A model for a large investor trading at market indifference prices. I: single-period case.

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Abstract

We develop a single-period model for a large economic agent who trades with market makers at their utility indifference prices. A key role is played by a pair of conjugate saddle functions associated with the description of Pareto optimal allocations in terms of the utility function of a representative market maker.

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Contents

1 Introduction

A typical financial model presumes that the prices of traded securities are not affected by an investor's buy and sell orders. From a practical viewpoint this assumption is justified as long as his trading volume remains small enough to be easily covered by market liquidity. An opposite situation occurs, for instance, when an economic agent has to sell a large block of shares over a short period of time; see, e.g., Almgren and Chriss [1] and Schied and Schöneborn [25]. This and other examples motivate the development of financial models for a "large" trader, where the dependence of market prices on his strategy, called a *price impact* or a *demand pressure*, is taken into account.

Hereafter, we assume that the interest rate is zero and, in particular, is not affected by investor's trading actions. As usual in mathematical finance we describe a (self-financing) strategy by a predictable process $Q = (Q_t)_{0\leq t\leq T}$ where Q_t is the number of stocks held *just before* time t and T is a finite time horizon. The role of a "model" is to define a predictable process $X(Q)$ representing the evolution of the cash balance for the strategy Q. We denote by $S(Q)$ the *marginal* price process of traded stocks, that is, $S_t(Q)$ is the price at which one can trade an infinitesimal quantity of stocks at time t. Recall that in the standard model of a "small" agent the price S does not depend on Q and

$$X_t(Q) = \int_0^t Q_u dS_u - Q_t S_t.$$

In mathematical finance a common approach is to specify the price impact of trades *exogenously*, that is, to postulate it as one of the inputs. For example, Frey and Stremme [12], Platen and Schweizer [23], Papanicolaou and Sircar [22], and Bank and Baum [4] choose a stochastic field of *reaction functions*, which explicitly state the dependence of the marginal prices on the investor's current holdings, Çetin, Jarrow, and Protter in [6] start with a stochastic field of *supply curves*, which define the prices in terms of traded quantities (*changes* in holdings), and Cvitanić and Ma [8] make the drift and the volatility of the price process dependent on a trading strategy; we refer the reader to the recent survey [15] by Gokay, Roch, and Soner for more details and additional references. Note that in all these models the processes $X(Q)$ and $S(Q)$, of the cash balance and of the marginal stock price, only depend on the "past" of the strategy Q, in the sense that,

$$X_t(Q) = X_t(Q^t), \quad S_t(Q) = S_t(Q^t), \tag{1.1}$$

where $Q^t \triangleq (Q_{\min(s,t)})_{0\leq s\leq T}$.

The exogenous nature of the above models facilitates their calibration to market data; see, e.g., [7] by Çetin, Jarrow, Protter, and Warachka. There are, however, some disadvantages. For example, the models in [12], [23], [22], [4], [6], and [7] do not satisfy the natural “closability” property:

$$|Q^n| \leq \frac{1}{n} \implies X_T(Q^n) \to 0, \quad n \to \infty,$$

while in [8] the stock price is not affected by a jump in investor's holdings: $S_t(Q_t + \Delta Q_t) = S_t(Q_t)$. More importantly, this direct modeling approach lacks a well-defined hierarchy of the small agent case, where any particular model is just a parameterization of the general semimartingale setup.

In our project we instead seek to derive the dependence of prices on strategies *endogenously* by relying on the framework developed in financial economics. A starting point here is the postulate that, at any given moment, a price reflects a balance between demand and supply or, more formally, it is an output of an *equilibrium*. In addition to the references cited below we refer the reader to the book [21] by O'Hara and the survey [2] by Amihud, Mendelson, and Pedersen.

To be more specific we denote by ψ the terminal price of the traded security, which we assume to be given exogenously, that is, $S_T(Q) = \psi$ for any strategy Q. Recall that in a small agent model the absence of arbitrage implies the existence of an equivalent probability measure $\mathbb{Q}$ such that

$$S_t = \mathbb{E}_{\mathbb{Q}}[\psi|\mathscr{F}_t], \quad 0 \leq t \leq T, \tag{1.2}$$

where $\mathscr{F}_t$ is the σ-field describing the information available at time t. This result is often called *the fundamental theorem of asset pricing*; in full generality, it has been proved by Delbaen and Schachermayer in [9, 10]. The economic nature of this *pricing measure* $\mathbb{Q}$ does not matter in the standard, small agent, setup. However, it becomes important in an equilibrium-based construction of models for a large trader where it typically originates from a Pareto optimal allocation of wealth; see Definition 2.3 below.

We shall consider an economy formed by M market participants, called hereafter the *market makers*, whose preferences for terminal wealth are defined by utility functions $u_m = u_m(x)$, $m = 1, \ldots, M$, and an identical subjective probability measure $\mathbb{P}$. It is well-known, see Theorem 4.3 for an exact statement, that the Pareto optimality of our market makers' terminal wealth allocation $\alpha = (\alpha^m)_{m=1,\ldots,M}$ yields the pricing measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = v^m u'_m(\alpha^m), \quad m = 1, \ldots, M, \tag{1.3}$$

where $v^m > 0$ is a normalizing constant.

It is natural to expect that in the case when the strategy Q is not anymore negligible an expression similar to (1.2) should still hold true for the *marginal* price process:

$$S_t(Q) = \mathbb{E}_{\mathbb{Q}_t(Q)}[\psi|\mathscr{F}_t(Q)], \quad 0 \leq t \leq T. \tag{1.4}$$

This indicates that the price impact at time t described by the mapping $Q \mapsto S_t(Q)$ may be attributed to two common aspects of market's microstructure:

1. *Information*: $Q \mapsto \mathscr{F}_t(Q)$. Models focusing on information aspects naturally occur in the presence of an insider, where $\mathscr{F}_t(Q)$, the information available to the market makers at time t, is usually generated by the sum of Q and the cumulative demand process of "noise" traders; see Glosten and Milgrom [14], Kyle [19], and Back and Baruch [3], among others.

2. *Inventory*: $Q \mapsto \mathbb{Q}_t(Q)$. In view of (1.3), this reflects how $\alpha_t(Q)$, the Pareto optimal allocation of the total wealth or "inventory" induced by Q, affects the valuation of marginal trades. Note that the random variable $\alpha_t(Q)$ is measurable with respect to the terminal σ-field $\mathscr{F}_T(Q)$ (not with respect to the current σ-field $\mathscr{F}_t(Q)$!).

In our study we shall focus on the inventory aspect of price formation and disregard the informational component. We assume that the market makers share the same exogenously given filtration $(\mathscr{F}_t)_{0\leq t\leq T}$ as the large trader and, in particular, their information flow is not affected by his strategy Q:

$$\mathscr{F}_t(Q) = \mathscr{F}_t, \quad 0 \leq t \leq T.$$

Note that this informational symmetry is postulated only regarding the externally given random outcome. As we shall discuss below, in inventory based models, the actual form of the map $Q \mapsto \mathbb{Q}_t(Q)$, or, equivalently, $Q \mapsto \alpha_t(Q)$ is implied by game-theoretical features of the interaction between the market makers and the investor. In particular, it depends on the knowledge the market makers possess at time t about the subsequent evolution $(Q_s)_{t\leq s\leq T}$ of the investor's strategy, conditionally to the forthcoming random outcome on $[t, T]$.

For example, the models in Grossman and Miller [16] and Garleanu, Pedersen, and Poteshman [13] rely on a setup inspired by the *Arrow-Debreu* *equilibrium.* Their framework implicitly assumes that already at initial time the market makers have *full knowledge* of the investor's future strategy Q (of course, contingent on the unfolding random scenario). In this case, the resulting pricing measures and the Pareto allocations do not depend on time:

$$\mathbb{Q}_t(Q) = \mathbb{Q}(Q), \quad \alpha_t(Q) = \alpha(Q), \quad 0 \leq t \leq T, \tag{1.5}$$

and are determined by the budget equations:

$$\mathbb{E}_{\mathbb{Q}(Q)}[\alpha^m(0)] = \mathbb{E}_{\mathbb{Q}(Q)}[\alpha^m(Q)], \quad m = 1, \ldots, M,$$

and the clearing condition:

$$\sum_{m=1}^{M} \alpha^m(Q) = \sum_{m=1}^{M} \alpha^m(0) + \int_0^T Q_t \, dS_t(Q).$$

Here $\mathbb{Q}(Q)$ and $S(Q)$ are defined in terms of $\alpha(Q)$ by (1.3) and (1.4). The positive sign in the clearing condition is due to our convenience convention to interpret Q as the number of stocks held by the market makers, see Remark 2.6. It is instructive to note that for the case of exponential utilities, when $u_m(x) = -\exp(-a_m x)$, $a_m > 0$, $m = 1, \ldots, M$, the stock price depends only on the "future" of the strategy:

$$S_t(Q) = S_t((Q_s)_{t \leq s \leq T}), \quad 0 \leq t \leq T,$$

which is just the opposite of (1.1).

In our model the interaction between the market makers and the investor takes place according to a *Bertrand competition*; a similar framework (but with a single market maker and only in a one-period setting) was used in Stoll [26]. The key economic assumptions can be summarized as follows:

1. After every trade the market makers can redistribute new income to form a Pareto allocation.

2. As a result of a trade, the expected utilities of the market makers do not change.

Indeed, in a Bertrand competition, market makers will quote the most aggressive prices that will not lower their utility levels; in the limit, this will leave these unchanged. In the process, the market makers are also supposed to find the most effective way to share among themselves the risk of the resulting total endowment, thus producing a Pareto optimal allocation.

This framework implicitly assumes that at any time t the market makers have *no a priori knowledge* about the subsequent trading strategy $(Q_s)_{t\leq s\leq T}$ of the economic agent (even conditional to the future random outcome). As a consequence, the marginal price process $S(Q)$ and the cash balance process $X(Q)$ are related to Q as in (1.1). Similarly, the dependence on Q of the pricing measures and of the Pareto optimal allocations is non-anticipative in the sense that

$$\mathbb{Q}_t(Q) = \mathbb{Q}_t(Q^t), \quad \alpha_t(Q) = \alpha_t(Q^t), \quad 0 \leq t \leq T,$$

which is quite opposite to (1.5).

For ease of exposition, the development of our model will be accomplished in two steps: in this paper we shall deal with a single-period case, while the companion paper [5] will study a continuous-time framework. A brief road map of the present paper will be given at the end of the next section.

2 Model

We consider a single-period financial model with initial time 0 and maturity 1 where $M \in \{1, 2, \ldots\}$ market makers quote prices for a finite number of traded assets. Uncertainty is modeled by a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. As usual, we identify random variables differing on a set of measure zero and use notations $\mathbf{L}^0(\mathbf{R}^d)$, for the metric space of such equivalence classes with values in $\mathbf{R}^d$ and convergence in probability, and $\mathbf{L}^p(\mathbf{R}^d)$, $p \geq 1$, for the Banach space of p-integrable random variables.

The way the market makers serve the incoming orders crucially depends on their attitude toward risk, which we model in the classical framework of expected utility. Thus, we interpret the probability measure $\mathbb{P}$ as a description of the common beliefs of our market makers (same for all) and denote by $u_m = (u_m(x))_{x\in\mathbf{R}}$ market maker m's utility function for terminal wealth.

Assumption 2.1. Each $u_m = u_m(x)$, $m = 1, \ldots, M$, is a strictly concave, strictly increasing, continuously differentiable, and bounded above function on the real line $\mathbf{R}$ satisfying

$$\lim_{x\to\infty} u_m(x) = 0. \tag{2.1}$$

The normalization to zero in (2.1) is added only for notational convenience. From Assumption 2.1 we clearly deduce

$$\lim_{x \to -\infty} u_m(x) = -\infty. \tag{2.2}$$

Many of our results will be derived under the following additional condition on the utility functions, which, in particular, implies their boundedness from above.

Assumption 2.2. Each utility function $u_m = u_m(x)$, $m = 1, \dots, M$, is twice continuously differentiable and its absolute risk aversion coefficient is bounded away from zero and infinity, that is, for some $c > 0$,

$$\frac{1}{c} \leq a_m(x) \triangleq -\frac{u_m''(x)}{u_m'(x)} \leq c, \quad x \in \mathbf{R}. \tag{2.3}$$

From Assumptions 2.1 and 2.2 we deduce

$$\frac{1}{c} \leq -\frac{u_m'(x)}{u_m(x)} \leq c, \quad x \in \mathbf{R}. \tag{2.4}$$

The prices quoted by the market makers are also influenced by their initial endowments $\alpha_0 = (\alpha_0^m)_{m=1,\dots,M} \in \mathbf{L}^0(\mathbf{R}^M)$, where α_0^m is an $\mathscr{F}$-measurable random variable describing the terminal wealth of the mth market maker (if the large investor, to be introduced later, chooses not to trade). We assume that the initial allocation α_0 is *Pareto optimal*, that is, there is no strictly better re-allocation of the same resources in the sense of the following

Definition 2.3. A vector of $\mathscr{F}$-measurable random variables $\alpha = (\alpha^m)_{m=1,\dots,M}$ is called a *Pareto optimal allocation* if

$$\mathbb{E}[|u_m(\alpha^m)|] < \infty, \quad m = 1, \dots, M, \tag{2.5}$$

and there is no other allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with the same total endowment,

$$\sum_{m=1}^{M} \beta^m = \sum_{m=1}^{M} \alpha^m, \tag{2.6}$$

which leaves all market makers not worse and at least one of them better off in the sense that

$$\mathbb{E}[u_m(\beta^m)] \geq \mathbb{E}[u_m(\alpha^m)] \text{ for all } m = 1, \dots, M,$$

and

$$\mathbb{E}[u_m(\beta^m)] > \mathbb{E}[u_m(\alpha^m)] \text{ for some } m \in \{1, \dots, M\}.$$

Remark 2.4. If there are $M \geq 2$ market makers, then, under Assumption 2.1, the integrability requirement (2.5) follows automatically from the other conditions of Pareto optimality. Conversely, in the case of a single market maker, (2.5) is, plainly, the only condition an allocation must satisfy to be Pareto optimal.

Finally, we consider an economic agent or investor who is going to trade the *marketed* contingent claims $\psi = (\psi^j)_{j=1,\dots,J} \in \mathbf{L}^0(\mathbf{R}^J)$, where ψ^j determines the cash payoff of the jth security at the common maturity 1. As a result of trading with the investor the total endowment of the market makers will change from $\Sigma_0 \triangleq \sum_{m=1}^M \alpha_0^m$ to

$$\Sigma(x,q) \triangleq \Sigma_0 + x + \langle q, \psi \rangle = \Sigma_0 + x + \sum_{j=1}^J q^j \psi^j, \tag{2.7}$$

where $x \in \mathbf{R}$ and $q \in \mathbf{R}^J$ are, respectively, the cash amount and the number of contingent claims acquired by the market makers from the investor. Here, $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. Our model will assume that $\Sigma(x,q)$ is re-allocated among the market makers in the form of a Pareto optimal allocation. For this to be possible we have to impose

Assumption 2.5. For any $x \in \mathbf{R}$ and $q \in \mathbf{R}^J$ there is an allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with total random endowment $\Sigma(x,q)$ defined in (2.7) such that

$$\mathbb{E}[u_m(\beta^m)] > -\infty, \quad m = 1, \dots, M. \tag{2.8}$$

As we shall show in Lemma 4.6 below, under Assumptions 2.1 and 2.2, Assumption 2.5 is equivalent to the existence of all exponential moments for ψ under the pricing measure $\mathbb{Q}_0$ associated with the initial Pareto optimal allocation α_0.

We specify an investment strategy of the agent by a vector $q \in \mathbf{R}^J$ of the number of contingent claims $\psi = (\psi^j)_{j=1,\dots,J}$ *sold* by the investor at time 0. For a strategy to be self-financing we have to complement q by a corresponding amount of cash $x \in \mathbf{R}$ *spent* by the investor.

Remark 2.6. Our description of trading strategies follows the standard practice of mathematical finance except for the sign: positive values of q or x now mean *short* positions for the economic agent in securities and cash, and, hence, total *long* positions for the market makers. This convention makes future notations more simple and intuitive.

The central assumptions of our model, which will allow us to identify uniquely the cash amount $x = x(q)$ associated with an order q, are that, as a result of the trade,

1. the random endowment $\Sigma(x, q)$ is redistributed between the market makers to form a new *Pareto* allocation α_1;
2. the market makers' expected utilities *do not change*:

$$\mathbb{E}[u_m(\alpha_1^m)] = \mathbb{E}[u_m(\alpha_0^m)], \quad m = 1, \dots, M. \tag{2.9}$$

The first condition accounts for the market makers' ability to trade among themselves so as to optimally share risk. Only in very special cases the market makers can achieve the required Pareto allocations by trading in ψ alone. This is the case, for example, if all utility functions are exponential, or if $\mathbf{L}^0(\mathbf{R})$ coincides with the linear space generated by 1 and $\psi = (\psi^j)_{j=1,\dots,J}$. In general, a larger set of contingent claims containing non-linear functions of ψ and of the initial endowment Σ_0 is needed.

The second item can be interpreted as a result of the competition among the market makers, where as long as one of them can improve in terms of utility by trading with the agent at a slightly lower price she will do so; in the limit, this leads to the indifference relation (2.9) for the pre- and post-transaction endowments. Note a similarity of this argument to the reasoning behind the famous *Bertrand model of competition* in economic theory.

