} \leq -u^m \frac{\partial g}{\partial u^m}(u,1,q) \leq c.$$

(G8) For any $b \in \mathbf{B}$ and any $z \in \mathbf{R}^M$,

$$\frac{1}{c}\langle z,z\rangle \leq \langle z, B(g)(b)z\rangle \leq c\langle z,z\rangle,$$

where the matrix $B(g)(b)$ is defined in (4.33).

(G9) For any $(u,q) \in (-\infty,0)^M \times \mathbf{R}^J$, the vector $z \in \mathbf{R}^M$ solving the linear equation:

$$B(g)(u,1,q)z = \mathbf{1},$$

satisfies

$$\frac{1}{c} \le z^m \le c, \quad m = 1,\dots,M.$$

Following [1], Section 3.2, we define the families of functions

$$\begin{aligned}
\mathbf{G}^1 &\triangleq \{g \text{ as in } (4.31) : \text{ (G1)–(G4) hold}\}, \\
\mathbf{G}^2 &\triangleq \{g \in \mathbf{G}^1 : \text{ (G5) holds}\}.
\end{aligned}$$

We also denote

$$\widetilde{\mathbf{G}}^1 \triangleq \{g \in \mathbf{G}^1 : \text{ (G6) holds}\},$$

and, for $c > 0$,

$$\widetilde{\mathbf{G}}^2(c) \triangleq \{g \in \mathbf{G}^2 : \text{ (G6)–(G9) hold for given } c\}.$$

As in the case with the spaces $\mathbf{F}^i$, $i = 1,2$, we shall view $\mathbf{G}^i$, $i = 1,2$, as the topological subspaces of the corresponding Fréchet spaces $\mathbf{C}^i(\mathbf{B})$. A similar convention will also be used for $\widetilde{\mathbf{G}}^1$ and $\widetilde{\mathbf{G}}^2(c)$.

Theorem 4.9. *Under Assumptions 3.1 and 3.4 the stochastic process $G = G(b,t)$ is well-defined by* (4.30) *and has sample paths in $\mathbf{D}(\widetilde{\mathbf{G}}^1,[0,T])$. If, in addition, Assumption 3.2 holds, then its sample paths belong to $\mathbf{D}(\widetilde{\mathbf{G}}^2(c),[0,T])$ with the same constant $c > 0$ as in Assumption 3.2.*

In view of Theorems 4.3 and 4.4, the proof of Theorem 4.9 is reduced to the study of the conjugacy relations between the spaces $\mathbf{F}$ (standing for $\mathbf{F}^1$, $\widetilde{\mathbf{F}}^1$, $\mathbf{F}^2$, and $\widetilde{\mathbf{F}}^2(c)$) and $\mathbf{G}$ (standing, respectively, for $\mathbf{G}^1$, $\widetilde{\mathbf{G}}^1$, $\mathbf{G}^2$, and $\widetilde{\mathbf{G}}^2(c)$). Many of these relations have been already established in [1], Section 3, and will be recalled below. For the proof of the RCLL structure of the sample paths of G we shall also require the continuity property of the conjugacy operator (4.34) below, mapping $\mathbf{F}$ onto $\mathbf{G}$, with respect to $\mathbf{C}^1$ or $\mathbf{C}^2$ topologies. This will be accomplished in Theorem 4.11.

For convenience of future references we begin by recalling some basic results from [1], Section 3, concerning the conjugacy relations between the spaces $\mathbf{F}$ and $\mathbf{G}$. We have that a function f is in $\mathbf{F}$ if and only if there exists $g \in \mathbf{G}$ such that, for any $b = (u, y, q) \in \mathbf{B}$,

$$g(b) = \sup_{v \in (0,\infty)^M} \inf_{x \in \mathbf{R}} [\langle v, u\rangle + xy - f(v, x, q)]. \tag{4.34}$$

The minimax value in (4.34) is attained at the unique saddle point (v, x) and, for $q \in \mathbf{R}^J$, the following relationships between $a = (v, x, q) \in \mathbf{A}$ and $b = (u, y, q) \in \mathbf{B}$ are equivalent:

1. Given (u, y), the minimax value in (4.34) is attained at (v, x).
2. We have $x = \frac{\partial g}{\partial y}(b) = g(u, 1, q)$ and $v = \frac{\partial g}{\partial u}(b)$.
3. We have $y = \frac{\partial f}{\partial x}(a)$ and $u = \frac{\partial f}{\partial v}(a)$.

Moreover, in this case, $f(a) = \langle u, v\rangle$ and $g(b) = xy$,

$$\frac{\partial g}{\partial q}(b) = -\frac{\partial f}{\partial q}(a),$$

the matrices $A(f)(a)$ and $B(g)(b)$, defined in (4.13) and (4.33), are inverse to each other:

$$B(g)(b) = (A(f)(a))^{-1}, \tag{4.35}$$

and the following matrices of second derivatives for f and g involving the differentiation with respect to q:

$$C^{mj}(f)(a) \triangleq \frac{v^m}{\frac{\partial f}{\partial x}} \left(\frac{\partial^2 f}{\partial v^m \partial q^j} - \frac{1}{\frac{\partial^2 f}{\partial x^2}} \frac{\partial^2 f}{\partial v^m \partial x} \frac{\partial^2 f}{\partial x \partial q^j} \right)(a), \tag{4.36}$$

$$D^{ij}(f)(a) \triangleq \frac{1}{\frac{\partial f}{\partial x}} \left(-\frac{\partial^2 f}{\partial q^i \partial q^j} + \frac{1}{\frac{\partial^2 f}{\partial x^2}} \frac{\partial^2 f}{\partial x \partial q^i} \frac{\partial^2 f}{\partial x \partial q^j} \right)(a), \tag{4.37}$$

and

$$E^{mj}(g)(b) \triangleq \frac{1}{\frac{\partial g}{\partial u^m}} \frac{\partial^2 g}{\partial u^m \partial q^j}(b) = \frac{1}{\frac{\partial g}{\partial u^m}} \frac{\partial^2 g}{\partial u^m \partial q^j}(u, 1, q), \tag{4.38}$$

$$H^{ij}(g)(b) \triangleq \frac{1}{y} \frac{\partial^2 g}{\partial q^i \partial q^j}(b) = \frac{\partial^2 g}{\partial q^i \partial q^j}(u, 1, q), \tag{4.39}$$

where $m = 1, \dots, M$ and $i, j = 1, \dots, J$, are related by

$$E(g)(b) = -(A(f)(a))^{-1}C(f)(a), \tag{4.40}$$
$$H(g)(b) = (C(f)(a))^T(A(f)(a))^{-1}C(f)(a) + D(f)(a). \tag{4.41}$$

In (4.38) and (4.39) we used the positive homogeneity property (4.32) of g with respect to y. Recall that for a matrix B the notations B^T and B^{-1} stand for its transpose and inverse, respectively.

Remark 4.10. The choice of the specific form for the matrices $A(f)(a)$, $C(f)(a)$, and $D(f)(a)$ and $B(g)(b)$, $E(g)(b)$, and $H(g)(b)$ was motivated by the fact that they remain invariant under the transformations: $(v, x, q) \mapsto (cv, x, q)$ and $(u, y, q) \mapsto (u, cy, q)$, $c > 0$, a natural requirement in light of the positive homogeneity conditions (4.10) and (4.32).

We remind the reader that, for $i = 1, 2$, the convergences in the spaces $\mathbf{F}^i$ and $\mathbf{G}^i$ are equivalent to the convergences in the corresponding Fréchet spaces $\mathbf{C}^i(\mathbf{A})$ and $\mathbf{C}^i(\mathbf{B})$ defined in Section 2.

Theorem 4.11. *Let $(f_n)_{n\geq 1}$ and f belong to $\mathbf{F}^1$ and $(g_n)_{n\geq 1}$ and g be their conjugate counterparts from $\mathbf{G}^1$. Then $(f_n)_{n\geq 1}$ converges to f in $\mathbf{F}^1$ if and only if $(g_n)_{n\geq 1}$ converges to g in $\mathbf{G}^1$.*

Proof. Recall that for convex or saddle functions the convergence in $\mathbf{C}^1$ is equivalent to the pointwise convergence. We also remind the reader that the conjugacy operations, as in (4.34), are, in general, not continuous under this convergence and, hence, the result does not hold automatically. A standard verification method in this case is to show the equivalence of the pointwise convergence and the *epi*-convergence (or its analogs such as *epi-hypo*-convergence), under which the conjugacy operations are continuous; see Rockafellar and Wets [6], Theorem 11.34. We find it simpler to give a direct argument.

Assume first that $(f_n)_{n\geq 1}$ converges to f in $\mathbf{F}^1$ (equivalently, in $\mathbf{C}^1(\mathbf{A})$). By the positive homogeneity condition (G4) and because they are saddle functions, it is sufficient to verify the pointwise convergence for $(g_n)_{n\geq 1}$ at $b = (u, y, q) \in \mathbf{B}$ with $y = 1$. Fix $\varepsilon > 0$ and find $u_i \in (-\infty, 0)^M$, $i = 1, 2$, such that $u_1 < u < u_2$ and

$$|g(b_2) - g(b_1)| < \varepsilon, \tag{4.42}$$

where $b_i \triangleq (u_i, 1, q)$. Denote, for $i = 1, 2$,

$$a_i = (v_i, x_i, q) \triangleq (\frac{\partial g}{\partial u}(b_i), g(b_i), q),$$

and, for $n \geq 1$,

$$b_{i,n} = (u_{i,n}, 1, q) \triangleq (\frac{\partial f_n}{\partial v}(a_i), 1, q).$$

The conjugacy relations between f_n and g_n and between f and g imply that $g_n(b_{i,n}) = x_i$ and $u_i = \frac{\partial f}{\partial v}(a_i)$. From the $\mathbf{C}^1$-convergence of $(f_n)_{n\geq 1}$ to f we deduce

$$\lim_{n\to\infty} u_{i,n} = \lim_{n\to\infty} \frac{\partial f_n}{\partial v}(a_i) = \frac{\partial f}{\partial v}(a_i) = u_i, \quad i = 1, 2,$$

and, hence, there is $n_0 > 1$ such that $u_{1,n} < u < u_{2,n}$ for $n \geq n_0$. Accounting for the monotonicity of the elements of $\mathbf{G}^1$ with respect to u we obtain

$$\begin{gathered} g(b_1) < g(b) < g(b_2), \\ g(b_1) = g_n(b_{1,n}) < g_n(b) < g_n(b_{2,n}) = g(b_2), \quad n \geq n_0, \end{gathered}$$

and then (4.42) yields

$$|g_n(b) - g(b)| < \varepsilon, \quad n \geq n_0,$$

thus proving the pointwise, hence, also the $\mathbf{G}^1$, convergence of $(g_n)_{n\geq 1}$ to g.

Assume now that $(g_n)_{n\geq 1}$ converges to g in $\mathbf{G}^1$ (equivalently, in $\mathbf{C}^1(\mathbf{B})$). We follow the same path as in the proof of the previous implication. Fix $\varepsilon > 0$, take $a = (v, x, q) \in \mathbf{A}$ and let $a_i = (v_i, x_i, q) \in \mathbf{A}$, $i = 1, 2$, be such that $v_1 > v > v_2$, $x_1 < x < x_2$, and

$$|f(a_2) - f(a_1)| < \varepsilon. \tag{4.43}$$

Denote, for $i = 1, 2$,

$$b_i = (u_i, y_i, q) \triangleq (\frac{\partial f}{\partial v}(a_i), \frac{\partial f}{\partial x}(a_i), q),$$

and, for $n \geq 1$,

$$a_{i,n} = (v_{i,n}, x_{i,n}, q) \triangleq (\frac{\partial g_n}{\partial u}(b_i), \frac{\partial g_n}{\partial y}(b_i), q).$$

From the conjugacy relations between f_n and g_n and between f and g we deduce that $f_n(a_{i,n}) = \langle u_i, v_{i,n}\rangle$, $f(a_i) = \langle u_i, v_i\rangle$, and $a_i = (\frac{\partial g}{\partial u}(b_i), \frac{\partial g}{\partial y}(b_i), q)$. As the $\mathbf{C}^1$-convergence of $(g_n)_{n\geq 1}$ to g implies the convergence of $(a_{i,n})_{n\geq 1}$ to a_i, there is $n_0 > 1$ such that, for $n \geq n_0$, $v_{1,n} > v > v_{2,n}$, $x_{1,n} < x < x_{2,n}$, and $|\langle u_i, v_{i,n}\rangle - \langle u_i, v_i\rangle| < \varepsilon$. Accounting for the monotonicity of the elements of $\mathbf{F}^1$ with respect to v and x, we deduce

$$
\begin{gathered}
f(a_1) < f(a) < f(a_2), \\
f(a_1) - \varepsilon < f_n(a_{1,n}) < f_n(a) < f_n(a_{2,n}) < f(a_2) + \varepsilon, \quad n \geq n_0,
\end{gathered}
$$

and, then, (4.43) implies

$$
|f_n(a) - f(a)| < 2\varepsilon, \quad n \geq n_0,
$$

proving the pointwise (hence, also $\mathbf{F}^1$) convergence of $(f_n)_{n\geq 1}$ to f. $\square$

Theorem 4.12. *Let $(f_n)_{n\geq 1}$ and f belong to $\mathbf{F}^2$ and $(g_n)_{n\geq 1}$ and g be their conjugate counterparts from $\mathbf{G}^2$. Then $(f_n)_{n\geq 1}$ converges to f in $\mathbf{F}^2$ if and only if $(g_n)_{n\geq 1}$ converges to g in $\mathbf{G}^2$.*

For the proof we need the following elementary

Lemma 4.13. *Let $f \in \mathbf{F}^2$. Then, for any $a = (v, x, q) \in \mathbf{A}$,*

$$
\sum_{m=1}^{M} C^{mj}(f)(a) = \left(\frac{1}{\frac{\partial f}{\partial x}} \frac{\partial f}{\partial q^j} - \frac{1}{\frac{\partial^2 f}{\partial x^2}} \frac{\partial^2 f}{\partial q^j \partial x} \right)(a), \quad j = 1, \ldots, J, \tag{4.44}
$$

where the matrix $C(f)$ is defined in (4.36).

Proof. From the positive homogeneity property (4.10) for f we deduce

$$
f(a) = \sum_{m=1}^{M} v^m \frac{\partial f}{\partial v^m}(a),
$$

which, in turn, implies

$$
\begin{aligned}
\frac{\partial f}{\partial x}(a) &= \sum_{m=1}^{M} v^m \frac{\partial^2 f}{\partial v^m \partial x}(a), \\
\frac{\partial f}{\partial q^j}(a) &= \sum_{m=1}^{M} v^m \frac{\partial^2 f}{\partial v^m \partial q^j}(a), \quad j = 1, \ldots, J,
\end{aligned}
$$

yielding (4.44). $\square$

Proof of Theorem 4.12. In view of Theorem 4.11, we only have to establish the uniform on compact sets convergences of second derivatives. Recall the notations $A(f)$, $C(f)$, and $D(f)$, for the matrices defined in (4.13), (4.36) and (4.37), and $B(g)$, $E(g)$, and $F(g)$, for the matrices defined in (4.33), (4.38), and (4.39).

Assume first that $(f_n)_{n\geq 1}$ converges to f in $\mathbf{F}^2$ or, equivalently, in $\mathbf{C}^2(\mathbf{A})$. Let $(b_n)_{n\geq 1}$ be a sequence in $\mathbf{B}$ that converges to $b \in \mathbf{B}$. By Theorem 4.11, the sequence $a_n \triangleq (\frac{\partial g_n}{\partial u}(b_n), \frac{\partial g_n}{\partial y}(b_n), q_n)$, $n \geq 1$, converges to $a \triangleq (\frac{\partial g}{\partial u}(b), \frac{\partial g}{\partial y}(b), q)$. The convergence of $(f_n)_{n\geq 1}$ to f in $\mathbf{F}^2$ then implies the convergence of the matrices $((A(f_n), C(f_n), D(F_n))(a_n))_{n\geq 1}$, to $(A(f), C(f), D(f))(a)$. By the identities (4.35), (4.40), and (4.41), this implies the convergence of the matrices $((B, E, F)(g_n)(b_n))_{n\geq 1}$, to $(B, E, F)(g)(b)$, which, by the construction of these matrices, yields the convergence of all second derivatives of g_n at b_n, $n \geq 1$, to the corresponding second derivatives of g at b. This, clearly, implies the uniform on compact sets convergence of the second derivatives of $(g_n)_{n\geq 1}$ to g.