By analogy with the popular *utility-based valuation* method in mathematical finance, one could view the resulting cash amount $x = x(q)$ as the *market indifference price* for the agent's order q.

Theorem 2.7. *Under Assumptions 2.1 and 2.5, every position $q \in \mathbf{R}^J$ yields a unique cash amount $x = x(q)$ and a unique Pareto optimal allocation $\alpha_1 = \alpha_1(q)$ of $\Sigma(x, q)$ preserving the market makers' expected utilities in the sense of* (2.9).

Proof. For a real number y denote by $\mathscr{B}(y)$ the family of allocations $\beta = (\beta^m)_{m=1,\dots,M}$ with total endowment less than $\Sigma(y, q)$ and such that

$$\mathbb{E}[u_m(\beta^m)] \geq \mathbb{E}[u_m(\alpha_0^m)], \quad m = 1, \dots, M.$$

By Assumptions 2.1 and 2.5, this set is non-empty for sufficiently large y and, by the concavity of utility functions, is a convex subset of $\mathbf{L}^0(\mathbf{R}^M)$. Denote

$$\widehat{y} \triangleq \inf\{y \in \mathbf{R} : \mathscr{B}(y) \neq \emptyset\},$$

let $(y_n)_{n\geq 1}$ be a strictly decreasing sequence of real numbers converging to $\widehat{y}$, and arbitrarily choose $\beta_n \in \mathscr{B}(y_n)$, $n \geq 1$.

From Assumption 2.1 we deduce the existence of $c > 0$ such that, for $m = 1, \dots, M$,

$$y^- \leq c(-u_m(y)), \quad y \in \mathbf{R},$$

where $y^- \triangleq \max(0, -y)$; for example, we can take

$$c = 1/\min_{m=1,\dots,M} u'_m(0).$$

It follows that

$$\mathbb{E}[(\beta_n^m)^-)] \leq c\mathbb{E}[(-u_m(\beta_n^m))] \leq c\mathbb{E}[(-u_m(\alpha_0^m))] < \infty, \quad n \geq 1,$$

and, therefore, the sequence $((\beta_n)^-)_{n\geq 1}$ is bounded in $\mathbf{L}^1(\mathbf{R}^M)$. Since, in addition,

$$\sum_{m=1}^{M} \beta_n^m \leq \Sigma(y_n, q) \leq \Sigma(y_1, q),$$

the family of all possible convex combinations of $(\beta_n)_{n\geq 1}$ is bounded in $\mathbf{L}^0(\mathbf{R}^M)$.

By Lemma A1.1 in [9] we can then choose convex combinations ζ_n of $(\beta_k)_{k\geq n}$, $n \geq 1$, converging almost surely to a random variable $\zeta \in \mathbf{L}^0(\mathbf{R}^M)$. Clearly,

$$\sum_{m=1}^{M} \zeta^m \leq \Sigma(\widehat{y}, q). \tag{2.10}$$

Since the utility functions are bounded from above, the Fatou lemma yields:

$$\mathbb{E}[u_m(\zeta^m)] \geq \limsup_{n\to\infty} \mathbb{E}[u_m(\zeta_n^m)] \geq \mathbb{E}[u_m(\alpha_0^m)], \tag{2.11}$$

where the second estimate holds because $\zeta_n \in \mathscr{B}(y_n)$ by the convexity of $\mathscr{B}(y_n)$. It follows that $\zeta \in \mathscr{B}(\widehat{y})$. The minimality of $\widehat{y}$ then immediately implies the equalities in (2.10) and (2.11) and the Pareto optimality of ζ. Hence, we can select $x = \widehat{y}$ and $\alpha_1 = \zeta$, thus proving their existence.

Finally, the uniqueness of x and α_1 follows from the strict concavity of utility functions. □

In the rest of the paper we shall study in depth the way the cash balances $x = x(q) : \mathbf{R}^J \to \mathbf{R}$ and the Pareto allocations $\alpha_1 = \alpha_1(q) : \mathbf{R}^J \to \mathbf{L}^0(\mathbf{R}^M)$, determined by Theorem 2.7, depend on the order q of the large investor. Our main results, Theorems 5.1 and 5.2, are stated in Section 5. Most notably, Theorem 5.1 characterizes $x(q)$ and $\alpha_1(q)$ in terms of the solution to a *finite-dimensional minimax* problem, while Theorem 5.2 computes the gradient vector and the Hessian matrix for the function $x = x(q)$. Note that this Hessian matrix provides a *quantitative description* of the local price impact effect in our model.

A key role in our analysis is played by a pair of conjugate saddle functions F_0 and G_0 defined in Section 4 and associated with the description of the Pareto allocations in our economy in terms of the utility function of a *representative market maker*. The corresponding conjugate spaces of saddle functions are studied in Section 3. Note that the results of these two sections are also extensively used in the construction of the continuous-time version of our model; see the companion paper [5].

3 Conjugate spaces of saddle functions

A key role in our study of random fields of Pareto optimal allocations in Section 4 will be played by the spaces of saddle functions $\mathbf{F}^i$ and $\mathbf{G}^i$, $i = 1, 2$, introduced in (3.6)–(3.7) and (3.15)–(3.16) below and by the conjugate relationships between these spaces established in Theorems 3.3 and 3.4 of this section.

Recall some standard notations. For a function $f = f(x, y)$, where $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$, we denote by $\frac{\partial f}{\partial x} \triangleq \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right)$ the vector of partial derivatives with respect to x and by $\nabla f \triangleq (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ its gradient. For a set $A \subset \mathbf{R}^d$, the notations ∂A and $\operatorname{cl} A$ stand for, respectively, its boundary and closure. Moreover, for $x, y, z \in \mathbf{R}^d$, $\langle x, y \rangle$ denotes the Euclidean scalar or inner product and $|z| \triangleq \sqrt{\langle z, z \rangle}$ the corresponding norm.

3.1 The spaces $\mathbf{F}^1$ and $\mathbf{F}^2$

For ease of notation denote

$$\mathbf{A} \triangleq (0, \infty)^M \times \mathbf{R} \times \mathbf{R}^J. \tag{3.1}$$

We shall often decompose $a \in \mathbf{A}$ as $a = (v, x, q)$, where $v \in (0, \infty)^M$, $x \in \mathbf{R}$, and $q \in \mathbf{R}^J$. In our later application, $\mathbf{A}$ will play the role of the parameter space of Pareto optimal allocations.

For a function $f : \mathbf{A} \to (-\infty, 0)$ define the following conditions:

(F1) The function f is continuously differentiable on $\mathbf{A}$.

(F2) For any $(x, q) \in \mathbf{R} \times \mathbf{R}^J$, the function $f(\cdot, x, q)$ is positively homogeneous:

$$f(cv, x, q) = cf(v, x, q), \text{ for all } c > 0 \text{ and } v \in (0, \infty)^M, \tag{3.2}$$

and strictly decreasing on $(0, \infty)^M$. Moreover, if $M > 1$ then $f(\cdot, x, q)$ is strictly convex on

$$\mathbf{S}^M \triangleq \{w \in (0, 1)^M : \sum_{m=1}^{M} w^m = 1\},$$

the interior of the simplex in $\mathbf{R}^M$, and for any sequence $(w_n)_{n \geq 1}$ in $\mathbf{S}^M$ converging to a boundary point of $\mathbf{S}^M$

$$\lim_{n \to \infty} f(w_n, x, q) = 0. \tag{3.3}$$

(F3) For any $v \in (0, \infty)^M$, the function $f(v, \cdot, \cdot)$ is concave on $\mathbf{R} \times \mathbf{R}^J$.

(F4) For any $(v, q) \in (0, \infty)^M \times \mathbf{R}^J$, the function $f(v, \cdot, q)$ is strictly concave and strictly increasing on $\mathbf{R}$ and

$$\lim_{x \to \infty} f(v, x, q) = 0. \tag{3.4}$$

(F5) The function f is twice continuously differentiable on $\mathbf{A}$ and, for any $a \in \mathbf{A}$,

$$\frac{\partial^2 f}{\partial x^2}(a) < 0,$$

and the matrix $A(f)(a) = (A^{lm}(f)(a))_{1 \leq l,m \leq M}$ given by

$$A^{lm}(f)(v, x, q) \triangleq \frac{v^l v^m}{\frac{\partial f}{\partial x}} \left(\frac{\partial^2 f}{\partial v^l \partial v^m} - \frac{1}{\frac{\partial^2 f}{\partial x^2}} \frac{\partial^2 f}{\partial v^l \partial x} \frac{\partial^2 f}{\partial v^m \partial x} \right) (v, x, q), \tag{3.5}$$

has full rank.

We now define the families of functions:

$$\mathbf{F}^1 \triangleq \{f : \mathbf{A} \to (-\infty, 0) : \text{(F1)–(F4) hold}\}, \tag{3.6}$$

$$\mathbf{F}^2 \triangleq \{f \in \mathbf{F}^1 : \text{(F5) holds}\}, \tag{3.7}$$

For convenience of future references we formulate some elementary identities for the matrix $A(f)$ from (3.5).

Lemma 3.1. *For $f \in \mathbf{F}^2$, the matrix $A(f)$ defined in* (3.5) *satisfies*

$$\sum_{m=1}^{M} A^{lm}(f) = -v^l \frac{\partial^2 f}{\partial v^l \partial x} / \frac{\partial^2 f}{\partial x^2}, \quad l = 1, \dots, M,$$

$$\sum_{l,m=1}^{M} A^{lm}(f) = -\frac{\partial f}{\partial x} / \frac{\partial^2 f}{\partial x^2}.$$

Proof. From the positive homogeneity condition (3.2) we deduce

$$\sum_{m=1}^{M} v^m \frac{\partial^2 f}{\partial v^l \partial v^m} = 0,$$

$$\sum_{m=1}^{M} v^m \frac{\partial^2 f}{\partial x \partial v^m} = \frac{\partial f}{\partial x},$$

and the result follows. □

Remark 3.2. Slightly abusing notations we shall use the same symbols $\mathbf{F}^i$, $i = 1, 2$, for the families of functions $f = f(v, x)$ defined on $(0, \infty)^M \times \mathbf{R}$ whose natural extensions $\widetilde{f}(v, x, q) \triangleq f(v, x)$ to functions defined on $\mathbf{A}$ belong to $\mathbf{F}^i$. Note that in this case (F3) follows trivially from (F4). A similar convention will also be used for other spaces of functions introduced below.

3.2 The spaces $\mathbf{G}^1$ and $\mathbf{G}^2$

For ease of notation denote

$$\mathbf{B} \triangleq (-\infty, 0)^M \times (0, \infty) \times \mathbf{R}^J.$$

We shall often decompose $b \in \mathbf{B}$ as $b = (u, y, q)$, where $u \in (-\infty, 0)^M$, $y \in (0, \infty)$, and $q \in \mathbf{R}^J$. In our future study of Pareto optimal allocations

$u = (u^m)_{m=1,\dots,M}$ will play the role of the vector of *indirect utilities* of the market makers.

For a function $g : \mathbf{B} \to \mathbf{R}$ define the following conditions:

(G1) The function g is continuously differentiable on $\mathbf{B}$.

(G2) For any $(y, q) \in (0, \infty) \times \mathbf{R}^J$, the function $g(\cdot, y, q)$ is strictly increasing and strictly convex on $(-\infty, 0)^M$. Moreover,

(a) If $(u_n)_{n\geq 1}$ is a sequence in $(-\infty, 0)^M$ converging to 0, then

$$\lim_{n\to\infty} g(u_n, y, q) = \infty. \tag{3.8}$$

(b) If $(u_n)_{n\geq 1}$ is a sequence in $(-\infty, 0)^M$ converging to a boundary point of $(-\infty, 0)^M$, then

$$\lim_{n\to\infty} \left|\frac{\partial g}{\partial u}(u_n, y, q)\right| = \infty. \tag{3.9}$$

(c) If $M > 1$ and $(u_n)_{n\geq 1}$ is a sequence in $(-\infty, 0)^M$ such that

$$\limsup_{n\to\infty} u_n^m < 0 \text{ for all } m = 1, \dots, M \tag{3.10}$$

and

$$\lim_{n\to\infty} u_n^{m_0} = -\infty \text{ for some } m_0 \in \{1, \dots, M\}, \tag{3.11}$$

then

$$\lim_{n\to\infty} g(u_n, y, q) = -\infty. \tag{3.12}$$

(G3) For any $y \in (0, \infty)$, the function $g(\cdot, y, \cdot)$ is convex on $(-\infty, 0)^M \times \mathbf{R}^J$.

(G4) For any $(u, q) \in (-\infty, 0)^M \times \mathbf{R}^J$, the function $g(u, \cdot, q)$ is positively homogeneous, that is,

$$g(u, y, q) = y g(u, 1, q), \quad y > 0. \tag{3.13}$$

(G5) The function g is twice continuously differentiable on $\mathbf{B}$ and, for any $b \in \mathbf{B}$, the matrix $B(g)(b) = (B^{lm}(g)(b))_{1\leq l,m\leq M}$ given by

$$B^{lm}(g)(u,y,q) \triangleq \frac{y}{\frac{\partial g}{\partial u^l}\frac{\partial g}{\partial u^m}} \frac{\partial^2 g}{\partial u^l \partial u^m}(u,y,q) \tag{3.14}$$

has full rank.

We define the families of functions

$$\mathbf{G}^1 \triangleq \{g : \mathbf{B} \to \mathbf{R} : \text{ (G1)–(G4) hold}\}, \tag{3.15}$$

$$\mathbf{G}^2 \triangleq \{g \in \mathbf{G}^1 : \text{ (G5) holds}\}. \tag{3.16}$$

3.3 Conjugacy relationships

The following two theorems establish conjugacy relationships between the families of functions $\mathbf{F}^i$ and $\mathbf{G}^i$, $i = 1, 2$. The detailed proofs of these results are rather long and are postponed until Section 6.

Theorem 3.3. *A function $f : \mathbf{A} \to (-\infty, 0)$ belongs to $\mathbf{F}^1$ if and only if there is $g \in \mathbf{G}^1$ which is conjugate to f in the sense that, for any $(u, y, q) \in \mathbf{B}$,*

$$\begin{aligned} g(u,y,q) &= \sup_{v\in(0,\infty)^M} \inf_{x\in\mathbf{R}} [\langle v,u\rangle + xy - f(v,x,q)] \\ &= \inf_{x\in\mathbf{R}} \sup_{v\in(0,\infty)^M} [\langle v,u\rangle + xy - f(v,x,q)], \end{aligned} \tag{3.17}$$

and, for any $(v, x, q) \in \mathbf{A}$,

$$\begin{aligned} f(v,x,q) &= \sup_{u\in(-\infty,0)^M} \inf_{y\in(0,\infty)} [\langle v,u\rangle + xy - g(u,y,q)], \\ &= \inf_{y\in(0,\infty)} \sup_{u\in(-\infty,0)^M} [\langle v,u\rangle + xy - g(u,y,q)]. \end{aligned} \tag{3.18}$$

The minimax values in (3.17) and (3.18) are attained at unique saddle points and, for any fixed $q \in \mathbf{R}^J$, the following conjugacy relationships between $(v, x) \in (0, \infty)^M \times \mathbf{R}$ and $(u, y) \in (-\infty, 0)^M \times (0, \infty)$ are equivalent:

1. *Given (u, y), the minimax values in (3.17) are attained at (v, x).*
2. *Given (v, x), the minimax values in (3.18) are attained at (u, y).*

3. We have $x = \frac{\partial g}{\partial y}(u, y, q) = g(u, 1, q)$ *and* $v = \frac{\partial g}{\partial u}(u, y, q)$.

4. We have $y = \frac{\partial f}{\partial x}(v, x, q)$ *and* $u = \frac{\partial f}{\partial v}(v, x, q)$.

Moreover, in this case, $f(v, x) = \langle u, v \rangle$, $g(u, y) = xy$, *and*

$$\frac{\partial g}{\partial q}(u, y, q) = -\frac{\partial f}{\partial q}(v, x, q). \tag{3.19}$$

For $f \in \mathbf{F}^2$ and $g \in \mathbf{G}^2$, in addition to the matrices $A(f)$ and $B(g)$ given by (3.5) and (3.14), define the following matrices of second derivatives, for $m = 1, \dots, M$ and $i, j = 1, \dots, J$:

$$C^{mj}(f)(v, x, q) \triangleq \frac{v^m}{\frac{\partial f}{\partial x}} \left(\frac{\partial^2 f}{\partial v^m \partial q^j} - \frac{1}{\frac{\partial^2 f}{\partial x^2}} \frac{\partial^2 f}{\partial v^m \partial x} \frac{\partial^2 f}{\partial x \partial q^j} \right) (v, x, q), \tag{3.20}$$

$$D^{ij}(f)(v, x, q) \triangleq \frac{1}{\frac{\partial f}{\partial x}} \left(-\frac{\partial^2 f}{\partial q^i \partial q^j} + \frac{1}{\frac{\partial^2 f}{\partial x^2}} \frac{\partial^2 f}{\partial x \partial q^i} \frac{\partial^2 f}{\partial x \partial q^j} \right) (v, x, q), \tag{3.21}$$

and

$$E^{mj}(g)(u, y, q) \triangleq \frac{1}{\frac{\partial g}{\partial u^m}} \frac{\partial^2 g}{\partial u^m \partial q^j}(u, y, q) = \frac{1}{\frac{\partial g}{\partial u^m}} \frac{\partial^2 g}{\partial u^m \partial q^j}(u, 1, q), \tag{3.22}$$

$$H^{ij}(g)(u, y, q) \triangleq \frac{1}{y} \frac{\partial^2 g}{\partial q^i \partial q^j}(u, y, q) = \frac{\partial^2 g}{\partial q^i \partial q^j}(u, 1, q), \tag{3.23}$$

where in (3.22) and (3.23) we used the positive homogeneity (3.13) of g with respect to y.