Similar arguments show that the $\mathbf{G}^2$-convergence of $(g_n)_{n\geq 1}$ to g implies that for any sequence $(a_n)_{n\geq 1}$ in $\mathbf{A}$ converging to $a \in \mathbf{A}$ the matrices $((A(f_n), C(f_n), D(F_n))(a_n))_{n\geq 1}$ converge to $(A(f), C(f), D(f))(a)$. This, in turn, implies the convergence of the second derivatives of f_n at a_n, $n \geq 1$, to the second derivatives of f at a if we account for the identity (4.44) for the matrix $C(f)$ and the following equalities for the matrix $A(f)$, see Lemma 3.1 in [1]:

$$\sum_{m=1}^{M} A^{lm}(f) = -\frac{v^l}{\frac{\partial^2 f}{\partial x^2}} \frac{\partial^2 f}{\partial v^l \partial x}, \quad l = 1, \ldots, M,$$
$$\sum_{l,m=1}^{M} A^{lm}(f) = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial^2 f}{\partial x^2}}.$$

□

After these preparations we are ready to prove Theorem 4.9.

Proof of Theorem 4.9. The fact that for every $t \in [0, T]$ the sample paths of the random field $G(\cdot, t) : \mathbf{B} \to \mathbf{L}^0(\mathbf{R}, \mathscr{F}_t)$ belong to $\widetilde{\mathbf{G}}^1$ is a corollary of the aforementioned conjugate relations between the spaces $\widetilde{\mathbf{G}}^1$ and $\widetilde{\mathbf{F}}^1$ and the properties of the sample paths for the random field $F(\cdot, t) : \mathbf{A} \to \mathbf{L}^0(\mathbf{R}, \mathscr{F}_t)$

established in Theorem 4.3. The RCLL properties of the sample paths of the stochastic process $G: [0,T] \to \mathbf{L}^0(\widetilde{\mathbf{G}}^1)$ follow from the RCLL properties of the stochastic process $F: [0,T] \to \mathbf{L}^0(\widetilde{\mathbf{F}}^1)$ and Theorem 4.11.

The assertions regarding the sample paths of G under Assumption 3.2 are similar consequences of the conjugate relations between the spaces $\widetilde{\mathbf{G}}^2(c)$ and $\widetilde{\mathbf{F}}^2(c)$ and Theorems 4.4 and 4.12. $\square$

To facilitate future references we conclude the section with the following direct corollary of Theorem 4.9 and the aforementioned conjugacy relations between $\mathbf{F}^1$ and $\mathbf{G}^1$.

Corollary 4.14. *Let Assumptions 3.1 and 3.4 hold, $w \in \mathbf{S}^M$, $x \in \mathbf{R}$, $q \in \mathbf{R}^J$, and $u \in (-\infty, 0)^M$. Denote $a = (w, x, q)$ and $b = (u, 1, q)$. Then, for any $t \in [0,T]$ we have the identities:*

$$
\begin{aligned}
w &= \frac{\frac{\partial G}{\partial u}(\frac{\partial F}{\partial v}(a,t), 1, q, t)}{\sum_{m=1}^M \frac{\partial G}{\partial u^m}(\frac{\partial F}{\partial v}(a,t), 1, q, t)}, \\
x &= G(\frac{\partial F}{\partial v}(a,t), 1, q, t), \\
u &= \frac{\partial F}{\partial v}(\frac{\partial G}{\partial u}(b,t), G(b,t), q, t) \\
&= \frac{\partial F}{\partial v}\left(\frac{\frac{\partial G}{\partial u}(b,t)}{\sum_{m=1}^M \frac{\partial G}{\partial u^m}(b,t)}, G(b,t), q, t\right).
\end{aligned}
$$

Remark 4.15. Recall that, according to (4.16), $\frac{\partial F}{\partial v^m}(a,t)$ represents the *indirect* or expected utility of market maker m at time t given the Pareto allocation $\pi(a)$. Hence, by the second identity in Corollary 4.14, the random variable $G(u, 1, q, t)$ defines the *collective cash amount* of the market makers at time t when their current indirect utilities are given by u and they jointly own q stocks.

5 Continuous-time strategies

We proceed now with the main topic of the paper, which is the construction of trading strategies with general continuous-time dynamics. Recall that the key economic assumption of our model is that the agent can rebalance his portfolio *without changing the expected utilities of the market makers.*

5.1 Simple strategies revisited

To facilitate the transition from the discrete evolution in Section 3.2 to the continuous dynamics below we begin by revisiting the case of a simple strategy

$$Q_t = \sum_{n=1}^{N} \theta_n 1_{(\tau_{n-1},\tau_n]}(t), \quad 0 \leq t \leq T, \tag{5.1}$$

with stopping times $0 = \tau_0 \leq \ldots \leq \tau_N = T$ and random variables $\theta_n \in \mathbf{L}^0(\mathscr{F}_{\tau_{n-1}}, \mathbf{R}^J)$, $n = 1, \ldots, N$.

The following result is an improvement over Theorem 3.7 in the sense that the forward induction for cash balances and Pareto optimal allocations is now made explicit through the use of the parameterization $\pi = \pi(a)$ of Pareto allocations from (4.2) and the stochastic processes $F = F(a,t)$ and $G = G(b,t)$ defined in (4.8) and (4.30).

Denote by $\lambda_0 \in \mathbf{S}^M$ the weight of the initial Pareto allocation α_0. This weight is uniquely determined by Theorem 4.1.

Theorem 5.1. *Let Assumptions 3.1 and 3.4 hold and consider a simple strategy Q given by (5.1). Then the sequence of conditionally Pareto optimal allocations $(\alpha_n)_{n=0,\ldots,N}$ constructed in Theorem 3.7 takes the form*

$$\alpha_n = \pi(\zeta_n), \quad n = 0, \ldots, N, \tag{5.2}$$

where $\zeta_0 \triangleq (\lambda_0, 0, 0)$ and the random vectors $\zeta_n \triangleq (\lambda_n, \xi_n, \theta_n) \in \mathbf{L}^0(\mathbf{S}^M \times \mathbf{R} \times \mathbf{R}^J, \mathscr{F}_{\tau_{n-1}})$, $n = 1, \ldots, N$, are uniquely determined by the recurrence relations:

$$\lambda_n = \frac{\frac{\partial G}{\partial u}(\frac{\partial F}{\partial v}(\zeta_{n-1}, \tau_{n-1}), 1, \theta_n, \tau_{n-1})}{\sum_{m=1}^{M} \frac{\partial G}{\partial u^m}(\frac{\partial F}{\partial v}(\zeta_{n-1}, \tau_{n-1}), 1, \theta_n, \tau_{n-1})}, \tag{5.3}$$

$$\xi_n = G(\frac{\partial F}{\partial v}(\zeta_{n-1}, \tau_{n-1}), 1, \theta_n, \tau_{n-1}). \tag{5.4}$$

Proof. The recurrence relations (5.3) and (5.4) clearly determine λ_n and ξ_n, $n = 1, \ldots, N$, uniquely. In view of the identity (4.16), for conditionally Pareto optimal allocations $(\alpha_n)_{n=0,\ldots,N}$ defined by (5.2) the indifference condition (3.11) can be expressed as

$$\frac{\partial F}{\partial v}(\zeta_n, \tau_{n-1}) = \frac{\partial F}{\partial v}(\zeta_{n-1}, \tau_{n-1}), \quad n = 1, \ldots, N, \tag{5.5}$$

which, by Corollary 4.14 and the fact that λ_n has values in $\mathbf{S}^M$, is, in turn, equivalent to (5.3) and (5.4). $\square$

In the setting of Theorem 5.1, let $A \triangleq (W, X, Q)$ where

$$W_t = \lambda_0 1_{[0]}(t) + \sum_{n=1}^{N} \lambda_n 1_{(\tau_{n-1},\tau_n]}(t), \tag{5.6}$$

$$X_t = \sum_{n=1}^{N} \xi_n 1_{(\tau_{n-1},\tau_n]}(t). \tag{5.7}$$

Then A is a simple predictable process with values in $\mathbf{A}$:

$$A_t = \zeta_0 1_{[0]}(t) + \sum_{n=1}^{N} \zeta_n 1_{(\tau_{n-1},\tau_n]}(t), \quad 0 \leq t \leq T, \tag{5.8}$$

with ζ_n belonging to $\mathbf{L}^0(\mathscr{F}_{\tau_{n-1}}, \mathbf{A})$ and defined in Theorem 5.1. It was shown in the proof of this theorem that the main condition (3.11) of the preservation of expected utilities is equivalent to (5.5). Observe now that (5.5) can also be expressed as

$$\frac{\partial F}{\partial v}(A_t, t) = \frac{\partial F}{\partial v}(A_0, 0) + \int_0^t \frac{\partial F}{\partial v}(A_s, ds), \quad 0 \leq t \leq T, \tag{5.9}$$

where, for a *simple* process A as in (5.8),

$$\int_0^t \frac{\partial F}{\partial v}(A_s, ds) \triangleq \sum_{n=1}^{N} \left(\frac{\partial F}{\partial v}(\zeta_n, \tau_n \wedge t) - \frac{\partial F}{\partial v}(\zeta_n, \tau_{n-1} \wedge t) \right)$$

denotes its nonlinear stochastic integral against the random field $\frac{\partial F}{\partial v}$. Note that, contrary to (3.11) and (5.5), the condition (5.9) also makes sense for predictable processes A which are not necessarily simple, provided that the nonlinear stochastic integral $\int \frac{\partial F}{\partial v}(A_s, ds)$ is well-defined. This will be a key for extending our model to general predictable strategies in the next section.

5.2 Extension to general predictable strategies

For a general predictable process A, the construction of $\int \frac{\partial F}{\partial v}(A_s, ds)$ requires additional conditions on the stochastic field $\frac{\partial F}{\partial v} = \frac{\partial F}{\partial v}(a, t)$; see, for example, Sznitman [7] and Kunita [4], Section 3.2. We choose to rely on [4], where the corresponding theory of stochastic integration is developed for continuous semimartingales. Hence, hereafter, we shall work in a Brownian setting. We assume that, for any $a \in \mathbf{A}$, the martingale $F(a, \cdot)$ of (4.8) admits an integral representation of the form

$$F(a,t) = F(a,0) + \int_0^t H_s(a) dB_s, \quad 0 \leq t \leq T, \tag{5.10}$$

where B is a d-dimensional Brownian motion and $H(a)$ is a predictable process with values in $\mathbf{R}^d$. Of course, the integral representation (5.10) holds automatically if the filtration $\mathbf{F}$ is generated by B. To use the construction of the stochastic integral $\int \frac{\partial F}{\partial v}(A_s, ds)$ from [4] we have to impose an additional regularity condition on the integrand H with respect to the parameter a.

Assumption 5.2. There exists a predictable process $H = (H_t)_{0 \leq t \leq T}$ with values in $\mathbf{C}^1(\mathbf{A}, \mathbf{R}^d)$ such that the integral representation (5.10) holds for any $a \in \mathbf{A}$ and, for any compact set $C \subset \mathbf{A}$,

$$\int_0^T \|H_t\|_{1,C}^2 dt < \infty.$$

For convenience of future references, recall the following elementary fact:

Lemma 5.3. *Let m be a non-negative integer, U be an open set in $\mathbf{R}^n$, and $\xi : U \to \mathbf{L}^0(\mathbf{R})$ be a stochastic field with sample paths in $\mathbf{C}^m = \mathbf{C}^m(U)$ such that for any compact set $C \subset U$*

$$\mathbb{E}[\|\xi\|_{m,C}] < \infty. \tag{5.11}$$

Assume that there are a Brownian motion B with values in $\mathbf{R}^d$ and a predictable process $H = (H_t)_{0 \leq t \leq T}$ with values in $\mathbf{C}^m(U, \mathbf{R}^d)$ such that, for $x \in U$ and $t \in [0, T]$,

$$M_t(x) \triangleq \mathbb{E}[\xi(x)|\mathscr{F}_t] = M(x,0) + \int_0^t H_s(x) dB_s,$$

and, for any compact set $C \subset U$,

$$\int_0^T \|H_t\|_{m,C}^2 dt < \infty. \tag{5.12}$$

Then, for $t \in [0,T]$, $x \in U$, and a multi-index $a = (a_1, \dots, a_n)$ with $|a| \le m$,

$$\partial^a M_t(x) = \partial^a M_0(x) + \int_0^t \partial^a H_s(x) dB_s,$$

where ∂^a is the differential operator with respect to x given by (2.2).

Proof. We shall use a modification of M with sample paths in $\mathbf{D}(\mathbf{C}^m(U), [0,T])$ which exists by Lemma 4.5.

It is sufficient to consider the case $m = 1$ and $a = (1, 0, \dots, 0)$. Denote $e_1 \triangleq (1, 0, \dots, 0) \in \mathbf{R}^n$. By (5.11),

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{1}{\varepsilon}|\xi(x + \varepsilon e_1) - \xi(x) - \varepsilon \frac{\partial \xi}{\partial x_1}(x)|\right] = 0$$

and then, by Doob's inequality,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(M(x + \varepsilon e_1) - M(x) - \varepsilon \frac{\partial M}{\partial x_1}(x)\right)^*_T = 0.$$

Observe also that by (5.12)

$$\lim_{\varepsilon \to 0} \int_0^T |\frac{1}{\varepsilon}(H(x + \varepsilon e_1) - H(x) - \varepsilon \frac{\partial H}{\partial x_1}(x))|^2 dt = 0.$$

The result now follows from the well-known fact that for a sequence of continuous local martingales $(N^n)_{n \ge 1}$

$$(N^n)^*_T \to 0 \quad \text{if and only if} \quad N^n_0 \to 0 \text{ and } \langle N^n \rangle_T \to 0, \quad n \to \infty,$$

where, for a continuous local martingale N, $\langle N \rangle$ denotes its quadratic variation. □

To simplify references we also state an immediate corollary of Theorems 4.3, 4.4, 4.11, and 4.12 and Assumption 5.2.

Corollary 5.4. *Under Assumptions 3.1, 3.4, and 5.2, $F = F(a,t)$ has sample paths in $\mathbf{C}(\widetilde{\mathbf{F}}^1, [0,T])$ and $G = G(b,t)$ has sample paths in $\mathbf{C}(\widetilde{\mathbf{G}}^1, [0,T])$. If, in addition, Assumption 3.2 holds, then $F = F(a,t)$ has sample paths in $\mathbf{C}(\widetilde{\mathbf{F}}^2(c), [0,T])$ and $G = G(b,t)$ has sample paths in $\mathbf{C}(\widetilde{\mathbf{G}}^2(c), [0,T])$ for the same constant $c > 0$ as in Assumption 3.2.*

Hereafter we shall work under Assumptions 3.1, 3.4, and 5.2. In this case, by Theorem 4.3 and Lemma 5.3,

$$\frac{\partial F}{\partial v}(a,t) = \frac{\partial F}{\partial v}(a,0) + \int_0^t \frac{\partial H_s}{\partial v}(a) dB_s.$$

Following Section 3.2 in [4], we say that a predictable process A with values in $\mathbf{A}$ is *integrable* with respect to the *kernel* $\frac{\partial F}{\partial v}(\cdot, dt)$ or, equivalently, that the stochastic integral $\int \frac{\partial F}{\partial v}(A_s, ds)$ is *well-defined* if

$$\int_0^T |\frac{\partial H_t}{\partial v}(A_t)|^2 dt < \infty.$$

In this case, we set

$$\int_0^t \frac{\partial F}{\partial v}(A_s, ds) \triangleq \int_0^t \frac{\partial H_s}{\partial v}(A_s) dB_s, \quad 0 \leq t \leq T.$$

We are now in a position to give a definition of a general trading strategy.

Definition 5.5. A predictable process Q with values in $\mathbf{R}^J$ is called a *strategy* if there are unique (in the sense of indistinguishability) predictable processes W and X with values in $\mathbf{S}^M$ and $\mathbf{R}$, respectively, such that, for $A \triangleq (W, X, Q)$, the initial Pareto allocation is given by

$$\alpha_0 = \pi(A_0), \tag{5.13}$$

the stochastic integral $\int \frac{\partial F}{\partial v}(A_s, ds)$ is well-defined and (5.9) holds.