We use standard notations of linear algebra: for a square matrix A of full rank, A^{-1} denotes its inverse, and, for a matrix B, B^T stands for its transpose.

Theorem 3.4. *A function* $f : \mathbf{A} \to (-\infty, 0)$ *belongs to* $\mathbf{F}^2$ *if and only if it is conjugate to a function* $g \in \mathbf{G}^2$ *in the sense that* (3.17) *and* (3.18) *hold.*

Moreover, if, for $q \in \mathbf{R}^J$, *the vectors* $a = (v, x, q) \in \mathbf{A}$ *and* $b = (u, y, q) \in \mathbf{B}$ *are conjugate in the sense of the equivalent conditions of items 1–4 of Theorem 3.3, then the matrices of the second derivatives for* f, $A(f)$, $C(f)$, *and* $D(f)$, *defined in* (3.5), (3.20), *and* (3.21), *and the matrices of the second* *derivatives for g, $B(g)$, $E(g)$, and $H(g)$, defined in* (3.14), (3.22), *and* (3.23), *are related by*

$$B(g)(b) = (A(f)(a))^{-1}, \tag{3.24}$$

$$E(g)(b) = -(A(f)(a))^{-1}C(f)(a), \tag{3.25}$$

$$H(g)(b) = (C(f)(a))^T(A(f)(a))^{-1}C(f)(a) + D(f)(a). \tag{3.26}$$

Remark 3.5. Our choice of the specific form for the matrices $A(f)(a)$, $C(f)(a)$, and $D(f)(a)$ and $B(g)(b)$, $E(g)(b)$, and $H(g)(b)$ was partially motivated by the fact that they are invariant under the transformations $(v, x, q) \to (cv, x, q)$ and $(u, y, q) \to (u, cy, q)$, $c > 0$, which are natural in light of the positive homogeneity conditions (3.2) and (3.13).

3.4 Additional conjugacy relations

If $f \in \mathbf{F}^1$ and $g \in \mathbf{G}^1$ are conjugate in the sense that (3.17) and (3.18) hold true, then any extra condition for f has its conjugate analog for g. Below we shall present several such extensions, which will prove useful in the study of Pareto optimal allocations.

Lemma 3.6. *Suppose $M > 1$. Let $f \in \mathbf{F}^1$ and $g \in \mathbf{G}^1$ be conjugate in the sense of* (3.17) *and* (3.18). *Then the following conditions are equivalent:*

(F6) *For any $(x, q) \in \mathbf{R} \times \mathbf{R}^J$ and any sequence $(w_n)_{n \geq 1}$ in $\mathbf{S}^M$ converging to a boundary point of $\mathbf{S}^M$ we have*

$$\lim_{n \to \infty} \sum_{m=1}^{M} \frac{\partial f}{\partial v^m}(w_n, x, q) = -\infty.$$

(G6) *For any $(y, q) \in (0, \infty) \times \mathbf{R}^J$ and any sequence $(u_n)_{n \geq 1}$ in $(-\infty, 0)^M$ converging to a boundary point of $(-\infty, 0)^M$ we have*

$$\lim_{n \to \infty} g(u_n, y, q) = \infty.$$

Note that (G6) implies (a) and (b) in (G2) and holds trivially when $M = 1$ by (3.8).

Proof. To simplify notations we shall omit the dependence of f and g on the irrelevant parameter q. Recall the notations ∂A and $\mathrm{cl}\, A$ for the boundary and the closure of a set A.

(F6) $\implies$ (G6). Let $(u_n)_{n\geq 1}$ be a sequence in $(-\infty, 0)^M$ converging to $\widehat{u} \in \partial(-\infty, 0)^M$. Denote $x_n \triangleq g(u_n, 1)$, $n \geq 1$, and, contrary to (G6), suppose

$$\liminf_{n\to\infty} g(u_n, 1) = \liminf_{n\to\infty} x_n < \infty.$$

As $g(\cdot, 1)$ is an increasing function on $(-\infty, 0)^M$ and the sequence $(u_n)_{n\geq 1}$ is bounded from below, the sequence $(x_n)_{n\geq 1}$ is also bounded from below. Hence, by passing to a subsequence, we can assume that $(x_n)_{n\geq 1}$ converges to some $\widehat{x} \in \mathbf{R}$.

Denoting $v_n \triangleq \frac{\partial g}{\partial u}(u_n, 1)$ and $w_n \triangleq \frac{v_n}{\sum_{m=1}^M v_n^m}$ we deduce from items 3 and 4 of Theorem 3.3 and the positive homogeneity condition (3.2) for f that

$$u_n = \frac{\partial f}{\partial v}(v_n, x_n) = \frac{\partial f}{\partial v}(w_n, x_n), \quad n \geq 1.$$

As $w_n \in \mathbf{S}^M$, passing to a subsequence, we can assume that $(w_n)_{n\geq 1}$ converges to $\widehat{w} \in \mathrm{cl}\, \mathbf{S}^M$. If $\widehat{w} \in \mathbf{S}^M$, then

$$\widehat{u} = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\partial f}{\partial v}(w_n, x_n) = \frac{\partial f}{\partial v}(\widehat{w}, \widehat{x}) \in (-\infty, 0)^M,$$

contradicting our choice of $\widehat{u}$. If, on the other hand, $\widehat{w} \in \partial\, \mathbf{S}^M$, then, by (3.3) and the monotonicity of $f = f(v, x)$ with respect to x,

$$\lim_{n\to\infty} f(w_n, x_n) = 0.$$

It follows that, for any $v \in (0, \infty)^M$,

$$\begin{aligned} 0 \geq f(v, \widehat{x}) = \lim_{n\to\infty} f(v, x_n) &\geq \lim_{n\to\infty} \left(f(w_n, x_n) + \left\langle \frac{\partial f}{\partial v}(w_n, x_n), v - w_n \right\rangle \right) \\ &= \lim_{n\to\infty} \left(f(w_n, x_n) + \langle u_n, v - w_n \rangle \right) = \langle \widehat{u}, v - \widehat{w} \rangle . \end{aligned}$$

Hence, if we extend, by continuity, the convex function $f(\cdot, \widehat{x})$ to the boundary of its domain by setting

$$f(v, \widehat{x}) = 0, \quad v \in \partial\,(0, \infty)^M,$$

then its subdifferential at $\widehat{w}$ is non-empty and contains $\widehat{u}$. In this case, there is a sequence $(\widetilde{v}_n)_{n\geq 1} \subset (0,\infty)^M$ which converges to $\widehat{w}$ and such that

$$\widehat{u} = \lim_{n\to\infty} \frac{\partial f}{\partial v}(\widetilde{v}_n, \widehat{x}),$$

see Theorem 25.6 in [24]. Denoting $\widetilde{w}_n \triangleq \frac{\widetilde{v}_n}{\sum_{m=1}^M \widetilde{v}_n^m}$, $n \geq 1$, and accounting for (3.2) we obtain

$$\widehat{u} = \lim_{n\to\infty} \frac{\partial f}{\partial v}(\widetilde{w}_n, \widehat{x}),$$

which contradicts (F6).

(G6) $\implies$ (F6). Fix $x \in \mathbf{R}$, let $(w_n)_{n\geq 1}$ be a sequence in $\mathbf{S}^M$ converging to $w \in \partial\,\mathbf{S}^M$, and denote

$$u_n \triangleq \frac{\partial f}{\partial v}(w_n, x), \quad y_n \triangleq \frac{\partial f}{\partial x}(w_n, x), \quad n \geq 1.$$

By (3.3), the concave functions $f(w_n, \cdot)$, $n \geq 1$, converge to 0. Hence, their derivatives also converge to 0, implying that

$$\lim_{n\to\infty} y_n = 0. \tag{3.27}$$

Contrary to (F6) suppose $(u_n)_{n\geq 1}$ contains a bounded subsequence. Passing to a subsequence we can then assume that $(u_n)_{n\geq 1}$ converges to $u \in (-\infty, 0]^M$.

By the equivalence of items 3 and 4 in Theorem 3.3,

$$w_n = y_n \frac{\partial g}{\partial u}(u_n, 1), \quad x = g(u_n, 1), \quad n \geq 1.$$

In view of (3.27), the first equality implies that $u \notin (-\infty, 0)^M$. Hence, $u \in \partial\,(-\infty, 0)^M$, contradicting (G6) and the second equality. □

Lemma 3.7. *Let $f \in \mathbf{F}^1$ and $g \in \mathbf{G}^1$ be conjugate as in* (3.17) *and* (3.18) *and let $c > 0$ be a constant. Then the following conditions are equivalent:*

(F7) *For any $a = (v, x, q) \in \mathbf{A}$ and $m = 1, \ldots, M$,*

$$\frac{1}{c}\frac{\partial f}{\partial x}(a) \leq -v^m \frac{\partial f}{\partial v^m}(a) \leq c \frac{\partial f}{\partial x}(a).$$

(G7) *For any* $(u, q) \in (-\infty, 0)^M \times \mathbf{R}^J$ *and* $m = 1, \dots, M$,

$$\frac{1}{c} \leq -u^m \frac{\partial g}{\partial u^m}(u, 1, q) \leq c.$$

Proof. Follows from the items 3 and 4 in the list of equivalent characterizations of saddle points in Theorem 3.3 and the positive homogeneity condition (3.13) for g. □

Lemma 3.8. *Let* $f \in \mathbf{F}^2$ *and* $g \in \mathbf{G}^2$ *be conjugate as in* (3.17) *and* (3.18) *and let* $c > 0$ *be a constant. Then the following conditions are equivalent:*

(F8) *For any* $a \in \mathbf{A}$ *and any* $z \in \mathbf{R}^M$,

$$\frac{1}{c} \langle z, z \rangle \leq \langle z, A(f)(a) z \rangle \leq c \langle z, z \rangle,$$

where the matrix $A(f)(a)$ *is defined in* (3.5).

(G8) *For any* $b \in \mathbf{B}$ *and any* $z \in \mathbf{R}^M$,

$$\frac{1}{c} \langle z, z \rangle \leq \langle z, B(g)(b) z \rangle \leq c \langle z, z \rangle,$$

where the matrix $B(g)(b)$ *is defined in* (3.14).

Proof. Follows from the inverse relation (3.24) between the matrices $A(f)(a)$ and $B(g)(b)$. □

Lemma 3.9. *Let* $f \in \mathbf{F}^2$ *and* $g \in \mathbf{G}^2$ *be conjugate as in* (3.17) *and* (3.18) *and let* $c > 0$ *be a constant. Then the following conditions are equivalent:*

(F9) *For any* $a = (v, x, q) \in \mathbf{A}$ *and* $m = 1, \dots, M$,

$$-\frac{1}{c} \frac{\partial^2 f}{\partial x^2}(a) \leq v^m \frac{\partial^2 f}{\partial v^m \partial x}(a) \leq -c \frac{\partial^2 f}{\partial x^2}(a).$$

(G9) *For any* $(u, q) \in (-\infty, 0)^M \times \mathbf{R}^J$*, the vector* $z \in \mathbf{R}^M$ *solving the linear equation:*

$$B(g)(u, 1, q) z = \mathbf{1},$$

where $\mathbf{1} \triangleq (1, \dots, 1) \in \mathbf{R}^M$*, satisfies*

$$\frac{1}{c} \leq z^m \leq c, \quad m = 1, \dots, M.$$

Proof. By Lemma 3.1, the condition (F9) can be equivalently stated as

$$\frac{1}{c} \leq (A(f)(a)\mathbf{1})^m \leq c, \quad m = 1, \ldots, M,$$

and the result follows from the inverse relation (3.24) between the matrices $A(f)(a)$ and $B(g)(b)$. $\square$

To simplify future references we define

$$\widetilde{\mathbf{F}}^1 \triangleq \{f \in \mathbf{F}^1 : \text{ (F6) holds}\}, \tag{3.28}$$

$$\widetilde{\mathbf{G}}^1 \triangleq \{g \in \mathbf{G}^1 : \text{ (G6) holds}\}, \tag{3.29}$$

and, for a constant $c > 0$,

$$\widetilde{\mathbf{F}}^2(c) \triangleq \{f \in \mathbf{F}^2 : \text{ (F6)–(F9) hold for given } c\}, \tag{3.30}$$

$$\widetilde{\mathbf{G}}^2(c) \triangleq \{g \in \mathbf{G}^2 : \text{ (G6)–(G9) hold for given } c\}. \tag{3.31}$$

Note that when $M = 1$ the conditions (F6) and (G6) hold trivially; in particular, $\widetilde{\mathbf{F}}^1 = \mathbf{F}^1$ and $\widetilde{\mathbf{G}}^1 = \mathbf{G}^1$.

From Theorems 3.3 and 3.4 and Lemmas 3.6–3.9 we immediately obtain

Theorem 3.10. *A function $f : \mathbf{A} \to (-\infty, 0)$ belongs to $\widetilde{\mathbf{F}}^1$ if and only if it is conjugate to a function $g \in \widetilde{\mathbf{G}}^1$ in the sense of* (3.17) *and* (3.18)*. Moreover, if $c > 0$, then $f \in \widetilde{\mathbf{F}}^2(c)$ if and only if it is conjugate to $g \in \widetilde{\mathbf{G}}^2(c)$.*

4 Random fields of Pareto allocations

A key role in the future quantitative analysis of our model from Section 2 will be played by a pair of conjugate saddle functions $F_0 = F_0(a)$ and $G_0 = G_0(b)$ introduced in Sections 4.3 and 4.4 below. These functions are closely related with the well-known *finite-dimensional* parameterization of Pareto allocations in terms of the utility function of a *representative agent*, see, e.g., Duffie [11] and Karatzas and Shreve [18]. In our setting this parameterization is studied in Sections 4.1 and 4.2.

4.1 Representative market maker

We begin with a study of the classical utility function of the *representative market maker*:

$$r(v,x) \triangleq \sup_{x^1+\dots+x^M=x} \sum_{m=1}^{M} v^m u_m(x^m), \quad v \in (0,\infty)^M, x \in \mathbf{R}, \tag{4.1}$$

which our Theorems 4.1 and 4.2 will identify as an element of $\widetilde{\mathbf{F}}^1$ and $\widetilde{\mathbf{F}}^2(c)$, respectively, under Assumptions 2.1 and 2.2. Note that throughout this section we interpret these families of functions, defined in (3.28) and (3.30), in the sense of Remark 3.2.

Theorem 4.1. *Under Assumption 2.1 the function $r = r(v,x)$ belongs to $\widetilde{\mathbf{F}}^1$. Moreover, for any $(v,x) \in (0,\infty)^M \times \mathbf{R}$, the supremum in* (4.1) *is attained at the vector $\widehat{x} \in \mathbf{R}^M$ uniquely determined by* (4.2) *or, equivalently,* (4.3) *below:*

$$v^m u_m'(\widehat{x}^m) = \frac{\partial r}{\partial x}(v,x), \tag{4.2}$$

$$u_m(\widehat{x}^m) = \frac{\partial r}{\partial v^m}(v,x), \quad m = 1,\dots,M. \tag{4.3}$$

Proof. Define the function $g = g(v,x,z): \ (0,\infty)^M \times \mathbf{R} \times \mathbf{R}^{M-1} \to \mathbf{R}$ by

$$g(v,x,z) \triangleq \sum_{m=1}^{M-1} v^m u_m(z^m) + v^M u_M(x - \sum_{m=1}^{M-1} z^m)$$

and observe that

$$r(v,x) = \sup_{z \in \mathbf{R}^{M-1}} g(v,x,z), \quad v \in (0,\infty)^M, x \in \mathbf{R}. \tag{4.4}$$

For any $v \in (0,\infty)^M$, the function $g(v,\cdot,\cdot)$ is strictly concave, continuously differentiable, and, by (2.1) and (2.2), for any $x \in \mathbf{R}$,

$$\lim_{|z|\to\infty} g(v,x,z) = -\infty.$$

It follows that the upper bound in (4.4) is attained at a unique $\widehat{z} = \widehat{z}(v,x)$ satisfying

$$0 = \frac{\partial g}{\partial z^m}(v,x,\widehat{z}) = v^m u_m'(\widehat{z}^m) - v^M u_M'(x - \sum_{k=1}^{M-1} \widehat{z}^k), \quad m = 1,\dots,M-1, \tag{4.5}$$

and, hence, the upper bound in (4.1) is attained at the unique $\widehat{x} = (\widehat{x}^m)_{m=1,\dots,M}$ given by

$$\begin{aligned}\widehat{x}^m &= \widehat{z}^m, \quad m = 1, \dots, M-1, \\ \widehat{x}^M &= x - \sum_{m=1}^{M-1} \widehat{z}^m.\end{aligned}$$

By Lemma A.3 in Appendix A the function $r(v, \cdot)$ is concave, differentiable, (hence, continuously differentiable), and

$$\frac{\partial r}{\partial x}(v, x) = \frac{\partial g}{\partial x}(v, x, \widehat{z}) = v^M u_M'(x - \sum_{m=1}^{M-1} \widehat{z}^k) = v^M u_M'(\widehat{x}^M),$$

which jointly with (4.5) proves (4.2). As $u_M' > 0$ we have $\frac{\partial r}{\partial x} > 0$ and, hence, $r(v, \cdot)$ is strictly increasing. From (2.1) we obtain

$$\lim_{x \to \infty} r(v, x) = 0.$$

Finally, the strict concavity of $r(v, \cdot)$ follows directly from the strict concavity of $(u_m)_{m=1,\dots,M}$ and the attainability of the upper bound in (4.1), thus finishing the verification of (F4).

For $(x, z) \in \mathbf{R}^M$, the function $g(\cdot, x, z)$ is affine on $(0, \infty)^M$ and, in particular, convex and continuously differentiable. Hence, by Lemma A.4 in Appendix A, the function $r(\cdot, x)$ is convex, differentiable, (hence, continuously differentiable), and

$$\frac{\partial r}{\partial v^m}(v, x) = \frac{\partial g}{\partial v^m}(v, x, \widehat{z}) = u_m(\widehat{x}^m), \quad m = 1, \dots, M,$$

proving (4.3). As $u_m < 0$, the function $r(\cdot, x)$ is strictly decreasing. It is, clearly, positively homogeneous. Moreover, if $M > 1$ then by (2.1)

$$\lim_{n \to \infty} r(w_n, x) = 0, \tag{4.6}$$

for any sequence $(w_n)_{n \geq 1}$ in $\mathbf{S}^M$ converging to $w \in \partial\, \mathbf{S}^M$. Hence, to complete the verification of (F2) we only need to show the strict convexity of this function on $\mathbf{S}^M$.

Let w_1 and w_2 be distinct elements of $\mathbf{S}^M$, w_3 be their midpoint, and $\widehat{x}_i$ be the points in $\mathbf{R}^M$ where the upper bound in (4.1) is attained for $r(x, w_i)$,

$i = 1, 2, 3$. From (4.2) we deduce that the points $(\widehat{x}_i)_{i=1,2,3}$ are distinct and, hence,

$$\begin{aligned} r(w_3, x) &= \sum_{m=1}^{M} w_3^m u_m(\widehat{x}_3^m) = \frac{1}{2}\left(\sum_{m=1}^{M} w_1^m u_m(\widehat{x}_3^m) + \sum_{m=1}^{M} w_2^m u_m(\widehat{x}_3^m)\right) \\ &< \frac{1}{2}\left(\sum_{m=1}^{M} w_1^m u_m(\widehat{x}_1^m) + \sum_{m=1}^{M} w_2^m u_m(\widehat{x}_2^m)\right) = \frac{1}{2}\left(r(x, w_1) + r(x, w_2)\right). \end{aligned}$$

This finishes the verification of (F2).

As we have already shown, $r = r(v, x)$ is a saddle function with well-defined partial derivatives at every point. In this case, r is continuously differentiable, see Theorem 35.8 and Corollary 35.7.1 in [24], and, hence, satisfies (F1).

With (F3) following trivially from (F4), to complete the proof, we only have to verify (F6). Assume $M > 1$ and let $(w_n)_{n\geq 1}$ be a sequence in $\mathbf{S}^M$ converging to $w \in \partial\,\mathbf{S}^M$. For $n \geq 1$ denote by $\widehat{x}_n \in \mathbf{R}^M$ the maximal allocation of x corresponding to w_n. In view of (4.6), $\lim_{n\to\infty} \widehat{x}_n^k = \infty$ for any index k with $w^k > 0$. As $\sum_{m=1}^M \widehat{x}_n^m = x$, there is an index m_0 such that $\lim_{n\to\infty} \widehat{x}_n^{m_0} = -\infty$ and, therefore, accounting for (4.3) and (2.2),

$$\lim_{n\to\infty} \sum_{m=1}^{M} \frac{\partial r}{\partial v^m}(w_n^m, x) \leq \lim_{n\to\infty} \frac{\partial r}{\partial v^{m_0}}(w_n^{m_0}, x) = \lim_{n\to\infty} u_{m_0}(\widehat{x}_n^{m_0}) = -\infty.$$

□

Denote by $t_m = t_m(x)$ the risk-tolerance coefficients of the utility functions $u_m = u_m(x)$:

$$t_m(x) \triangleq -\frac{u_m'(x)}{u_m''(x)} = \frac{1}{a_m(x)}, \quad x \in \mathbf{R},\ m = 1, \dots, M.$$

The same symbol $\widehat{x} = \widehat{x}(v, x)$ will be used to define the functional dependence of the maximal vector $\widehat{x} = (\widehat{x}^m)_{m=1,\dots,M}$ from Theorem 4.1 on v and x.

Theorem 4.2. *Under Assumptions 2.1 and 2.2 the function $r = r(v, x)$ belongs to $\widetilde{\mathbf{F}}^2(c)$ with the same constant $c > 0$ as in (2.3), the function*

$\widehat{x} = \widehat{x}(v,x)$ is continuously differentiable, and, for $l, m = 1, \ldots, M$,

$$\frac{\partial \widehat{x}^m}{\partial x}(v,x) = \frac{t_m(\widehat{x}^m)}{\sum_{k=1}^M t_k(\widehat{x}^k)}, \tag{4.7}$$

$$v^l \frac{\partial \widehat{x}^m}{\partial v^l}(v,x) = v^m \frac{\partial \widehat{x}^l}{\partial v^m}(v,x) = t_m(\widehat{x}^m)\left(\delta_{lm} - \frac{t_l(\widehat{x}^l)}{\sum_{k=1}^M t_k(\widehat{x}^k)}\right), \tag{4.8}$$

where $\delta_{lm} \triangleq 1_{\{l=m\}}$ is the Kronecker delta,

$$\frac{\partial^2 r}{\partial x^2}(v,x) = -\frac{\partial r}{\partial x}(v,x)\frac{1}{\sum_{k=1}^M t_k(\widehat{x}^k)}, \tag{4.9}$$

$$v^m \frac{\partial^2 r}{\partial v^m \partial x}(v,x) = \frac{\partial r}{\partial x}(v,x)\frac{t_m(\widehat{x}^m)}{\sum_{k=1}^M t_k(\widehat{x}^k)}, \tag{4.10}$$

$$v^l v^m \frac{\partial^2 r}{\partial v^l \partial v^m}(v,x) = \frac{\partial r}{\partial x}(v,x) t_l(\widehat{x}^l)\left(\delta_{lm} - \frac{t_m(\widehat{x}^m)}{\sum_{k=1}^M t_k(\widehat{x}^k)}\right), \tag{4.11}$$

and, for the matrix $A(r)$ in (F5),

$$\begin{aligned} A^{lm}(r)(v,x) &\triangleq \frac{v^l v^m}{\frac{\partial r}{\partial x}}\left(\frac{\partial^2 r}{\partial v^l \partial v^m} - \frac{1}{\frac{\partial^2 r}{\partial x^2}}\frac{\partial^2 r}{\partial v^l \partial x}\frac{\partial^2 r}{\partial v^m \partial x}\right)(v,x) \\ &= t_l(\widehat{x}^l)\delta_{lm}. \end{aligned} \tag{4.12}$$

Proof. The proof relies on the Implicit Function Theorem. Define the function $h = h(v,x,y,z) : (0,\infty)^M \times \mathbf{R} \times \mathbf{R}^M \times \mathbf{R} \to \mathbf{R}^{M+1}$ by

$$\begin{aligned} h^m(v,x,y,z) &= z - v^m u'_m(y^m), \quad m = 1, \ldots, M, \\ h^{M+1}(v,x,y,z) &= \sum_{m=1}^M y^m - x, \end{aligned}$$

and observe that, by Theorem 4.1,

$$h\left(v, x, \widehat{x}(v,x), \frac{\partial r}{\partial x}(v,x)\right) = 0, \quad (v,x) \in (0,\infty)^M \times \mathbf{R}.$$

Fix (v_0, x_0), set $y_0 \triangleq \widehat{x}(v_0,x_0)$, $z_0 \triangleq \frac{\partial r}{\partial x}(v_0,x_0)$, and denote by $B = (B^{kl})_{k,l=1,\ldots,M+1}$ the Jacobian of $h(v_0,x_0,\cdot,\cdot)$ evaluated at (y_0,z_0). Accounting for the fact that

$$v_0^m u'_m(y_0^m) = z_0, \quad m = 1, \ldots, M,$$

we deduce

$$\begin{aligned}
B^{kl} &= B^{lk} = -v_0^k u_k''(y_0^k)\delta_{kl} = \frac{z_0}{t_k(y_0^k)}\delta_{kl}, \quad k,l = 1,\dots,M,\\
B^{(M+1)m} &= B^{m(M+1)} = 1, \quad m = 1,\dots,M,\\
B^{(M+1)(M+1)} &= 0.
\end{aligned}$$

Direct computations show that the inverse matrix $C \triangleq B^{(-1)}$ is given by

$$\begin{aligned}
C^{kl} &= C^{lk} = \frac{t_k(y_0^k)}{z_0}\left(\delta_{kl} - \frac{t_l(y_0^l)}{\sum_{i=1}^M t_i(y_0^i)}\right), \quad k,l = 1,\dots,M,\\
C^{(M+1)m} &= C^{m(M+1)} = \frac{t_m(y_0^m)}{\sum_{i=1}^M t_i(y_0^i)}, \quad m = 1,\dots,M,\\
C^{(M+1)(M+1)} &= -\frac{z_0}{\sum_{i=1}^M t_i(y_0^i)}.
\end{aligned}$$

Since, for $m = 1,\dots,M+1$ and $l = 1,\dots,M$,

$$\begin{aligned}
&\frac{\partial h^m}{\partial x}(v_0,x_0,y_0,z_0) = -\delta_{m(M+1)},\\
&v^l\frac{\partial h^m}{\partial v^l}(v_0,x_0,y_0,z_0) = -v^m u_m'(y_0^m)\delta_{lm} = -z_0\delta_{lm},
\end{aligned}$$

the Implicit Function Theorem implies the continuous differentiability of the functions $\widehat{x} = \widehat{x}(v,x)$ and $\frac{\partial r}{\partial x} = \frac{\partial r}{\partial x}(v,x)$ in the neighborhood of (v_0,x_0) and the identities:

$$\begin{aligned}
\frac{\partial \widehat{x}^m}{\partial x}(v_0,x_0) &= -\sum_{k=1}^{M+1} C^{mk}\frac{\partial h^k}{\partial x}(v_0,x_0,y_0,z_0) = C^{m(M+1)},\\
v^l\frac{\partial \widehat{x}^m}{\partial v^l}(v_0,x_0) &= -\sum_{k=1}^{M+1} C^{mk}v^l\frac{\partial h^k}{\partial v^l}(v_0,x_0,y_0,z_0) = z_0C^{ml},\\
\frac{\partial^2 r}{\partial x^2}(v_0,x_0) &= -\sum_{k=1}^{M+1} C^{(M+1)k}\frac{\partial h^k}{\partial x}(v_0,x_0,y_0,z_0) = C^{(M+1)(M+1)},\\
v^l\frac{\partial^2 r}{\partial x\partial v^l}(v_0,x_0) &= -\sum_{k=1}^{M+1} C^{(M+1)k}v^l\frac{\partial h^k}{\partial v^l}(v_0,x_0,y_0,z_0) = z_0C^{(M+1)l},
\end{aligned}$$

proving (4.7)–(4.8) and (4.9)–(4.10).

The continuous differentiability of $\frac{\partial r}{\partial v} = \frac{\partial r}{\partial v}(v, x)$ with respect to v and the identity (4.11) follow from (4.3) and (4.8). Direct computations relying on (4.9), (4.10), and (4.11) lead to the expression (4.12) for $A(r)$, which jointly with (4.9) implies the validity of (F5) for $r = r(v, x)$.

Finally, accounting for (2.4) and observing that (2.3) can be equivalently stated as

$$\frac{1}{c} \leq t_m(x) \leq c, \quad x \in \mathbf{R},\ m = 1, \ldots, M,$$

we deduce that, for the function $r = r(v, x)$, the property (F7) follows from (4.2) and (4.3), (F8) is implied by (4.12), and (F9) follows from (4.9) and (4.10). $\square$

4.2 Parameterization of Pareto allocations

A well-known use of the utility function of the representative market maker defined in (4.1) is the following characterization of Pareto optimal allocations.

Theorem 4.3. *Under Assumption 2.1 the following statements are equivalent for a random vector $\alpha \in \mathbf{L}^0(\mathbf{R}^M)$:*

1. *The allocation $\alpha = (\alpha^m)_{m=1,\ldots,M}$ is Pareto optimal.*
2. *The random variables $\alpha = (\alpha^m)_{m=1,\ldots,M}$ satisfy the integrability condition (2.5) and there is a (deterministic) vector $\lambda = (\lambda^m)_{m=1,\ldots,M} \in \mathbf{S}^M$ such that*

$$\lambda^m u'_m(\alpha^m) = \frac{\partial r}{\partial x}(\lambda, \sum_{m=1}^{M} \alpha^m), \quad m = 1, \ldots, M. \tag{4.13}$$

Moreover, such a vector λ is defined uniquely.

Proof. Denote by $\mathscr{B}$ the family of allocations $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with the same total endowment as α, that is, satisfying (2.6). Clearly, $\mathscr{B}$ is a convex set.

$1 \implies 2$: The integrability condition (2.5) holds by the definition of a Pareto optimal allocation. By the concavity of the utility functions, the set

$$C \triangleq \{z \in \mathbf{R}^M :\ z^m \leq \mathbb{E}[u_m(\beta^m)],\ m = 1, \ldots, M,\ \text{for some } \beta \in \mathscr{B}\}$$

is convex and, by the Pareto optimality of α, the point

$$\widehat{z}^m \triangleq \mathbb{E}[u_m(\alpha^m)], \quad m = 1 \ldots, M,$$

belongs to the boundary of C. Hence, there is a non-zero $v \in \mathbf{R}^M$ such that

$$\langle v, \widehat{z} \rangle \geq \langle v, z \rangle, \quad z \in C,$$

or, equivalently,

$$\mathbb{E}[\sum_{m=1}^{M} v^m u_m(\alpha^m)] = \sup_{\beta \in \mathscr{B}} \mathbb{E}[\sum_{m=1}^{M} v^m u_m(\beta^m)].$$

As $v \neq 0$, the properties of the utility functions in Assumption 2.1 imply that $v \in (0, \infty)^M$ and, then, by normalization, we can choose $\lambda \triangleq v \in \mathbf{S}^M$. By Theorem 4.1, the fact that the upper bound above is attained at α is equivalent to (4.13).

2 $\implies$ 1: By Theorem 4.1, for any $\beta \in \mathscr{B}$

$$\sum_{m=1}^{M} \lambda^m u_m(\beta^m) \leq r(\lambda, \sum_{m=1}^{M} \alpha^m) = \sum_{m=1}^{M} \lambda^m u_m(\alpha^m).$$

Given the integrability requirement (2.5), this clearly implies the Pareto optimality of α.

Finally, we note that (4.13) and the normalization requirement $\lambda \in \mathbf{S}^M$ determine λ uniquely. □

The following result allows us to parameterize the Pareto allocations in our economy by the set $\mathbf{A}$ defined in (3.1).

Lemma 4.4. *Let Assumption 2.1 hold. Then Assumption 2.5 is equivalent to*

$$\mathbb{E}[r(v, \Sigma(x, q))] > -\infty, \quad (v, x, q) \in \mathbf{A}. \tag{4.14}$$

In this case, for $a = (v, x, q) \in \mathbf{A}$, the random vector $\pi(a) \in \mathbf{L}^0(\mathbf{R}^M)$ defined by

$$v^m u_m'(\pi^m(a)) = \frac{\partial r}{\partial x}(v, \Sigma(x, q)), \quad m = 1, \ldots, M, \tag{4.15}$$

forms a Pareto optimal allocation. Conversely, for $(x, q) \in \mathbf{R} \times \mathbf{R}^J$, any Pareto allocation of the total endowment $\Sigma(x, q)$ is given by (4.15) for some $v \in (0, \infty)^M$.

Proof. Under Assumption 2.5, there is an allocation $\beta = (\beta^m)_{m=1,\dots,M}$ of $\Sigma(x,q)$ satisfying (2.8). Since,

$$\sum_{m=1}^{M} v^m u_m(\beta^m) \leq r(v, \Sigma(x,q)) \leq 0,$$

we obtain (4.14).

Assume now that (4.14) holds. By Theorem 4.1, the condition (4.15) is equivalent to

$$\sum_{m=1}^{M} v^m u_m(\pi^m(a)) = r(v, \Sigma(x,q)),$$

which, jointly with (4.14), implies that $u_m(\pi^m(a)) \in \mathbf{L}^1$, $m = 1, \dots, M$, and, in particular, implies Assumption 2.5. The Pareto optimality of $\pi(a)$ is now an immediate corollary of Theorem 4.3.

Finally, the fact that any Pareto allocation of $\Sigma(x,q)$ is given by (4.15), for some $v \in (0,\infty)^M$, follows from Theorem 4.3. □

Hereafter we shall denote by $\pi : \mathbf{A} \to \mathbf{L}^0(\mathbf{R}^M)$ the random field of Pareto allocations defined in Lemma 4.4.