Remark 5.6. From now on, the term "strategy" will always be used in the sense of Definition 5.5. Note that, at this point, it is still an open question whether a simple predictable process Q is a (valid) strategy, as in Theorem 5.1 the uniqueness of W and X, such that $A \triangleq (W, X, Q)$ solves (5.9), was proved only in the class of *simple* processes. The affirmative answer to this question will be given in Theorem 5.18 below, where, in addition to the standing Assumptions 3.1, 3.4, and 5.2, we shall also require Assumptions 3.2 and 5.14.

The predictable processes W and X in Definition 5.5 will be called the *Pareto weights* and *cash balance* processes for the strategy Q. We remind the reader, that the bookkeeping in our model is done from the collective point of view of the market makers, see Remark 3.6. In other words, for a strategy

Q, the number of shares and the amount of cash *owned by the large investor* at time t are given by $-Q_t$ and $-X_t$.

Accounting for (4.16) we call

$$U_t \triangleq \frac{\partial F}{\partial v}(A_t, t), \quad 0 \leq t \leq T, \tag{5.14}$$

the process of *indirect utilities* for the market makers. Observe that, as $U < 0$ and $U - U_0$ is a stochastic integral with respect to a Brownian motion, U is a local martingale and a (global) submartingale. From Corollary 4.14 we obtain the following expressions for W and X in terms of U and Q:

$$W_t = \frac{\frac{\partial G}{\partial u}(U_t, 1, Q_t, t)}{\sum_{m=1}^{M} \frac{\partial G}{\partial u^m}(U_t, 1, Q_t, t)}, \tag{5.15}$$

$$X_t = G(U_t, 1, Q_t, t), \quad 0 \leq t \leq T. \tag{5.16}$$

We also call

$$V_t \triangleq -G(U_t, 1, 0, t) = -G(\frac{\partial F}{\partial v}(A_t, t), 1, 0, t), \quad 0 \leq t \leq T, \tag{5.17}$$

the *cumulative gain* process for the large trader. This term is justified as, by (5.16), V_t represents the cash amount the agent will hold at t if he liquidates his position in stocks. Of course, at maturity

$$V_T = -(X_T + \langle Q_T, \psi \rangle).$$

It is interesting to observe that, contrary to the standard, small agent, model of mathematical finance no further "admissibility" conditions on a strategy Q are needed to exclude an arbitrage.

Lemma 5.7. *Let Assumptions 3.1, 3.4, and 5.2 hold and Q be a strategy such that the terminal gain of the large trader is nonnegative: $V_T \geq 0$. Then, in fact, $V_T = 0$.*

Proof. Recall the notations $\lambda_0 \in \mathbf{S}^M$ for the weights and $\Sigma_0 \in \mathbf{L}^0(\mathbf{R}^M)$ for the total endowment of the initial Pareto allocation α_0 and $r = r(v, x)$ for the utility function of the representative market maker from (4.1). Denote by α_1 the terminal wealth distribution between the market makers at maturity resulting from strategy Q. From the characterization of Pareto allocations in Theorem 4.1 and the submartingale property of the process U of indirect utilities we obtain

$$\begin{aligned}\mathbb{E}[r(\lambda_0, \Sigma_0)] &= \mathbb{E}[\sum_{m=1}^{m} \lambda_0^m u_m(\alpha_0^m)] = \langle \lambda_0, U_0 \rangle \leq \mathbb{E}[\langle \lambda_0, U_T \rangle] \\ &= \mathbb{E}[\sum_{m=1}^{m} \lambda_0^m u_m(\alpha_1^m)] \leq \mathbb{E}[r(\lambda_0, \Sigma_0 - V_T)].\end{aligned}$$

Since $r(\lambda_0, \cdot)$ is a strictly increasing function, the result follows. □

We state now a key result of the paper where we reduce the question whether a predictable process Q is a strategy to the unique solvability of a stochastic differential equation parameterized by Q.

Theorem 5.8. *Under Assumptions 3.1, 3.4, and 5.2, a predictable process Q with values in $\mathbf{R}^J$ is a strategy if and only if the stochastic differential equation*

$$U_t = U_0 + \int_0^t K_s(U_s, Q_s) dB_s, \tag{5.18}$$

has a unique strong solution U with values in $(-\infty, 0)^M$ on $[0, T]$, where

$$U_0^m \triangleq \mathbb{E}[u_m(\alpha_0^m)], \quad m = 1, \ldots, M,$$

and, for $u \in (-\infty, 0)^M$, $q \in \mathbf{R}^J$, $t \in [0, T]$,

$$K_t(u, q) \triangleq \frac{\partial H_t}{\partial v}(\frac{\partial G}{\partial u}(u, 1, q, t), G(u, 1, q, t), q). \tag{5.19}$$

In this case, U is the process of indirect utilities, and the processes of Pareto weights W and cash balance X are given by (5.15) *and* (5.16).

Proof. The result follows directly from the definition of a strategy and Corollary 4.14, if we observe that, by the positive homogeneity property (4.10) of the elements of $\mathbf{F}^1$, for any $(v, x, q) \in \mathbf{A}$, $c > 0$, and $t \in [0, T]$,

$$\frac{\partial H_t}{\partial v}(cv, x, q) = \frac{\partial H_t}{\partial v}(v, x, q),$$

and, hence, the process K from (5.19) can also be written as

$$K_t(u, q) = \frac{\partial H_t}{\partial v} \left(\frac{\frac{\partial G}{\partial u}(u, 1, q, t)}{\sum_{m=1}^M \frac{\partial G}{\partial u^m}(u, 1, q, t)}, G(u, 1, q, t), q \right).$$

□

In the follow-up paper [2] we provide sufficient conditions for a locally bounded predictable process Q with values in $\mathbf{R}^J$ to be a strategy, or, equivalently, for (5.18) to have a unique strong solution, in terms of the "original" inputs to the model: the utility functions $(u_m)_{m=1,\dots,M}$, the initial endowment Σ_0, and the contingent claims ψ. In particular, these conditions will also imply Assumptions 5.2 and 5.14 on $H = H_t(a)$.

As an illustration, we give an example where (5.18) is a linear equation, and, hence, can be solved explicitly.

Example 5.9 (Bachelier model with price impact)**.** Consider an economy with a single market maker and one stock. The market maker's utility function is exponential:

$$u(x) = -\frac{1}{a}e^{-ax}, \quad x \in \mathbf{R},$$

where $a > 0$ is the absolute risk-aversion coefficient. The initial endowment of the market maker and the payoff of the stock are given by

$$\begin{aligned}\Sigma_0 = \alpha_0 &= b + \frac{\mu}{a\sigma}B_T, \\ \psi &= s + \mu T + \sigma B_T,\end{aligned}$$

where $b, \mu, s \in \mathbf{R}$ and $\sigma > 0$. Note that the initial Pareto pricing measure $\mathbb{Q} = \mathbb{Q}_0$ and the stock price S, see (3.16) and (3.15), have the expressions:

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}} &\triangleq \text{const}\; u'(\Sigma_0) = e^{-\frac{\mu}{\sigma}B_T - \frac{\mu^2}{2\sigma^2}T}, \\ S_t &\triangleq \mathbb{E}_{\mathbb{Q}_0}[\psi|\mathscr{F}_t] = s + \mu t + \sigma B_t, \quad t \in [0,T],\end{aligned}$$

and coincide with the martingale measure and the stock price in the classical Bachelier model for a "small" investor.