Lemma 4.5. *Under Assumptions 2.1 and 2.5, for any $a = (v,x,q) \in \mathbf{A}$, there is a probability measure $\mathbb{Q}(a)$ such that*

$$\frac{d\mathbb{Q}(a)}{d\mathbb{P}} = \frac{\frac{\partial r}{\partial x}(v, \Sigma(x,q))}{\mathbb{E}[\frac{\partial r}{\partial x}(v, \Sigma(x,q))]} = \frac{u'_m(\pi^m(a))}{\mathbb{E}[u'_m(\pi^m(a))]}, \quad m = 1, \dots, M, \tag{4.16}$$

where $\pi(a) = (\pi^m(a))_{m=1,\dots,M}$ is the Pareto optimal allocation defined in (4.15).

Proof. In view of Lemma 4.4 it is sufficient to verify that

$$\mathbb{E}[\frac{\partial r}{\partial x}(v, \Sigma(x,q))] < \infty.$$

This follows from (4.14) and the inequality

$$\frac{\partial r}{\partial x}(v, \Sigma(x,q)) \leq r(v, \Sigma(x,q)) - r(v, \Sigma(x-1,q)),$$

which holds by the concavity of $r(v,\cdot)$. □

The probability measure $\mathbb{Q}(a)$ defined in (4.16) is called the *pricing measure* of the Pareto optimal allocation $\pi(a)$. Denote by $\mathbb{Q}_0$ the pricing measure of the initial Pareto allocation $\alpha_0 = (\alpha_0^m)_{m=1,\dots,M}$:

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}} = \frac{u'_m(\alpha_0^m)}{\mathbb{E}[u'_m(\alpha_0^m)]}, \quad m = 1, \dots, M. \tag{4.17}$$

Lemma 4.6. *Under Assumptions 2.1 and 2.2 the pricing measure $\mathbb{Q}_0$ of the initial Pareto optimal allocation is well-defined by* (4.17) *and Assumption 2.5 is equivalent to the existence of all exponential moments of $\psi = (\psi^j)_{j=1,\dots,J}$ under $\mathbb{Q}_0$:*

$$\mathbb{E}_{\mathbb{Q}_0}[e^{z|\psi|}] < \infty, \quad z \in \mathbf{R}. \tag{4.18}$$

Proof. For $\mathbb{Q}_0$ to be well-defined we have to verify that

$$\mathbb{E}[u'_m(\alpha_0^m)] < \infty, \quad m = 1, \dots, M.$$

This follows from (2.4) and the inequality

$$\mathbb{E}[u_m(\alpha_0^m)] > -\infty, \quad m = 1, \dots, M,$$

which holds by the definition of a Pareto optimal allocation.

Let $\lambda_0 \in \mathbf{S}^M$ denote the weight of α_0 defined in Theorem 4.3. The positive homogeneity and the monotonicity properties of $r(\cdot, x)$ imply the equivalence of (4.14) and

$$\mathbb{E}[r(\lambda_0, \Sigma(x, q))] > -\infty, \quad (x, q) \in \mathbf{R} \times \mathbf{R}^J. \tag{4.19}$$

From (4.2) and (2.4) we deduce

$$\frac{1}{c}\frac{\partial r}{\partial x}(v, x) \leq -r(v, x) \leq c\frac{\partial r}{\partial x}(v, x), \quad (v, x) \in (0, \infty)^M \times \mathbf{R},$$

where the constant $c > 0$ is taken from Assumption 2.2. Hence, by Gronwall's inequality, for $(x, q) \in \mathbf{R} \times \mathbf{R}^J$,

$$\begin{aligned}
&\frac{1}{c}\frac{\partial r}{\partial x}(\lambda_0, \Sigma_0) \exp\left(\frac{1}{c}(x + \langle q, \psi\rangle)^+ - c(x + \langle q, \psi\rangle)^-\right) \leq -r(\lambda_0, \Sigma(x, q)) \\
&\leq c\frac{\partial r}{\partial x}(\lambda_0, \Sigma_0) \exp\left(c(x + \langle q, \psi\rangle)^+ - \frac{1}{c}(x + \langle q, \psi\rangle)^-\right),
\end{aligned}$$

where $x^+ \triangleq \max(x, 0)$ and $x^- \triangleq \max(-x, 0)$. The equivalence of (4.18) and (4.19) follows now from the definition of $\mathbb{Q}_0$. $\square$

4.3 Stochastic process of indirect utilities

A key role in the "quantitative" analysis of investment strategies presented in Section 5 will be played by the *indirect utility* function of the representative market maker defined by

$$F_0(a) \triangleq \mathbb{E}[r(v, \Sigma(x, q))], \quad a = (v, x, q) \in \mathbf{A}. \tag{4.20}$$

It will be convenient to view $F_0(a)$ as the initial value of the martingale-process of indirect utility, $F(a) = (F_0(a), F_1(a))$, with

$$F_1(a) \triangleq r(v, \Sigma(x, q)), \quad a = (v, x, q) \in \mathbf{A}. \tag{4.21}$$

Some important properties of the function $F_0 : \mathbf{A} \to (-\infty, 0)$ and of the random field $F_1 : \mathbf{A} \to \mathbf{L}^0$ are collected in Theorems 4.7 and 4.13 below.

To state the results we need to introduce some standard notations. Let m, n, and l be non-negative integers and U be an open subset of $\mathbf{R}^n$. We denote by $\mathbf{C}^m = \mathbf{C}^m(U, \mathbf{R}^l)$ the Fréchet space of m-times continuously differentiable maps $f : U \to \mathbf{R}^l$ with the topology generated by the semi-norms

$$\|f\|_{m,C} \triangleq \sum_{0 \le |\mathbf{k}| \le m} \sup_{x \in C} |\frac{\partial^{\mathbf{k}} f}{\partial x^{\mathbf{k}}}(x)|,$$

where C is a compact subset of U, $\mathbf{k} = (k_1, \dots, k_n)$ is a multi-index of non-negative integers, $|\mathbf{k}| \triangleq \sum_{i=1}^n k_i$, and

$$\frac{\partial^{\mathbf{k}} f}{\partial x^{\mathbf{k}}} \triangleq \frac{\partial^{|\mathbf{k}|} f}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}.$$

In particular, for $m = 0$, $\frac{\partial^0}{\partial x^0}$ is the identity operator and $\|f\|_{0,C} \triangleq \sup_{x \in C} |f(x)|$.

It is natural to view a random field $\xi = \xi(x) : U \to \mathbf{L}^0$ as a function $\xi = \xi(x, \omega)$ defined on $U \times \Omega$. For $\omega \in \Omega$, the function $\xi(\cdot, \omega) : U \to \mathbf{R}$ is called the *sample path* of ξ at ω.

Theorem 4.7. *Under Assumptions 2.1 and 2.5 the function* $F_0 : \mathbf{A} \to (-\infty, 0)$ *defined in* (4.20) *and the sample paths of the random field* $F_1 : \mathbf{A} \to \mathbf{L}^0$ *defined in* (4.21) *belong to the space* $\widetilde{\mathbf{F}}^1$ *defined in* (3.28). *Moreover, for any compact set* $C \subset \mathbf{A}$

$$\mathbb{E}[\|F_1(\cdot)\|_{1,C}] < \infty, \tag{4.22}$$

and for any $a = (v, x, q) \in \mathbf{A}$ *and* $i = 1, \dots, M + 1 + J$

$$\frac{\partial F_0}{\partial a_i}(a) = \mathbb{E}[\frac{\partial F_1}{\partial a_i}(a)]. \tag{4.23}$$

The proof will rely on several lemmas.

Lemma 4.8. *Let* U *be an open set in* $\mathbf{R}^n$*,* m *be a non-negative integer, and* $\xi = \xi(x) : U \to \mathbf{L}^1$ *be a random field with sample paths in* $\mathbf{C}^m = \mathbf{C}^m(U)$ *such that for any compact set* $C \subset U$

$$\mathbb{E}[\|\xi\|_{m,C}] < \infty.$$

Then the function $f(x) \triangleq \mathbb{E}[\xi(x)]$*,* $x \in U$*, belongs to* $\mathbf{C}^m$ *and for any multi-index* $\mathbf{k} = (k_1, \dots, k_n)$ *with* $|\mathbf{k}| \le m$

$$\frac{\partial^{\mathbf{k}} f}{\partial x^{\mathbf{k}}}(x) = \mathbb{E}[\frac{\partial^{\mathbf{k}} \xi}{\partial x^{\mathbf{k}}}(x)], \quad x \in U.$$

Proof. By induction, it is sufficient to consider the case when $\mathbf{k} = (1, 0, \dots, 0)$ and, hence, $\frac{\partial^{\mathbf{k}}}{\partial x^{\mathbf{k}}} = \frac{\partial}{\partial x_1}$. However, then the result is an immediate consequence of the Fubini Theorem. □

Lemma 4.9. *Let* U *be an open set in* $\mathbf{R}^d$*,* $f : U \to \mathbf{R}$ *be a convex function,* C *be a compact subset of* U*, and* $\varepsilon > 0$ *be such that*

$$C(\varepsilon) \triangleq \{x \in \mathbf{R}^d : \inf_{y \in C} |x - y| \le \varepsilon\} \subset U. \tag{4.24}$$

Then for any $y \in C$ *we have*

$$\min_{x \in C} f(x) \ge f(y) + \frac{\sup_{x \in C} |x - y|}{\varepsilon} \left(f(y) - \max_{x \in C(\varepsilon)} f(x) \right). \tag{4.25}$$

Proof. Fix $y \in C$. For any $x \in C$ there is $z \in \partial C(\varepsilon)$ such that y is a convex combination of x and z: $y = tx + (1 - t)z$ for some $t \in (0, 1)$. Using the fact that $|y - z| \ge \varepsilon$ we obtain

$$\frac{1 - t}{t} = \frac{|x - y|}{|y - z|} \le \frac{\sup_{x \in C} |x - y|}{\varepsilon}. \tag{4.26}$$

The convexity of f implies

$$f(y) \le t f(x) + (1 - t) f(z),$$

or, equivalently,

$$f(x) \geq f(y) + \frac{1-t}{t}(f(y) - f(z)),$$

which, in view of (4.26), yields (4.25). $\square$

Lemma 4.10. *In addition to the conditions of Lemma 4.9 assume that* $f \in \mathbf{C}^1(U)$. *Then*

$$\|f\|_{1,C} \leq \left(\frac{2\sqrt{d}}{\varepsilon} + 1\right) \|f\|_{0,C(\varepsilon)}. \tag{4.27}$$

Proof. For $y \in C$ and $x \in C(\varepsilon)$ we obtain from the convexity of f:

$$f(x) - f(y) \geq \langle x - y, \nabla f(y) \rangle .$$

It follows that

$$2\|f\|_{0,C(\varepsilon)} \geq \sup_{y \in C} \sup_{\{x:\, |x-y| \leq \varepsilon\}} (\langle x - y, \nabla f(y)\rangle) = \varepsilon \sup_{y \in C} |\nabla f(y)|.$$

Since,

$$|\nabla f(y)| \triangleq \sqrt{\sum_{i=1}^{d} \left(\frac{\partial f}{\partial x_i}(y)\right)^2} \geq \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |\frac{\partial f}{\partial x_i}(y)|,$$

we obtain

$$|f(y)| + \sum_{i=1}^{d} |\frac{\partial f}{\partial x_i}(y)| \leq \left(\frac{2\sqrt{d}}{\varepsilon} + 1\right) \|f\|_{0,C(\varepsilon)},$$

which clearly implies (4.27). $\square$

Lemma 4.11. *Let* U *be an open set in* $\mathbf{R}^d$, $\xi = \xi(x) : U \to \mathbf{L}^1$ *be a random field with sample paths in the space of convex functions on* U. *Then, for any compact set* $C \subset U$,

$$\mathbb{E}[\|\xi\|_{0,C}] < \infty. \tag{4.28}$$

If, in addition, the sample paths of ξ *belong to* $\mathbf{C}^1$ *then*

$$\mathbb{E}[\|\xi\|_{1,C}] < \infty. \tag{4.29}$$

Proof. Let us first show that for any compact set $C \subset U$

$$\mathbb{E}[\max_{x \in C} \xi(x)] < \infty. \tag{4.30}$$

Without restricting generality we can assume that C is the closed convex hull of a finite family $(x_i)_{i=1,\dots,I}$ in U. From the convexity of ξ we then deduce

$$\max_{x \in C} \xi(x) = \max_{i=1,\dots,I} \xi(x_i),$$

and (4.30) follows from the assumption $\xi(x) \in \mathbf{L}^1$, $x \in U$.

Since C is a compact set in U, for sufficiently small $\varepsilon > 0$ the set $C(\varepsilon)$ defined in (4.24) is also a compact subset of U. By (4.30) and Lemma 4.9 we obtain

$$\mathbb{E}[\min_{x \in C} \xi(x)] > -\infty,$$

which implies (4.28).

Finally, if $f \in \mathbf{C}^1$, then (4.29) follows from (4.28) and Lemma 4.10. $\square$

Lemma 4.12. *Let $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$ be open sets, $\xi : U \times V \to \mathbf{L}^1$ be a random field with sample paths in the space of concave-convex functions on $U \times V$, and*

$$f(x, y) \triangleq \mathbb{E}[\xi(x, y)], \quad (x, y) \in U \times V.$$

Then $f = f(x, y)$ is a concave-convex function on $U \times V$ and for any compact set $C \subset U \times V$

$$\mathbb{E}[\|\xi\|_{0,C}] < \infty. \tag{4.31}$$

If, in addition, the sample paths of ξ belong to $\mathbf{C}^1$, then $f \in \mathbf{C}^1$ and

$$\mathbb{E}[\|\xi\|_{1,C}] < \infty. \tag{4.32}$$

Proof. It is sufficient to consider the case $C = C_1 \times C_2$, where C_1 and C_2 are compact subsets of U and V, respectively. To prove (4.31) it is enough to show that

$$\sup_{x \in C_1} \sup_{y \in C_2} \xi(x, y) \in \mathbf{L}^1. \tag{4.33}$$

Indeed, having established (4.33) for any random field ξ and any pair of open sets U and V satisfying the conditions of the lemma we deduce

$$\inf_{x \in C_1} \inf_{y \in C_2} \xi(x, y) = -\sup_{x \in C_1} \sup_{y \in C_2} (-\xi(x, y)) \in \mathbf{L}^1,$$

which, jointly with (4.33), implies (4.31). To verify (4.33) observe that the random field

$$\eta(y) \triangleq \sup_{x \in C_1} \xi(x,y), \quad y \in V,$$

has sample paths in the space of convex functions and, by Lemma 4.11, $\eta(y) \in \mathbf{L}^1$. Another application of Lemma 4.11 yields $\|\eta\|_{0,C_2} \in \mathbf{L}^1$, which clearly implies (4.33).

To verify (4.32), choose $\varepsilon > 0$ so that the sets $C_1(\varepsilon)$ and $C_2(\varepsilon)$ defined by (4.24) are still in U and V. Then, by Lemma 4.10, there is $c = c(\varepsilon) > 0$ such that for any $x \in C_1$ and $y \in C_2$

$$\begin{aligned} \|\xi(x,\cdot)\|_{1,C_2} + \|\xi(\cdot,y)\|_{1,C_1} &\leq c(\|\xi(x,\cdot)\|_{0,C_2(\varepsilon)} + \|\xi(\cdot,y)\|_{0,C_1(\varepsilon)}) \\ &\leq 2c\|\xi\|_{0,C_1(\varepsilon)\times C_2(\varepsilon)}, \end{aligned}$$

and the result follows. $\square$

Proof of Theorem 4.7. The assertions concerning the sample paths of $F_1 = F_1(a)$ are immediate corollaries of the corresponding properties of $r = r(v,x)$ established in Theorem 4.1. From Lemma 4.4 we deduce $F_1(a) \in \mathbf{L}^1$, $a \in \mathbf{A}$, and then, by Lemma 4.12, obtain (4.22).

The continuous differentiability of $F_0 = F_0(a)$ and the identities (4.23) follow from Lemma 4.8. The validity of (3.3) in (F2) for F_0 follows from the corresponding property for F_1 and the Dominated Convergence Theorem if we observe that for any allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ of $\Sigma(x,q)$

$$0 \geq \inf_{w \in \mathbf{S}^M} F_1(w,x,q) \geq \sum_{m=1}^{M} u_m(\beta^m),$$

and use β from Assumption 2.5. Similarly, every other assertion regarding F_0 is an elementary corollary of the corresponding feature for F_1. $\square$

For later use let us compute some of the first derivatives of $F_0 = F_0(a)$ and $F_1 = F_1(a)$. From Theorem 4.1 we deduce, for any $a = (v,x,q) \in \mathbf{A}$, $m = 1,\dots,M$, and $j = 1,\dots,J$,

$$\begin{aligned} \frac{\partial F_1}{\partial v^m}(a) &= u_m(\pi^m(a)), \\ \frac{\partial F_1}{\partial q^j}(a) &= \frac{\partial F_1}{\partial x}(a)\psi^j, \end{aligned}$$

and thus

$$\frac{\partial F_0}{\partial v^m}(a) = \mathbb{E}[u_m(\pi^m(a))], \tag{4.34}$$

$$\frac{\partial F_0}{\partial q^j}(a) = \frac{\partial F_0}{\partial x}(a)\mathbb{E}_{\mathbb{Q}(a)}[\psi^j]. \tag{4.35}$$

In particular, $\frac{\partial F_0}{\partial v^m}(a)$ coincides with the indirect or expected utility of the mth market maker at Pareto optimal allocation $\pi(a)$. This fact will play a key role in the quantitative analysis of investment strategies in Section 5.

We now study the second derivatives of F_0 and F_1.

Theorem 4.13. *Under Assumptions 2.1, 2.2, and 2.5 the function $F_0 : \mathbf{A} \to (-\infty, 0)$ from* (4.20) *and the sample paths of the random field $F_1 : \mathbf{A} \to \mathbf{L}^0$ from* (4.21) *belong to the space $\widetilde{\mathbf{F}}^2(c)$ defined in* (3.30) *with the same constant $c > 0$ as in Assumption 2.2. Moreover, for any compact set $C \subset \mathbf{A}$*

$$\mathbb{E}[\|F_1\|_{2,C}] < \infty, \tag{4.36}$$

and for any $a = (v, x, q) \in \mathbf{A}$ and $i, j = 1, \dots, M + 1 + J$

$$\frac{\partial^2 F_0}{\partial a^i \partial a^j}(a) = \mathbb{E}[\frac{\partial^2 F_1}{\partial a^i \partial a^j}(a)]. \tag{4.37}$$

Proof. The assertions regarding the sample paths of F_1 follow from Theorem 4.2. The validity of (F7) for F_0 follows from (4.23) and the corresponding property for F_1.

To verify (4.36) fix a compact set $C \subset \mathbf{A}$ and suppose $a_0 \triangleq (\lambda_0, 0, 0) \in C$, where $\lambda_0 \in \mathbf{S}^M$ is the weight vector of the initial Pareto optimal allocation α_0. From the formulas for the second derivatives of $r = r(v, x)$ in Theorem 4.2 and Assumption 2.2 we deduce the existence of $b = b(C) > 0$ such that, for $i, j = 1, \dots, M + 1 + J$,

$$|\frac{\partial^2 F_1}{\partial a^i \partial a^j}(a)| \leq b\frac{\partial F_1}{\partial x}(a)(1 + |\psi|^2), \tag{4.38}$$

$$|\frac{\partial^2 F_1}{\partial a^i \partial x}(a)| \leq b\frac{\partial F_1}{\partial x}(a)(1 + |\psi|), \quad a \in C. \tag{4.39}$$

By Gronwall's inequality, we deduce from (4.39) the existence of $b > 0$ such that

$$\frac{\partial F_1}{\partial x}(a) \leq b\frac{\partial F_1}{\partial x}(a_0)e^{b|\psi|}, \quad a \in C. \tag{4.40}$$

Using the fact that the Radon-Nikodym derivative of the pricing measure $\mathbb{Q}_0$ of the initial Pareto optimal allocation α_0 is given by

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}} = \frac{\partial F_1}{\partial x}(a_0)/\frac{\partial F_0}{\partial x}(a_0),$$

we deduce from (4.38) and (4.40) the existence of $b > 0$ such that

$$|\frac{\partial^2 F_1}{\partial a^i \partial a^j}(a)| \leq b \frac{d\mathbb{Q}_0}{d\mathbb{P}} e^{b|\psi|}, \quad a \in C.$$

The inequality (4.36) now follows from Lemma 4.6 and the corresponding inequality (4.22) in Theorem 4.7.

Given (4.36), the two-times continuous differentiability of $F_0 = F_0(a)$ and the formulas (4.37) follow from Lemma 4.8. In particular,

$$\frac{\partial^2 F_0}{\partial x^2}(a) = \mathbb{E}[\frac{\partial^2 F_1}{\partial x^2}(a)] < 0, \quad a \in \mathbf{A},$$

proving the first condition in (F5).

The matrix $A(F_0) = A(F_0)(a)$ from (3.5) is computed in (4.50), in Lemma 4.14 below. By Lemma 4.15, this representation for $A(F_0)$ implies (F8), with the same $c > 0$ as in Assumption 2.2, and, hence, also the second condition in (F5). Finally, (F9) for F_0 follows from the corresponding property for F_1 and (4.37). □

For a two-times continuously differentiable $f = f(a) : \mathbf{A} \to \mathbf{R}$ recall the notations $A(f)$, $C(f)$, and $D(f)$ for the matrices defined in (3.5), (3.20), and (3.21). Our next goal is to compute these matrices for $F_1 = F_1(a)$ and $F_0 = F_0(a)$.

Some additional notations are needed. For $a \in \mathbf{A}$ define the probability measure $\mathbb{R}(a)$ by

$$\frac{d\mathbb{R}(a)}{d\mathbb{P}} \triangleq \frac{\partial^2 F_1}{\partial x^2}(a)/\frac{\partial^2 F_0}{\partial x^2}(a), \tag{4.41}$$

and the stochastic process

$$R_i(a) \triangleq -\frac{\partial F_i}{\partial x}(a)/\frac{\partial^2 F_i}{\partial x^2}(a), \quad i = 0, 1, \tag{4.42}$$

representing the absolute risk-tolerance of the indirect utility F_i at the Pareto allocation $\pi(a)$. Notice that $R(a) = (R_0(a), R_1(a))$ is a martingale under $\mathbb{R}(a)$ and, for the pricing measure $\mathbb{Q}(a)$ defined in (4.16),

$$\frac{d\mathbb{Q}(a)}{d\mathbb{R}(a)} = \frac{R_1(a)}{R_0(a)}. \tag{4.43}$$

Denote also

$$\tau^m(a) \triangleq t_m(\pi^m(a)), \quad m = 1, \dots, M, \tag{4.44}$$

where $t_m = t_m(x)$ is the risk-tolerance of $u_m = u_m(x)$. Observe that, by (4.9),

$$\sum_{m=1}^{M} \tau^m(a) = R_1(a), \tag{4.45}$$

and, by Assumption 2.2,

$$\frac{1}{c} \leq \tau^m \leq c, \quad m = 1, \dots, M. \tag{4.46}$$

For a probability measure $\mathbb{R}$ and random variables ξ and η denote

$$\mathrm{Cov}_{\mathbb{R}}[\xi, \eta] \triangleq \mathbb{E}_{\mathbb{R}}[\xi\eta] - \mathbb{E}_{\mathbb{R}}[\xi]\mathbb{E}_{\mathbb{R}}[\eta],$$

the covariance of ξ and η under $\mathbb{R}$.

Lemma 4.14. *Under Assumptions 2.1, 2.2, and 2.5, for the stochastic field $F_1 = F_1(a)$, the matrices $A(F_1)$, $C(F_1)$, and $D(F_1)$ defined in* (3.5), (3.20), *and* (3.21) *are given by, for $l, m = 1, \dots, M$ and $i, j = 1, \dots, J$,*

$$A^{lm}(F_1)(a) = \delta_{lm}\tau^m(a), \tag{4.47}$$

$$C^{mj}(F_1)(a) = 0, \tag{4.48}$$

$$D^{ij}(F_1)(a) = 0, \tag{4.49}$$

where the random variable $\tau^m(a)$ is defined in (4.44).

The corresponding matrices for the function $F_0 = F_0(a)$ are given by

$$
\begin{aligned}
A^{lm}(F_0)(a) &= \frac{1}{R_0(a)}\mathbb{E}_{\mathbb{R}(a)}[\tau^l(a)(\delta_{lm}\sum_{k=1}^{M}\tau^k(a) - \tau^m(a))] \\
&\quad + \frac{1}{R_0(a)}\mathbb{E}_{\mathbb{R}(a)}[\tau^l(a)]\mathbb{E}_{\mathbb{R}(a)}[\tau^m(a)],
\end{aligned}
\tag{4.50}
$$

$$
C^{mj}(F_0)(a) = \frac{1}{R_0(a)}\operatorname{Cov}_{\mathbb{R}(a)}[\tau^m(a), \psi^j], \tag{4.51}
$$

$$
D^{ij}(F_0)(a) = \frac{1}{R_0(a)}\operatorname{Cov}_{\mathbb{R}(a)}[\psi^i, \psi^j], \tag{4.52}
$$

where $\mathbb{R}(a)$ and $R_0(a)$ are defined in (4.41) *and* (4.42).

Note that in (4.50) the matrix $A(F_0)$ is represented as the sum of two symmetric non-negative semi-definite matrices.

Proof. From Theorem 4.2 we deduce

$$
\begin{aligned}
v^m\frac{\partial^2 F_1}{\partial v^m \partial x}(a) &= -\frac{\partial^2 F_1}{\partial x^2}(a)\tau^m(a), \\
v^l v^m\frac{\partial^2 F_1}{\partial v^l \partial v^m}(a) &= -\frac{\partial^2 F_1}{\partial x^2}(a)\tau^l(a)(\delta_{lm}\sum_{k=1}^{M}\tau^k(a) - \tau^m(a)), \\
v^m\frac{\partial^2 F_1}{\partial v^m \partial q^j}(a) &= -\frac{\partial^2 F_1}{\partial x^2}(a)\tau^m(a)\psi^j, \\
\frac{\partial^2 F_1}{\partial x \partial q^j}(a) &= \frac{\partial^2 F_1}{\partial x^2}(a)\psi^j, \\
\frac{\partial^2 F_1}{\partial q^i \partial q^j}(a) &= \frac{\partial^2 F_1}{\partial x^2}(a)\psi^i\psi^j,
\end{aligned}
$$

and the formulas (4.47)–(4.49) and (4.50)–(4.52) follow by direct computations, where in the latter case we account for the definitions of $R_0(a)$ and $\mathbb{R}(a)$. $\square$

To complete the proof of Theorem 4.13 we still have to verify (F8) for the matrix $A(F_0)$.

Lemma 4.15. *Under Assumptions 2.1, 2.2, and 2.5, for any $a \in \mathbf{A}$, the matrix $A(F_0)(a)$ defined in* (4.50) *has full rank and, for any $z \in \mathbf{R}^n$,*

$$\frac{1}{c}|z|^2 \leq \langle z, A(F_0)(a)z\rangle \leq c|z|^2, \tag{4.53}$$

with the same constant $c > 0$ as in Assumption 2.2.

Proof. To simplify notations we omit the dependence on $a \in \mathbf{A}$. Elementary calculations show that

$$\begin{aligned}\langle z, A(F_0)z\rangle &= \frac{1}{R_0}\left(\mathbb{E}_{\mathbb{R}}\left[R_1\sum_{m=1}^{M}\tau^m z_m^2 - \langle \tau, z\rangle^2\right] + \langle \mathbb{E}_{\mathbb{R}}[\tau], z\rangle^2\right) \\ &= \frac{1}{R_0}\mathbb{E}_{\mathbb{R}}\left[R_1\sum_{m=1}^{M}\tau^m z_m^2 - \langle \tau - \mathbb{E}_{\mathbb{R}}[\tau], z\rangle^2\right],\end{aligned}$$

where we used (4.45). This immediately implies the upper bound in (4.53):

$$\langle z, A(F_0)z\rangle \leq \frac{1}{R_0}\mathbb{E}_{\mathbb{R}}[R_1\sum_{m=1}^{M}\tau^m z_m^2] = \mathbb{E}_{\mathbb{Q}}[\sum_{m=1}^{M}\tau^m z_m^2] \leq c|z|^2,$$

where we used the measure $\mathbb{Q}$ from (4.43) and the inequality (4.46) for τ.

To verify the lower bound we use (4.46) to write τ as

$$\tau = \frac{1}{c} + \theta, \quad \theta \in \mathbf{L}^0(\mathbf{R}^M_+),$$

and obtain

$$\begin{aligned}\langle z, A(F_0)z\rangle &= \frac{1}{R_0}\mathbb{E}_{\mathbb{R}}\left[R_1\sum_{m=1}^{M}(\frac{1}{c} + \theta^m) z_m^2 - \langle \theta - \mathbb{E}_{\mathbb{R}}[\theta], z\rangle^2\right] \\ &= \frac{1}{c}|z|^2 + \frac{1}{R_0}\mathbb{E}_{\mathbb{R}}\left[R_1\sum_{m=1}^{M}\theta^m z_m^2 - \langle \theta - \mathbb{E}_{\mathbb{R}}[\theta], z\rangle^2\right].\end{aligned}$$

As $R_1 = \langle \tau, \mathbf{1}\rangle \geq \langle \theta, \mathbf{1}\rangle$, where $\mathbf{1} \triangleq (1, \dots, 1)$, we deduce

$$\begin{aligned}R_0\left(\langle z, A(F_0)z\rangle - \frac{1}{c}|z|^2\right) &\geq \mathbb{E}_{\mathbb{R}}\left[\langle \theta, \mathbf{1}\rangle\sum_{m=1}^{M}\theta^m z_m^2 - \langle \theta - \mathbb{E}_{\mathbb{R}}[\theta], z\rangle^2\right] \\ &= \mathbb{E}_{\mathbb{R}}\left[\frac{1}{\langle \theta, \mathbf{1}\rangle}\sum_{m=1}^{M}\theta^m (z_m\langle \theta, \mathbf{1}\rangle - \langle \theta, z\rangle)^2\right] + \langle \mathbb{E}_{\mathbb{R}}[\theta], z\rangle^2 \geq 0.\end{aligned}$$

□

4.4 Stochastic process of cash balances

We now construct a stochastic process (G_0, G_1) with values in $\widetilde{\mathbf{G}}^1$ related to (F_0, F_1) in the sense of the conjugacy relations of Theorem 3.3. The function $G_0 = G_0(b)$ will play later a crucial role in the quantitative description of strategies.

Theorem 4.16. *Under Assumptions 2.1 and 2.5, there are a function* $G_0 = G_0(b): \mathbf{B} \to \mathbf{R}$ *and a random field* $G_1 = G_1(b): \mathbf{B} \to \mathbf{L}^0$ *such that, for* $i = 0, 1$,

$$G_i(u, y, q) \triangleq \sup_{v \in (0,\infty)^M} \inf_{x \in \mathbf{R}} [\langle v, u \rangle + xy - F_i(v, x, q)], \quad (u, y, q) \in \mathbf{B}. \tag{4.54}$$

The function G_0 *and the sample paths of* G_1 *belong to the space* $\widetilde{\mathbf{G}}^1$ *defined in* (3.29).

If, in addition, Assumption 2.2 holds, then G_0 *and the sample paths of* G_1 *belong to the space* $\widetilde{\mathbf{G}}^2(c)$ *defined in* (3.31) *with the same constant* $c > 0$ *as in Assumption 2.2.*

Proof. The results are immediate consequences of Theorems 4.7, 4.13, and 3.10. □

From Theorem 4.16 and the equivalence of items 3 and 4 in Theorem 3.3 we deduce the identity

$$x = G_i(\frac{\partial F_i}{\partial v}(a), 1, q), \quad i = 0, 1, \ a = (v, x, q) \in \mathbf{A}.$$

Recall that, according to (4.34), $\frac{\partial F_0}{\partial v^m}(a, t)$ represents the *indirect* or expected utility of the mth market maker at time 0 given the Pareto allocation $\pi(a)$. Hence, $G_0(u, 1, q)$ defines the *collective cash amount* of the market makers at time 0 when their indirect utilities are given by u and they jointly own q stocks. Note that at maturity 1 a similar interpretation is apparent from the following representation for the random field $G_1 = G_1(b)$, obtained by direct computations:

$$G_1(z, y, q) = y \left(\sum_{m=1}^{M} u_m^{-1}(z^m) - \langle q, \psi \rangle - \Sigma_0 \right), \quad (z, y, q) \in \mathbf{B},$$

where u_m^{-1} is the inverse function to u_m.

5 Quantitative analysis of strategies

We continue with the study of investment strategies initiated in Section 2. Theorems 5.1 and 5.2 below improve Theorem 2.7 and constitute the main results of this paper.

Recall the notations $\pi = \pi(a)$, for the random field of Pareto allocations defined in Lemma 4.4, $\mathbb{Q}(a)$, for the pricing measure of $\pi(a)$ defined in (4.16), and $G_0 = G_0(b)$, for the saddle function defined in (4.54).

Theorem 5.1. *Under Assumptions 2.1 and 2.5, for every position $q \in \mathbf{R}^J$ there is a unique cash amount $x(q)$ and a unique Pareto optimal allocation $\alpha_1(q)$ with total endowment $\Sigma(x(q), q)$ such that*

$$U_0^m \triangleq \mathbb{E}[u_m(\alpha_0^m)] = \mathbb{E}[u_m(\alpha_1^m(q))], \quad m = 1, \dots, M. \tag{5.1}$$

The Pareto optimal allocation $\alpha_1(q)$ has the form

$$\alpha_1(q) = \pi(a(q)), \quad a(q) = (w(q), x(q), q), \tag{5.2}$$

where the weights $w(q) \in \mathbf{S}^M$ and the cash amount $x(q) \in \mathbf{R}$ are given by

$$w^m(q) = \frac{\partial G_0}{\partial u^m}(U_0, 1, q) / \sum_{k=1}^{M} \frac{\partial G_0}{\partial u^k}(U_0, 1, q), \quad m = 1, \dots, M. \tag{5.3}$$

$$x(q) = G_0(U_0, 1, q). \tag{5.4}$$

The function $x = x(q) : \mathbf{R}^J \to \mathbf{R}$ is convex, continuously differentiable, and, for $q \in \mathbf{R}^J$,

$$\frac{\partial x}{\partial q^j}(q) = -\mathbb{E}_{\mathbb{Q}(a(q))}[\psi^j], \quad j = 1, \dots, M. \tag{5.5}$$

Proof. The uniqueness of the cash amount and of the Pareto optimal allocation with the desired properties follows directly from the definition of Pareto optimality and the strict concavity of utility functions.