Direct computations show that, for $a = (v, x, q) \in \mathbf{A}$,

$$F(a,t) = ve^{-ax}N_t(q),$$

where the martingale $N(q)$ evolves as

$$dN_t(q) = -(\frac{\mu}{\sigma} + a\sigma q)N_t(q)dB_t. \tag{5.20}$$

For the integrand $H = H_t(a)$ in (5.10) and the stochastic process $G = G(b,t)$ we obtain

$$\begin{aligned}\frac{\partial H_t}{\partial v}(a) &= -(\frac{\mu}{\sigma} + a\sigma q)e^{-ax}N_t(q), \\ u &= e^{-aG(u,1,q,t)}N_t(q), \quad u \in (-\infty, 0),\end{aligned}$$

where the second equality follows from the last identity in Corollary 4.14. The process $K = K_t(u,q)$ in (5.19) is then given by

$$K_t(u,q) = -(\frac{\mu}{\sigma} + a\sigma q)u, \quad u \in (-\infty, 0).$$

From Theorem 5.8 we obtain that a predictable process Q is a strategy if and only if

$$\int_0^T Q_t^2 dt < \infty,$$

and that, in this case, the indirect utility process U for the market maker evolves as

$$dU_t = -(\frac{\mu}{\sigma} + a\sigma Q_t)U_t dB_t. \tag{5.21}$$

Observe now that, by (5.17), the cumulative gain V_t of the large trader satisfies

$$U_t = e^{aV_t} N_t(0).$$

From (5.20) and (5.21) and the fact that $V_0 = 0$ we deduce

$$\begin{aligned} V_t &= \int_0^t \left[(-Q_r)(\mu dr + \sigma dB_r) - \frac{a\sigma^2}{2} Q_r^2 dr \right] \\ &= \int_0^t \left[(-Q_r) dS_r - \frac{a\sigma^2}{2} Q_r^2 dr \right]. \end{aligned}$$

Recall that $-Q$ denotes the number of shares owned by the large investor and then observe that the first, linear with respect to Q, term yields the wealth evolution in the classical Bachelier model. The second, quadratic, term thus describes the feedback effect of the large trader's actions on stock prices, with the risk-aversion coefficient $a > 0$ playing the role of a *price impact coefficient.*

5.3 Maximal local strategies

For a stochastic process X and a stopping time σ with values in $[0, T]$ recall the notation $X^\sigma \triangleq (X_{t\wedge\sigma})_{0\leq t\leq T}$ for X "stopped" at σ. The following simple localization fact for strategies will be used later on several occasions.

Lemma 5.10. *Let Assumptions 3.1, 3.4, and 5.2 hold, σ be a stopping time with values in $[0,T]$, Q be a strategy, and W, X, V, and U be its processes of Pareto weights, cash balance, cumulative gain, and indirect utilities. Then Q^σ is also a strategy and W^σ and X^σ are its processes of Pareto weights and cash balance. The processes of cumulative gain, $V(Q^\sigma)$, and of indirect utilities, $U(Q^\sigma)$, for the strategy Q^σ coincide with V and U on $[0,\sigma]$, while on $(\sigma,T]$ they are given by*

$$\begin{aligned} U(Q^\sigma)_t &= \frac{\partial F}{\partial v}(W_\sigma, X_\sigma, Q_\sigma, t), \\ V(Q^\sigma)_t &= -G(U(Q^\sigma)_t, 1, 0, t). \end{aligned}$$

Proof. Follows directly from Definition 5.5 and the construction of U and V in (5.14) and (5.17). $\square$

Let τ be a stopping time with values in $(0,T] \cup \{\infty\}$ and U be a process with values in $(-\infty,0)^M$ defined on $[0,\tau) \cap [0,T]$. Recall that, for the equation (5.18), τ and U are called the *explosion* time and the *maximal local solution* if for any stopping time σ with values in $[0,\tau) \cap [0,T]$ the process U^σ is the unique solution to (5.18) on $[0,\sigma]$ and

$$\limsup_{t\uparrow\tau} |\log(-U_t)| = \infty \text{ on } \{\tau < \infty\}. \tag{5.22}$$

Observe that, for $m = 1,\dots,M$, the submartingale property of $U^m < 0$ insures the existence of the limit: $\lim_{t\uparrow\tau} U^m_t$ and prevents it from being $-\infty$. Hence, (5.22) is equivalent to

$$\lim_{t\uparrow\tau} \max_{m=1,\dots,M} U^m_t = 0 \text{ on } \{\tau < \infty\}.$$

For convenience of future references we introduce a similar localized concept for strategies.

Definition 5.11. A predictable process Q with values in $\mathbf{R}^J$ is called a *maximal local strategy* if there are a stopping time τ with values in $(0,T] \cup \{\infty\}$ and processes V, W, and X on $[0,\tau) \cap [0,T]$ with values in $\mathbf{R}$, $\mathbf{S}^M$, and $\mathbf{R}$, respectively, such that

$$\lim_{t\uparrow\tau} V_t = -\infty \text{ on } \{\tau < \infty\} \tag{5.23}$$

and for any stopping time σ with values in $[0,\tau) \cap [0,T]$ the process Q^σ is a strategy with Pareto weights W^σ and cash balance X^σ whose cumulative gain equals V on $[0,\sigma]$.

Similar to the "global" case we shall call V, W, and X from Definition 5.11 the processes of cumulative gain, Pareto weights, and cash balance, respectively; the process U of indirect utilities is defined on $[0,\tau) \cap [0,T]$ as in (5.14). In view of (5.23), we shall call τ the *explosion* time for V. Note that, by Lemma 5.10, the class of maximal local strategies contains the class of (global) strategies.

Theorem 5.12. *Let Assumptions 3.1, 3.4, and 5.2 hold and τ be a stopping time with values in $(0,T] \cup \{\infty\}$. A predictable process Q with values in $\mathbf{R}^J$ is a maximal local strategy and τ is the explosion time for its cumulative gain process if and only if the stochastic differential equation* (5.18) *admits the unique maximal local solution U with the explosion time τ.*

If, in addition, Q is locally bounded, then τ is also the explosion time for its cash balance process:

$$\lim_{t \uparrow \tau} X_t = \infty \text{ on } \{\tau < \infty\}.$$

For the proof we need a uniform version of (G6) for the elements of $\widetilde{\mathbf{G}}^1$.

Lemma 5.13. *Let $(g_n)_{n\geq 1}$ converge to g in $\widetilde{\mathbf{G}}^1$. Then, for any compact set $C \subset \mathbf{R}^J$ and any sequence $(u_n)_{n\geq 1}$ in $(-\infty,0)^M$ converging to a boundary point of $(-\infty,0)^M$ we have*

$$\lim_{n\to\infty} \inf_{q\in C} g_n(u_n,1,q) = \infty.$$

Proof. Since the elements of $\widetilde{\mathbf{G}}^1$ are concave with respect to q, it is sufficient to consider the case when C is a singleton. Moreover, as the elements of $\widetilde{\mathbf{G}}^1$ are increasing with respect to u, we can assume that the sequence $(u_n)_{n\geq 1}$ is increasing. In this case, for $q \in \mathbf{R}^J$,

$$\liminf_{n\to\infty} g_n(u_n,1,q) \geq \lim_{k\to\infty} \liminf_{n\to\infty} g_n(u_k,1,q) = \lim_{k\to\infty} g(u_k,1,q) = \infty,$$

where the last equality follows from (G6). $\square$

Proof of Theorem 5.12. By Corollary 5.4, $G = G(b,t)$ has its sample paths in $\mathbf{C}(\widetilde{\mathbf{G}}^1,[0,T])$. The result now follows directly from Theorem 5.8 and Lemma 5.13. $\square$

To establish the existence of a maximal local strategy or, equivalently, the existence and uniqueness of a maximal local solution to (5.18) we shall also require Assumption 3.2 and a stronger version of Assumption 5.2.

Assumption 5.14. The predictable process H from Assumption 5.2 has values in $\mathbf{C}^2(\mathbf{A}, \mathbf{R}^d)$ and, for any compact set $C \subset \mathbf{A}$,

$$\int_0^T \|H_t\|_{2,C}^2 dt < \infty.$$

The role of these additional assumptions is to guarantee the local Lipschitz property with respect to u for the stochastic process K in (5.19).