For the existence, consider the cash amount $x(q)$, the weights $w(q)$, and the Pareto optimal allocation $\alpha_1(q)$ defined by (5.2)–(5.4). Clearly, by the construction of the random field $\pi = \pi(a)$, the total endowment of $\alpha_1(q)$ equals $\Sigma(x(q), q)$. By Theorem 3.3, the identities (5.4) and (5.3) imply

$$\frac{\partial F_0}{\partial v^m}(a(q)) = U_0^m, \quad m = 1, \dots, M,$$

which, by (4.34), is exactly (5.1).

The convexity and the continuous differentiability of $x = x(q)$ hold as $G_0 \in \mathbf{G}^1$. Finally, (5.5) follows from the relation (3.19) in Theorem 3.3 and (4.35). $\square$

Hereafter, we shall denote by $x = x(q) : \mathbf{R}^J \to \mathbf{R}$ and $w = w(q) : \mathbf{R}^J \to \mathbf{S}^M$ the cash amount and the Pareto weights defined by (5.4) and (5.3), respectively. Recall that, according to the sign convention of our model, the large investor *pays* the cash amount $x(-q)$ for the ownership of q securities.

For our sensitivity analysis of these maps we shall need some extra notation. For a random variable ξ and a probability measure $\mathbb{R}$ denote

$$\mathrm{Var}_{\mathbb{R}}[\xi] \triangleq \mathrm{Cov}_{\mathbb{R}}[\xi, \xi],$$

the variance of ξ under $\mathbb{R}$. For vectors $\mu \in \mathbf{S}^M$ and $z \in \mathbf{R}^M$ we shall use a similar notation:

$$\mathrm{Var}_{\mu}[z] \triangleq \sum_{i=1}^{M} \mu[i](z^i)^2 - (\sum_{i=1}^{M} \mu[i] z^i)^2 \tag{5.6}$$

interpreted as the variance of the random variable z defined on the sample space $\{1, \dots, M\}$ with respect to the probability measure μ. For $a \in \mathbf{A}$ define $\rho(a) \in \mathbf{L}^0(\mathbf{S}^M)$ by

$$\rho^m(a) \triangleq \frac{\tau^m(a)}{\sum_{k=1}^{M} \tau^k(a)} = \frac{\tau^m(a)}{R_1(a)}, \quad m = 1, \dots, M, \tag{5.7}$$

where $\tau(a) = (\tau^m(a))_{m=1,\dots,M}$ and $R_1(a)$ are given by (4.44) and (4.42), respectively. Finally, for $g \in \mathbf{G}^2$, recall the notations $E(g)$ and $H(g)$ for the matrices defined in (3.22) and (3.23).

Theorem 5.2. *Let Assumptions 2.1, 2.2, and 2.5 hold. Then the Pareto weights $w = w(q)$ are continuously differentiable, the cash amount $x = x(q)$ is two-times continuously differentiable and, for $m = 1, \dots, M$, $i, j = 1, \dots, J$, and $q \in \mathbf{R}^J$,*

$$Z^{mj}(q) \triangleq \frac{1}{w^m} \frac{\partial w^m}{\partial q^j}(q) = E^{mj}(G_0)(b(q)) - \sum_{k=1}^{M} w^k(q) E^{kj}(G_0)(b(q)), \tag{5.8}$$

$$\frac{\partial^2 x}{\partial q^i \partial q^j}(q) = H^{ij}(G_0)(b(q)), \tag{5.9}$$

where $b(q) \triangleq (U_0, 1, q)$. The matrices $E = E(G_0)(b(q))$ and $H = H(G_0)(b(q))$ admit the representations:

$$E = -A^{-1}C, \tag{5.10}$$

$$H = C^T A^{-1} C + D, \tag{5.11}$$

where the matrices $A \triangleq A(F_0)(a(q))$, $C \triangleq C(F_0)(a(q))$, and $D \triangleq D(F_0)(a(q))$ are computed in (4.50), (4.51), *and* (4.52) *and $a(q) \triangleq (w(q), x(q), q)$.*

Moreover, the second-order expansion for $x = x(q)$ can be written as

$$\begin{aligned} x(q+\Delta q) - x(q) = {} & -\mathbb{E}_{\mathbb{Q}}[\langle \Delta q, \psi\rangle] + \frac{R_0}{2}\mathbb{E}_{\mathbb{R}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{R}}\right)^2 \operatorname{Var}_\rho[Z\Delta q]\right] \\ & + \frac{1}{2R_0}\left\{\left(\operatorname{Cov}_{\mathbb{R}}[\frac{d\mathbb{Q}}{d\mathbb{R}}, \langle \Delta q, \psi\rangle]\right)^2 + \operatorname{Var}_{\mathbb{R}}[\langle \Delta q, \psi\rangle]\right\} \\ & + o(|\Delta q|^2), \quad \Delta q \to 0, \end{aligned} \tag{5.12}$$

where $Z = Z(q)$ is defined in (5.8), *we omitted the argument $a(q)$ for $\mathbb{Q}$, $\mathbb{R}$, R_0, and $\rho = (\rho^m)_{m=1,\dots,M}$ from* (4.16), (4.41), (4.42), *and* (5.7), *and the notation Var_ρ is explained in* (5.6).

Proof. By Theorem 4.16, $G_0 \in \mathbf{G}^2$. The representations (5.3), for $w(q)$, and (5.4), for $x(q)$, then imply that $w = w(q) \in \mathbf{C}^1$, $x = x(q) \in \mathbf{C}^2$, and (5.8) and (5.9) hold. The formulas (5.10) and (5.11) for the matrices $E = E(G_0)(b(q))$ and $H = H(G_0)(b(q))$ follow from Theorem 3.4. Hence, to complete the proof we only have to verify (5.12).

The linear term in (5.12) follows from (5.5). To verify the second-order part we decompose the matrix $A = A(F_0)(a(q))$ from (4.50) as

$$A = R_0(S + ss^T),$$

where, for $l, m = 1, \dots, M$,

$$\begin{aligned} S^{lm} &\triangleq \frac{1}{R_0^2}\mathbb{E}_{\mathbb{R}}[\tau^l(\delta_{lm}\sum_{k=1}^M \tau^k - \tau^m)], \\ s^l &\triangleq \frac{1}{R_0}\mathbb{E}_{\mathbb{R}}[\tau^l]. \end{aligned}$$

Recall the notation $\mathbf{1} \triangleq (1, \dots, 1)$ and observe that

$$S\mathbf{1} = (\mathbf{1}^T S)^T = 0, \tag{5.13}$$

$$\langle s, \mathbf{1} \rangle = \frac{1}{R_0} \mathbb{E}_{\mathbb{R}}[\sum_{k=1}^{M} \tau^k] = 1, \tag{5.14}$$

where at the last step we accounted for (4.45) and the martingale property of R_i under $\mathbb{R}$.

As $H = H(G_0)(b(q))$ is the Hessian matrix for $x = x(q)$ and accounting for (5.10) and (5.11), we deduce the following expression for the second-order part in (5.12):

$$\begin{aligned}\frac{1}{2} \langle \Delta q, H \Delta q \rangle &= \frac{1}{2} \langle E\Delta q, AE\Delta q \rangle + \frac{1}{2} \langle \Delta q, D\Delta q \rangle \\ &= \frac{R_0}{2} \langle E\Delta q, SE\Delta q \rangle + \frac{R_0}{2} (\langle E\Delta q, s \rangle)^2 + \frac{1}{2} \langle \Delta q, D\Delta q \rangle .\end{aligned}$$

Since, by (5.8), $(E - Z)\Delta q$ is the product of some scalar on the vector $\mathbf{1}$, (5.13) implies that $\langle E\Delta q, SE\Delta q \rangle = \langle Z\Delta q, SZ\Delta q \rangle$. Observe that, in view of (4.43) and (4.45), the matrix S can be written as

$$S^{lm} = \frac{1}{R_0^2} \mathbb{E}_{\mathbb{R}}[R_1^2 \rho^l (\delta_{lm} - \rho^m)] = \mathbb{E}_{\mathbb{R}}[\left(\frac{d\mathbb{Q}}{d\mathbb{R}}\right)^2 \rho^l (\delta_{lm} - \rho^m)].$$

It follows that

$$\langle Z\Delta q, SZ\Delta q \rangle = \mathbb{E}_{\mathbb{R}}[\left(\frac{d\mathbb{Q}}{d\mathbb{R}}\right)^2 \mathrm{Var}_\rho[Z\Delta q]],$$

giving the first quadratic term in (5.12).

By (5.10), $\langle \mathbf{1}, (AE + C)\Delta q \rangle = 0$, which, in view of (5.13) and (5.14), implies that

$$R_0 \langle s, E\Delta q \rangle + \langle \mathbf{1}, C\Delta q \rangle = 0.$$

From the construction of C in (4.51) and accounting again for (4.43) and (4.45) we deduce the second quadratic term in (5.12):

$$\begin{aligned}\langle s, E\Delta q \rangle &= -\frac{1}{R_0} \langle \mathbf{1}, C\Delta q \rangle = -\frac{1}{R_0^2} \mathrm{Cov}_{\mathbb{R}}[R_1, \langle \Delta q, \psi \rangle] \\ &= -\frac{1}{R_0} \mathrm{Cov}_{\mathbb{R}}[\frac{d\mathbb{Q}}{d\mathbb{R}}, \langle \Delta q, \psi \rangle].\end{aligned}$$

Finally, the expression (4.52) for D yields the last term:

$$\langle \Delta q, D\Delta q\rangle = \frac{1}{R_0}\operatorname{Var}_{\mathbb{R}}[\langle \Delta q, \psi\rangle].$$

$\square$

Remark 5.3. The linear term in (5.12) corresponds to the "standard" model of mathematical finance, where a *"small" investor* can trade *any number* of securities ψ at "fixed" or *exogenous* prices $\mathbb{E}_{\mathbb{Q}}[\psi]$. The second, quadratic, component can thus be viewed as a *price impact correction* to this model. Note that all three terms of the quadratic part are *non-negative* and the last term, $\operatorname{Var}_{\mathbb{R}}[\langle \Delta q, \psi\rangle]$, equals zero iff $\langle \Delta q, \psi\rangle = \text{const}$. Hence, for any non-trivial transaction our large investor will have to pay a *strictly positive penalty* due to his price impact in comparison with a hypothetical small agent trading at $\mathbb{E}_{\mathbb{Q}}[\psi]$.

A common technique in economic theory is to replace a collection of economic agents with a single, *representative*, agent whose utility function is given by (4.1) with a *fixed* or "frozen" weight $w \in \mathbf{S}^M$. In our case, this simplification yields the linear term in (5.12), as well as the last two components of the quadratic part. It is interesting to obtain conditions for the first term of the quadratic part to vanish, since, then, the representative agent approximation leads to the identical expression for the price impact coefficient as our original model with many market makers. This is accomplished in the following

Lemma 5.4. *Let the conditions of Theorem 5.2 hold and $q, r \in \mathbf{R}^J$. Then the following assertions are equivalent:*

1. $\mathbb{E}_{\mathbb{R}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{R}}\right)^2 \operatorname{Var}_\rho[Z(q)r]\right] = 0$;
2. $Z(q)r = 0$;
3. $\mathbb{E}_{\mathbb{R}^m}[\langle r, \psi\rangle] = \mathbb{E}_{\mathbb{Q}}[\langle r, \psi\rangle]$, $m = 1, \dots, M$,

where, for $a \in \mathbf{A}$,

$$\frac{d\mathbb{R}^m(a)}{d\mathbb{Q}(a)} \triangleq \frac{\rho^m(a)}{\mathbb{E}_{\mathbb{Q}}[\rho^m(a)]}, \quad m = 1, \dots, M,$$

and we omitted the argument $a(q)$ for $\mathbb{R}$, $\mathbb{Q}$, ρ, and $\mathbb{R}^m$.

Proof. Denote $\xi \triangleq \langle r, \psi \rangle \in \mathbf{L}^0(\mathbf{R})$ and $z \triangleq Z(q)r \in \mathbf{R}^M$.

$1 \iff 2$: Clearly, item 1 holds if and only if $\mathrm{Var}_\rho[z] = 0$, which, in turn, is equivalent to $z = y\mathbf{1}$ for some $y \in \mathbf{R}$. From the construction of the matrix $Z(q)$ in (5.8) we deduce that $\langle w(q), z \rangle = 0$, where the Pareto weights $w(q)$ take values in $\mathbf{S}^M$. It follows that $y = 0$.

$2 \iff 3$: From the definition of the measures $\mathbb{R}^m$, $m = 1, \ldots, M$, we deduce the equivalence of item 3 to

$$\mathbb{E}_{\mathbb{R}^m}[\xi] = \mathbb{E}_{\mathbb{R}^1}[\xi], \quad m = 2, \ldots, M. \tag{5.15}$$

From (5.8) we deduce that $Z(q)r = 0$ if and only if $Er = y\mathbf{1}$ for some $y \in \mathbf{R}$, where the matrix $E = E(G_0)(b(q))$ satisfies (5.10). Hence, item 2 is equivalent to the existence of a constant $y \in \mathbf{R}$ such that

$$yA\mathbf{1} + Cr = 0, \tag{5.16}$$

where $A = A(F_0)(a(q))$ and $C = C(F_0)(a(q))$. From the expressions (4.50) and (4.51) for the matrices A and C we obtain

$$\begin{aligned}
(A\mathbf{1})^m &= \mathbb{E}_{\mathbb{R}}[\tau^m] = R_0 \mathbb{E}_{\mathbb{Q}}[\rho^m], \\
(Cr)^m &= \frac{1}{R_0} \mathrm{Cov}_{\mathbb{R}}[\tau^m, \xi] = \frac{1}{R_0} \left(\mathbb{E}_{\mathbb{R}}[\tau^m \xi] - \mathbb{E}_{\mathbb{R}}[\tau^m] \mathbb{E}_{\mathbb{R}}[\xi] \right) \\
&= \mathbb{E}_{\mathbb{Q}}[\rho^m] \left(\mathbb{E}_{\mathbb{R}^m}[\xi] - \mathbb{E}_{\mathbb{R}}[\xi] \right), \quad m = 1, \ldots, M,
\end{aligned}$$

which clearly implies the equivalence of (5.15) and (5.16). $\square$

The condition of item 3 is clearly satisfied when the random weights ρ defined in (5.7) are deterministic. This is the case, for instance, if all market makers have exponential utilities: $u_m(x) = -\exp(-a_m x)$, with constant $a_m > 0$, $m = 1, \ldots, M$. Moreover, if the securities ψ form a *complete* model in the sense that, jointly with the constant security paying 1 they span all random variables, than the validity of item 3 for any $r \in \mathbf{R}^J$ is in fact equivalent to ρ being deterministic.

6 Proofs of Theorems 3.3 and 3.4

The proofs will rely on the theory of saddle functions presented in Part VII of the classical book [24] by Rockafellar.

6.1 Proof of Theorem 3.3

The proof will be divided into a number of lemmas. To simplify notations we omit the dependence on q, where it is not important, and then interpret the classes $\mathbf{F}^1$ and $\mathbf{G}^1$ in the sense of Remark 3.2.

Lemma 6.1. *Let $f = f(v,x): (0,\infty)^M \times \mathbf{R} \to (-\infty, 0)$ be in $\mathbf{F}^1$. Then there exists a continuously differentiable $g = g(u,y): (-\infty,0)^M \times (0,\infty) \to \mathbf{R}$, which is conjugate to f in the sense that, for any $u \in (-\infty,0)^M$ and $y \in (0,\infty)$,*

$$\begin{aligned} g(u,y) &= \sup_{v\in(0,\infty)^M} \inf_{x\in\mathbf{R}} [\langle v,u\rangle + xy - f(v,x)] \\ &= \inf_{x\in\mathbf{R}} \sup_{v\in(0,\infty)^M} [\langle v,u\rangle + xy - f(v,x)], \end{aligned} \tag{6.1}$$

and, for any $v \in (0,\infty)^M$ and $x \in \mathbf{R}$,

$$\begin{aligned} f(v,x) &= \sup_{u\in(-\infty,0)^M} \inf_{y\in(0,\infty)} [\langle v,u\rangle + xy - g(u,y)], \\ &= \inf_{y\in(0,\infty)} \sup_{u\in(-\infty,0)^M} [\langle v,u\rangle + xy - g(u,y)]. \end{aligned} \tag{6.2}$$

Moreover, the minimax values in (6.1) *and* (6.2) *are attained at unique saddle points.*

Proof. To facilitate the references to Section 37 in [24] we define f on the whole Euclidean space $\mathbf{R}^{M+1}$ by setting its values outside of $(0,\infty)^M \times \mathbf{R}$ as

$$f(v,x) = \begin{cases} 0, & v \in \partial\,\mathbf{R}_+^M \\ \infty, & v \notin \mathbf{R}_+^M \end{cases}, \quad x \in \mathbf{R}, \tag{6.3}$$

where $\mathbf{R}_+^M \triangleq [0,\infty)^M$. By (3.2) and (3.3), after this extension f becomes a *closed* saddle function (according to the definition in Section 34 of [24]) with *effective domain*

$$\operatorname{dom} f \triangleq \operatorname{dom}_1 f \times \operatorname{dom}_2 f = \mathbf{R}_+^M \times \mathbf{R},$$

where

$$\begin{aligned} \operatorname{dom}_1 f &\triangleq \{v \in \mathbf{R}^M : \ f(v,x) < \infty, \quad \forall x \in \mathbf{R}\} = \mathbf{R}_+^M, \\ \operatorname{dom}_2 f &\triangleq \{x \in \mathbf{R} : \ f(v,x) > -\infty, \quad \forall v \in \mathbf{R}^M\} = \mathbf{R}. \end{aligned}$$

Using the extended version of f we introduce the saddle functions

$$\begin{aligned}
\underline{g}(u,y) &\triangleq \sup_{v\in\mathbf{R}^M}\inf_{x\in\mathbf{R}}[\langle v,u\rangle + xy - f(v,x)] \\
&= \sup_{v\in\mathbf{R}^M_+}\inf_{x\in\mathbf{R}}[\langle v,u\rangle + xy - f(v,x)], \\
\overline{g}(u,y) &\triangleq \inf_{x\in\mathbf{R}}\sup_{v\in\mathbf{R}^M}[\langle v,u\rangle + xy - f(v,x)] \\
&= \inf_{x\in\mathbf{R}}\sup_{v\in\mathbf{R}^M_+}[\langle v,u\rangle + xy - f(v,x)]
\end{aligned}$$

defined for $u \in \mathbf{R}^M$ and $y \in \mathbf{R}$ and taking values in $[-\infty,\infty]$. By the duality theory for conjugate saddle functions, see [24], Theorem 37.1 and Corollaries 37.1.1 and 37.1.2, the functions $\underline{g}$ and $\overline{g}$ have a common *effective* domain, which we denote $C \times D$, and coincide on $(\operatorname{int} C \times D) \cup (C \times \operatorname{int} D)$, where $\operatorname{int} A$ denotes the interior of a set A.