Lemma 5.15. *Under Assumptions 3.1, 3.2, 3.4, 5.2, and 5.14 the predictable process K defined in* (5.19) *has values in $\mathbf{C}^1((-\infty, 0)^M \times \mathbf{R}^J, \mathbf{R}^{M\times d})$ and, for any compact set $C \subset (-\infty, 0)^M \times \mathbf{R}^J$,*

$$\int_0^T \|K_t\|_{1,C}^2 dt < \infty.$$

Proof. Follows from Assumption 5.14 and the fact, that, by Corollary 5.4, the process $G = G(b, t)$ has sample paths in $\mathbf{C}(\widetilde{\mathbf{G}}^2(c), [0, T]) \subset \mathbf{C}(\mathbf{C}^2(\mathbf{B}), [0, T])$. □

Theorem 5.16. *Let Assumptions 3.1, 3.2, 3.4, 5.2, and 5.14 hold and Q be a predictable process with values in $\mathbf{R}^J$ such that, for any compact set $C \subset (-\infty, 0)^M$,*

$$\int_0^T \|K_t(\cdot, Q_t)\|_{1,C}^2 dt < \infty. \tag{5.24}$$

Then Q is a maximal local strategy.

Proof. It is well-known, see, for example, Theorem 3.4.5 in [4], that (5.24) implies the existence of a unique *maximal local solution* to (5.18). The result now follows from Theorem 5.12. □

Theorem 5.17. *Under Assumptions 3.1, 3.2, 3.4, 5.2, and 5.14 any locally bounded predictable process Q is a maximal local strategy.*

Proof. Follows from Theorem 5.16 if we observe that, by Lemma 5.15, a locally bounded Q satisfies (5.24). □

The preceding result allows us to finally reconcile Definition 5.5 with the construction of simple strategies in Theorems 3.7 and 5.1 since it resolves the uniqueness issue raised in Remark 5.6.

Theorem 5.18. *Under Assumptions 3.1, 3.2, 3.4, 5.2, and 5.14 any simple predictable process Q with values in $\mathbf{R}^J$ is a strategy and its processes of Pareto weights W and cash balance X are simple and given by* (5.6)–(5.7) *and* (5.3)–(5.4).

Proof. The fact, that, for W and X given by (5.6)–(5.7) and (5.3)–(5.4), the process $A \triangleq (W, X, Q)$ satisfies (5.13) and (5.9) has been already established in our discussion following Theorem 5.1. The uniqueness follows from Theorem 5.17. □

6 Approximation by simple strategies

In this final section we provide a justification for the construction of the general strategies in Definition 5.5 by discussing approximations based on simple strategies. To simplify the presentation we restrict ourselves to the case of locally bounded processes.

For measurable stochastic processes, in addition to the ucp convergence, we also consider the convergence in $\mathbf{L}^0(d\mathbb{P} \times dt)$ defined by the metric

$$d(X, Y) \triangleq \mathbb{E}[\int_0^T (|X_t - Y_t| \wedge 1)dt].$$

We call a sequence of stochastic processes $(X^n)_{n\geq 1}$ *uniformly locally bounded from above* if there is an increasing sequence of stopping times $(\sigma_n)_{n\geq 1}$ such that $\mathbb{P}[\sigma_n < T] \to 0$, $n \to \infty$ and $X^k_t \leq n$ on $[0, \sigma_n]$ for $k \geq 1$. The sequence $(X^n)_{n\geq 1}$ is called *uniformly locally bounded* if the sequence of its absolute values $(|X^n|)_{n\geq 1}$ is uniformly locally bounded from above.

We begin with a general convergence result:

Theorem 6.1. *Let Assumptions 3.1, 3.2, 3.4, 5.2, and 5.14 hold and consider a sequence of strategies $(Q^n)_{n\geq 1}$ which is uniformly locally bounded and converges to a strategy Q in $\mathbf{L}^0(d\mathbb{P} \times dt)$.*

Then the processes $(U^n, V^n)_{n\geq 1}$, of indirect utilities and cumulative gains, converge to (U, V) in ucp, the processes $(W^n, X^n)_{n\geq 1}$, of Pareto weights and cash balance, converge to (W, X) in $\mathbf{L}^0(d\mathbb{P} \times dt)$, and the sequence $(X^n)_{n\geq 1}$

is uniformly locally bounded. If, in addition, the sequence $(Q^n)_{n\geq 1}$ converges to Q in ucp, then the sequence $(W^n, X^n)_{n\geq 1}$ also converges to (W, X) in ucp.

Proof. By standard localization arguments, we can assume the existence of constants $a > 0$ and $b > 0$ such that

$$\max(|\ln(-U)|, |Q|, \sup_{n\geq 1}|Q^n|) \leq a,$$

and, in view of Lemma 5.15, such that

$$\int_0^T \|K_s(\cdot)\|^2_{1,C(a)} ds \leq b, \tag{6.1}$$

where

$$C(a) \triangleq \{(u, q) \in (-\infty, 0)^M \times \mathbf{R}^J : \max(|\ln(-u)|, |q|) \leq 2a\}.$$

Define the stopping times

$$\sigma_n \triangleq \inf\{t \in [0, T] : |\ln(-U^n_t)| \geq 2a\} \wedge T, \quad n \geq 1,$$

where we follow the convention that $\inf \varnothing \triangleq \infty$. Observe that the ucp convergence of $(U^n)_{n\geq 1}$ to U holds if

$$(U - U^n)^*_{T\wedge\sigma_n} \to 0, \quad n \to \infty. \tag{6.2}$$

To prove (6.2), note first that for any two stopping times $0 \leq \tau_* \leq \tau^* \leq \sigma_n$ we have using Doob's inequality

$$\begin{aligned}
&\mathbb{E}[\sup_{\tau_*\leq t\leq\tau^*} |U_t - U^n_t|^2] \\
&\leq \mathbb{E}[2|U_{\tau_*} - U^n_{\tau_*}|^2 + 2\sup_{\tau_*\leq t\leq\tau^*} |\int_{\tau_*}^t (K_s(U_s, Q_s) - K_s(U^n_s, Q^n_s))dB_s|^2] \\
&\leq 2\mathbb{E}|U_{\tau_*} - U^n_{\tau_*}|^2 + 8\mathbb{E}[\int_{\tau_*}^{\tau^*} |K_s(U_s, Q_s) - K_s(U^n_s, Q^n_s)|^2 ds] \\
&\leq 2\mathbb{E}|U_{\tau_*} - U^n_{\tau_*}|^2 + 8\mathbb{E}[\int_{\tau_*}^{\tau^*} \|K_s(\cdot)\|^2_{1,C(a)}(|U_s - U^n_s|^2 + |Q_s - Q^n_s|^2)ds] \\
&\leq 2\mathbb{E}|U_{\tau_*} - U^n_{\tau_*}|^2 + 8\mathbb{E}[\int_{\tau_*}^{\tau^*} \|K_s(\cdot)\|^2_{1,C(a)} ds \sup_{\tau_*\leq t\leq\tau^*} |U_t - U^n_t|^2] \\
&\quad + 8\mathbb{E}[\int_{\tau_*}^{\tau^*} \|K_s(\cdot)\|^2_{1,C(a)} |Q_s - Q^n_s|^2 ds].
\end{aligned}$$

Rearranging terms we thus obtain

$$
\begin{aligned}
(6.3) \qquad & \mathbb{E}[(1 - 8\int_{\tau_*}^{\tau^*} \|K_s(\cdot)\|^2_{1,C(a)} ds) \sup_{\tau_* \leq t \leq \tau^*} |U_t - U^n_t|^2] \\
& \leq 2\mathbb{E}|U_{\tau_*} - U^n_{\tau_*}|^2 + 8\mathbb{E}[\int_{\tau_*}^{\tau^*} \|K_s(\cdot)\|^2_{1,C(a)} |Q_s - Q^n_s|^2 ds].
\end{aligned}
$$

Now choose $\tau_0 \triangleq 0$ and, for $i = 1, 2, \ldots$, let

$$\tau_i \triangleq \inf\{t \geq \tau_{i-1} :\ 8\int_{\tau_{i-1}}^{t} \|K_s(\cdot)\|^2_{1,C(a)} ds \geq \frac{1}{2}\} \wedge T.$$

Note that because of (6.1) we have $\tau_i = T$ for $i \geq i_0$, where i_0 is the smallest integer greater than $16b$. Hence, to establish (6.2), it suffices to prove

$$\mathbb{E}[\sup_{\tau_{i-1}\wedge\sigma^n \leq s \leq \tau_i \wedge \sigma^n} |U_s - U^n_s|^2] \to 0, \quad n \to \infty \quad \text{for } i = 1, \ldots, i_0.$$

For $i = 1$ this follows from estimate (6.3) with $\tau_* \triangleq \tau_0 = 0$ and $\tau^* \triangleq \tau_1 \wedge \sigma^n$ because $U_0 = U^n_0$ and because of our assumption on the sequence $(Q^n)_{n\geq 1}$. For $i = 2, 3, \ldots$ this convergence holds by induction, since with $\tau_* \triangleq \tau_{i-1} \wedge \sigma^n$ and $\tau^* \triangleq \tau_i \wedge \sigma^n$ the first term on the right hand side of (6.3) vanishes for $n \to \infty$ because of the validity of our claim for $i - 1$ and the second term disappears again by assumption on $(Q^n)_{n\geq 1}$. This finishes the proof of the ucp convergence of $(U^n)_{n\geq 1}$ to U.

The rest of the assertions follows from the representations (5.15), (5.16), and (5.17) for Pareto weights, cash balances, and cumulative gains in terms of the stochastic field $G = G(b, t)$ and the fact that, by Corollary 5.4, G has sample paths in $\mathbf{C}(\mathbf{C}^1(\mathbf{B}), [0, T])$. □

Theorem 6.2. *Under Assumptions 3.1, 3.2, 3.4, 5.2, and 5.14, a predictable locally bounded process Q with values in $\mathbf{R}^J$ is a strategy if and only if there is a sequence $(Q^n)_{n\geq 1}$ of simple strategies, which is uniformly locally bounded, converges to Q in $\mathbf{L}^0(d\mathbb{P} \times dt)$, and for which the sequence of associated cash balances $(X^n)_{n\geq 1}$ is uniformly locally bounded from above.*

Lemma 6.3. *Under Assumptions 3.1, 3.2, 3.4, 5.2, and 5.14, for any strategy Q and any $t \in [0,T]$*

$$\begin{aligned}
&\sum_{m=1}^{M} \left(\frac{1}{c} \log((-U^m_t) \vee 1) + c \log((-U^m_t) \wedge 1) \right) \\
&\quad \leq G(-\mathbf{1}, 1, Q_t, t) - X_t \\
&\quad \leq \sum_{m=1}^{M} \left(\frac{1}{c} \log((-U^m_t) \wedge 1) + c \log((-U^m_t) \vee 1) \right),
\end{aligned} \tag{6.4}$$

where $c > 0$ is taken from Assumption 3.2 and X and U are the processes of cash balance and indirect utilities for Q.

Proof. Recall that, by Corollary 5.4, $G = G(b,t)$ has trajectories in $\mathbf{C}(\widetilde{\mathbf{G}}^2(c), [0,T])$ and, hence, by the property (G7) of the elements of $\widetilde{\mathbf{G}}^2(c)$,

$$\frac{1}{c} \leq -u^m \frac{\partial G}{\partial u^m}(u, 1, q, t) \leq c, \quad m = 1, \dots, M.$$

This implies the result if we account for the representation (5.16) for X. $\square$

Proof of Theorem 6.2. The "only if" part follows from Theorem 6.1 and the fact that any locally bounded predictable process Q can be approximated in $\mathbf{L}^0(d\mathbb{P} \times dt)$ by a sequence of simple predictable processes $(Q^n)_{n \geq 1}$ which is uniformly locally bounded. Hereafter we shall focus on sufficiency.

By Theorem 5.17, Q is a maximal local strategy. Denote by U and X its processes of indirect utilities and cash balance and by τ the explosion time of X, see Theorem 5.12. We have to show that $\tau = \infty$.

For $a > 0$ and $b > a$ define the stopping times

$$\begin{aligned}
\tau(a) &\triangleq \inf\{t \in [0,T] : \max_{m=1,\dots,M} U^m_t > -a\}, \\
\tau_n(a) &\triangleq \inf\{t \in [0,T] : \sup_{k \geq n} \max_{m=1,\dots,M} U^{k,m}_t > -a\}, \quad n \geq 1, \\
\sigma(b) &\triangleq \inf\{t \in [0,T] : \min_{m=1,\dots,M} U^m_t < -b\}, \\
\sigma_n(b) &\triangleq \inf\{t \in [0,T] : \inf_{k \geq n} \min_{m=1,\dots,M} U^{k,m}_t < -b\}, \quad n \geq 1,
\end{aligned}$$

where U^n is the process of indirect utilities for Q^n and where we let $\inf \varnothing \triangleq \infty$. Note that, by Theorem 5.12, $\tau(a) \to \tau$, $a \to 0$, and, hence, $\tau = \infty$ if and only if

$$\lim_{a\to 0} \mathbb{P}[\tau(a) \le T] = 0. \tag{6.5}$$

From Theorem 5.12 and Lemma 5.10 we deduce that $Q^{\tau(a)\wedge T}$ is a strategy whose indirect utility process coincides with U on $[0, \tau(a) \wedge T]$. Hence, by Theorem 6.1,

$$(U^n - U)^*_{\tau(a)\wedge T} \to 0, \quad n \to \infty. \tag{6.6}$$

Hereafter, we shall assume that a is rational and that, for every such a, the convergence above takes place almost surely. This can always be arranged by passing to a subsequence.

Since

$$\{\tau(a) < \tau_n(2a)\} \subset \bigcap_{k\ge n} \{(U^k - U)^*_{\tau(a)\wedge T} \ge a\},$$

we obtain

$$\lim_{n\to\infty} \mathbb{P}[\tau(a) < \tau_n(2a)] = 0. \tag{6.7}$$

Similarly, as

$$\{\sigma_n(2b) \wedge \tau(a) < \sigma(b) \wedge \tau(a)\} \subset \bigcup_{k\ge n} \{(U^k - U)^*_{\tau(a)\wedge T} \ge b\},$$

and since the convergence in (6.6) takes place almost surely, we deduce

$$\lim_{n\to\infty} \mathbb{P}[\sigma_n(2b) \wedge \tau(a) < \sigma(b) \wedge \tau(a)] = 0.$$

The latter convergence implies that

$$\limsup_{n\to\infty} \mathbb{P}[\sigma_n(2b) < \tau(a)] \le \mathbb{P}[\sigma(b) < \tau(a)] \le \mathbb{P}[\sigma(b) < \tau]. \tag{6.8}$$

From (6.7) and (6.8) we deduce

$$\mathbb{P}[\tau(a) \le T] \le \mathbb{P}[\sigma(b) < \tau] + \limsup_{n\to\infty} \mathbb{P}[\tau_n(2a) \le \sigma_n(2b) \wedge T].$$

Therefore, (6.5) holds if

$$\lim_{b\to\infty} \mathbb{P}[\sigma(b) < \tau] = 0, \tag{6.9}$$

and, for any $b > 0$,

$$\lim_{a \to 0} \limsup_{n \to \infty} \mathbb{P}[\tau_n(a) \leq \sigma_n(b) \wedge T] = 0. \tag{6.10}$$

The verification of (6.9) is straightforward due to the submartingale property of U. The uniform local boundedness conditions on $(Q^n)_{n \geq 1}$ and $(X^n)_{n \geq 1}$ (from above) and the fact that G has trajectories in $\mathbf{C}(\mathbf{C}(\mathbf{B}), [0, T])$ imply that the process

$$Y_t \triangleq \inf_{n \geq 1} \left(G(-\mathbf{1}, 1, Q^n_t, t) - X^n_t \right), \quad 0 \leq t \leq T,$$

is locally bounded from below. The convergence (6.10) follows now from the second inequality in (6.4) of Lemma 6.3. □

We conclude this section with affirmative answers to our Questions 3.9 and 3.10 from Section 3.2. Recall that the acronym LCRL means left-continuous with right limits.

Theorem 6.4. *Under Assumptions 3.1, 3.2, 3.4, 5.2, and 5.14, a predictable process Q be with values in $\mathbf{R}^J$ and LCRL trajectories is a strategy if and only if there is a predictable process X with values in $\mathbf{R}$ and a sequence of simple strategies $(Q^n)_{n \geq 1}$ converging to Q in ucp such that the sequence of its cash balances $(X^n)_{n \geq 1}$ converges to X in ucp. In this case, X is the cash balance process for Q.*

Proof. Follows from Theorems 6.1 and 6.2 and the fact that any predictable process with LCRL trajectories is a limit in ucp of a sequence of simple processes which then necessarily is also uniformly locally bounded. □

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