Hence, on $(\operatorname{int} C \times D) \cup (C \times \operatorname{int} D)$ we can define a finite saddle function $g = g(u,y)$ such that

$$\begin{aligned}
g(u,y) &= \sup_{v\in\mathbf{R}^M_+}\inf_{x\in\mathbf{R}}[\langle v,u\rangle + xy - f(v,x)] \\
&= \inf_{x\in\mathbf{R}}\sup_{v\in\mathbf{R}^M_+}[\langle v,u\rangle + xy - f(v,x)].
\end{aligned} \tag{6.4}$$

Moreover, from the same Theorem 37.1, Corollaries 37.1.1 and 37.1.2 in [24], and since (6.3) is the unique closed extension of f to $\mathbf{R}^{M+1}$ we deduce

$$\begin{aligned}
f(v,x) &= \sup_{u\in C}\inf_{y\in\operatorname{int} D}[\langle v,u\rangle + xy - g(u,y)], \\
&= \inf_{y\in D}\sup_{u\in\operatorname{int} C}[\langle v,u\rangle + xy - g(u,y)], \quad (v,x) \in \mathbf{R}^{M+1}.
\end{aligned} \tag{6.5}$$

Noting that the continuous differentiability of g on $(-\infty,0)^M \times (0,\infty)$ is an immediate consequence of the existence and the uniqueness of the saddle points for (6.1), see [24], Theorem 35.8 and Corollary 37.5.3, we obtain that the result holds if

1. the interiors of the sets C and D are given by

$$\operatorname{int} C = (-\infty,0)^M, \tag{6.6}$$

$$\operatorname{int} D = (0,\infty); \tag{6.7}$$

2. for $(u,y) \in (-\infty,0)^M \times (0,\infty)$, the minimax values in (6.4) are attained at a unique $(v,x) \in (0,\infty)^M \times \mathbf{R}$;

3. for $(v,x) \in (0,\infty)^M \times \mathbf{R}$, the minimax values in (6.5) are attained at a unique $(u,y) \in (-\infty,0)^M \times (0,\infty)$.

For the set C we have

$$\begin{aligned} C &\triangleq \{u \in \mathbf{R}^M : \ \overline{g}(u,y) < \infty \text{ for all } y \in \mathbf{R}\} \\ &= \{u \in \mathbf{R}^M : \ \sup_{v \in \mathbf{R}^M_+} [\langle u,v\rangle - f(v,x)] < \infty \text{ for some } x \in \mathbf{R}\} \\ &= \{u \in \mathbf{R}^M : \ \sup_{w \in \mathbf{S}^M} [\langle u,w\rangle - f(w,x)] \le 0 \text{ for some } x \in \mathbf{R}\}, \end{aligned}$$

where at the last step we used (3.2). As $f \le 0$ on $\mathbf{R}^M_+ \times \mathbf{R}$, we have $C \subset (-\infty,0]^M$. On the other hand, by (3.3) and (3.4), and, since, for any $w \in \mathbf{S}^M$, the function $f(w,\cdot)$ is increasing,

$$\lim_{x\to\infty} \inf_{w \in \mathbf{S}^M} f(w,x) = 0.$$

It follows that $(-\infty,0)^M \subset C$, proving (6.6).

For the set D we obtain

$$\begin{aligned} D &\triangleq \{y \in \mathbf{R} : \ \underline{g}(u,y) > -\infty \text{ for all } u \in \mathbf{R}^M\} \\ &= \{y \in \mathbf{R} : \ \inf_{x \in \mathbf{R}} [xy - f(v,x)] > -\infty \text{ for some } v \in \mathbf{R}^M\}. \end{aligned}$$

From (3.4) we deduce that $D \subset \mathbf{R}_+$. As $f \le 0$ on $\mathbf{R}^M_+ \times \mathbf{R}$, the point $y=0$ belongs to D. If $y>0$, then (3.2) implies the existence of $v \in (0,\infty)^M$ such that

$$y = \frac{\partial f}{\partial x}(v,1),$$

and, therefore, for such y and v,

$$\inf_{x \in \mathbf{R}} [xy - f(v,x)] = y - f(v,1) > -\infty.$$

Hence, $D = \mathbf{R}_+$, implying (6.7).

Fix $v \in (0,\infty)^M$ and $x \in \mathbf{R}$. By the properties of f,

$$\nabla f(v,x) \in (-\infty,0)^M \times (0,\infty) = \operatorname{int} C \times \operatorname{int} D,$$

implying that $(u,y) \triangleq \nabla f(v,x)$ is the unique saddle point of (6.5), see Corollary 37.5.3 in [24].

Fix now $u \in (-\infty,0)^M$ and $y \in (0,\infty)$. As f (viewed as a function on $\mathbf{R}^{M+1}$) is a closed saddle function and (u,y) belongs to the *interior* of the effective domain of g, the minimax values in (6.4) are attained on a closed convex set of saddle points, namely, the subdifferential of g evaluated at (u,y), see Corollary 37.5.3 in [24]. To complete the proof it remains to be shown that this set is a singleton in $(0,\infty)^M \times \mathbf{R}$.

If $(\widehat{v},\widehat{x})$ is a saddle point of (6.4), then $(\widehat{v},\widehat{x}) \in \operatorname{dom} f = \mathbf{R}^M_+ \times \mathbf{R}$, and

$$
\begin{aligned}
g(u,y) &= \widehat{x}y + \langle \widehat{v}, u\rangle - f(\widehat{v},\widehat{x}) = \widehat{x}y + \sup_{v\in\mathbf{R}^M_+} [\langle v,u\rangle - f(v,\widehat{x})] \\
&= \langle \widehat{v}, u\rangle + \inf_{x\in\mathbf{R}} [xy - f(\widehat{v},x)].
\end{aligned}
$$

Accounting for the positive homogeneity property (3.2) of $f(\cdot,\widehat{x})$ we deduce that

$$
\widehat{x}y = g(u,y), \tag{6.8}
$$

$$
\langle \widehat{v},u\rangle - f(\widehat{v},\widehat{x}) = \sup_{v\in\mathbf{R}^M_+} [\langle v,u\rangle - f(v,\widehat{x})] = 0, \tag{6.9}
$$

$$
\widehat{x}y - f(\widehat{v},\widehat{x}) = \inf_{x\in\mathbf{R}} [xy - f(\widehat{v},x)]. \tag{6.10}
$$

The equality (6.8) defines $\widehat{x}$ uniquely. To show the uniqueness of $\widehat{v}$ we observe first that $\widehat{v} \in (0,\infty)^M$. Indeed, otherwise, we would have $f(\widehat{v},x) = 0$, $x \in \mathbf{R}$, and the right side of (6.10) would be $-\infty$. Hence, $\widehat{v}$ can be decomposed as a product of $\widehat{w} \in \mathbf{S}^M$ and $\widehat{z} > 0$. By (3.2) and (6.9),

$$
\langle \widehat{w},u\rangle - f(\widehat{w},\widehat{x}) = \sup_{w\in\mathbf{S}^M} [\langle w,u\rangle - f(w,\widehat{x})] = 0.
$$

As $f(\cdot,\widehat{x})$ is strictly convex on $\mathbf{S}^M$, this identity determines $\widehat{w}$ uniquely. Finally, from (6.10) and the continuous differentiability of $f(\widehat{v},\cdot)$ on $\mathbf{R}$ we deduce that

$$
y = \frac{\partial f}{\partial x}(\widehat{v},\widehat{x}) = \widehat{z}\frac{\partial f}{\partial x}(\widehat{w},\widehat{x}),
$$

proving the uniqueness of $\widehat{z}$. □

Lemma 6.2. *Let f and g be as in Lemma 6.1. Then g satisfies the positive homogeneity property* (3.13) *and for $(v,x) \in (0,\infty)^M \times \mathbf{R}$ and $(u,y) \in (-\infty,0)^M \times (0,\infty)$ the relations below are equivalent:*

1. *Given* (u,y) *the minimax values in* (6.1) *are attained at* (v,x).

2. *Given* (v,x) *the minimax values in* (6.2) *are attained at* (u,y).

3. *We have* $x = \frac{\partial g}{\partial y}(u,y) = g(u,1)$ *and* $v = \frac{\partial g}{\partial u}(u,y)$.

4. *We have* $y = \frac{\partial f}{\partial x}(v,x)$ *and* $u = \frac{\partial f}{\partial v}(v,x)$.

Moreover, in this case, $f(v,x) = \langle u,v\rangle$ *and* $g(u,y) = xy$.

Proof. First, we observe that (3.13) for g follows from the corresponding feature (3.2) for f and the construction of g in (6.1). The equivalence of items 1–4 follows from the characterization of saddle points in terms of the subdifferentials of conjugate functions, see Theorem 37.5 and Corollary 37.5.3 in [24]. The last assertion is straightforward. □

Lemma 6.3. *Let* f *and* g *be as in Lemma 6.1. Then* $g \in \mathbf{G}^1$.

Proof. The continuous differentiability of $g = g(u,y)$ and the positive homogeneity condition (G4) have been already established in Lemmas 6.1 and 6.2, while (G3) (for g not depending on q) follows trivially from (G2). Hence, (G2) is the only remaining property to be proved. In view of (G4) it is sufficient to verify it only for $y = 1$.

The function $g(\cdot,1)$ is strictly increasing because its gradient is strictly positive by item 3 of Lemma 6.2. To show the strict convexity of $g(\cdot,1)$, select u_1 and u_2, distinct elements of $(-\infty,0)^M$, denote by u_3 their midpoint, and, for $i = 1,2,3$, set $v_i \triangleq \frac{\partial g}{\partial u}(u_i,1)$, $x_i \triangleq g(u_i,1)$. Since each $v_i \in (0,\infty)^M$, we can represent it as the product $v_i = z_i w_i$ of $z_i \in (0,\infty)$ and $w_i \in \mathbf{S}^M$. By Lemma 6.2,

$$\langle u_i, w_i\rangle - f(w_i,x_i) = \sup_{w\in\mathbf{S}^M} [\langle u_i, w\rangle - f(w,x_i)] = 0, \tag{6.11}$$

$$1 = z_i \frac{\partial f}{\partial x}(w_i,x_i). \tag{6.12}$$

Since $(u_i)_{i=1,2,3}$ are distinct, so are $(x_i,v_i)_{i=1,2,3}$ and, hence, by (6.12), so are $(x_i,w_i)_{i=1,2,3}$. As $f(v,\cdot)$ is strictly concave on $\mathbf{R}$, we deduce from (6.11) that

$$\begin{aligned} f(w_3,x_3) = \langle u_3,w_3\rangle &= \frac{1}{2}((\langle u_1,w_3\rangle - f(w_3,x_1)) + (\langle u_2,w_3\rangle - f(w_3,x_2))) \\ &+ \frac{1}{2}(f(w_3,x_1) + f(w_3,x_2)) < f(w_3,\frac{1}{2}(x_1+x_2)). \end{aligned}$$

Since $f(w_3, \cdot)$ is strictly increasing on $\mathbf{R}$, we deduce that $x_3 < (x_1 + x_2)/2$, implying the strict convexity of $g(\cdot, 1)$.

The assertion (3.8) of (G2)(a) follows from the monotonicity of $g(\cdot, 1)$ and the fact that, by Lemmas 6.1 and 6.2, for any $x \in \mathbf{R}$ one can find $u \in (-\infty, 0)^M$ such that $x = g(u, 1)$.

For the proof of (G2)(b) and (G2)(c) we shall argue by contradiction. Let $(u_n)_{n\geq 1}$ be a sequence in $(-\infty, 0)^M$ converging to a boundary point of $(-\infty, 0)^M$. Denote $x_n \triangleq g(u_n, 1)$, $v_n \triangleq \frac{\partial g}{\partial u}(u_n, 1)$, $n \geq 1$, and, contrary to (3.9), assume that the sequence $(v_n)_{n\geq 1}$ is bounded. Then, by the convexity of $g(\cdot, 1)$ and the boundedness of $(u_n)_{n\geq 1}$, the sequence $(x_n)_{n\geq 1}$ is also bounded. Hence, by passing to a subsequence, we can assume that the sequences $(v_n)_{n\geq 1}$ and $(x_n)_{n\geq 1}$ converge to *finite* limits $\widehat{v} \in \mathbf{R}^M_+$ and $\widehat{x} \in \mathbf{R}$, respectively. From Lemma 6.2 we deduce that

$$u_n = \frac{\partial f}{\partial v}(v_n, x_n), \quad 1 = \frac{\partial f}{\partial x}(v_n, x_n), \quad n \geq 1. \tag{6.13}$$

If $\widehat{v} \in (0, \infty)^M$, then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\partial f}{\partial v}(v_n, x_n) = \frac{\partial f}{\partial v}(\widehat{v}, \widehat{x}) \in (-\infty, 0)^M,$$

contradicting our choice of $(u_n)_{n\geq 1}$. If, on the other hand, $\widehat{v} \in \partial\, \mathbf{R}^M_+$, then, by (3.2) and (3.3),

$$\lim_{n\to\infty} f(v_n, x) = 0, \quad x \in \mathbf{R}.$$

Since the functions $f(v_n, \cdot)$ are concave, their pointwise convergence to 0 implies the convergence to 0 of its derivatives, uniformly on compact sets in $\mathbf{R}$, see Theorem 25.7 in [24]. It follows that

$$\lim_{n\to\infty} \frac{\partial f}{\partial x}(v_n, x_n) = 0,$$

contradicting the second equality in (6.13). This finishes the proof of (G2)(b).

Let now $(u_n)_{n\geq 1}$ be a sequence in $(-\infty, 0)^M$ satisfying the conditions (3.10) and (3.11) of (G2)(c). Denote $x_n \triangleq g(u_n, 1)$, $n \geq 1$ and, contrary to (3.12), assume that

$$\limsup_{n\to\infty} g(u_n, 1) = \limsup_{n\to\infty} x_n > -\infty.$$

As $g(\cdot, 1)$ is an increasing function on $(-\infty, 0)^M$, (3.10) implies the boundedness of the sequence $(x_n)_{n\geq 1}$ from above. Hence, by passing, if necessary, to a subsequence, we can assume that it converges to $\widehat{x} \in \mathbf{R}$. Define a sequence $(w_n)_{n\geq 1}$ in $\mathbf{S}^M$ by

$$w_n = \frac{\partial g}{\partial u}(u_n, y_n) = y_n \frac{\partial g}{\partial u}(u_n, 1),$$

for appropriate normalizing constants y_n, $n \geq 1$. By passing to a subsequence, we can assume that $(w_n)_{n\geq 1}$ converges to $\widehat{w} \in \operatorname{cl} \mathbf{S}^M$. From Lemma 6.2 we deduce that

$$u_n = \frac{\partial f}{\partial v}(w_n, x_n), \quad n \geq 1.$$

If $\widehat{w} \in \mathbf{S}^M$, then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\partial f}{\partial v}(w_n, x_n) = \frac{\partial f}{\partial v}(\widehat{w}, \widehat{x}) \in (-\infty, 0)^M,$$

contradicting (3.11). If $\widehat{w} \in \partial\, \mathbf{S}^M$, then, by (3.3) and Lemma 6.2,

$$\lim_{n\to\infty} \langle u_n, w_n \rangle = \lim_{n\to\infty} f(w_n, x_n) = 0,$$

contradicting (3.10). This finishes the proof of (G2)(c) and, with it, the proof of the lemma. $\square$

Lemma 6.4. *Let $g = g(u, y) : (-\infty, 0)^M \times (0, \infty) \to \mathbf{R}$ be in $\mathbf{G}^1$. Then there is a continuously differentiable function $f = f(v, x) : (0, \infty)^M \times \mathbf{R} \to (-\infty, 0)$ such that the minimax relations* (6.1) *and* (6.2) *hold and have unique saddle points.*

Proof. We follow the same path as in the proof of Lemma 6.1. To use the results of Section 37 in [24] we need to define the values of $g = g(u, y)$ at the boundary of the original domain by an appropriate closure operation. For $u \in (-\infty, 0)^M$ we set, by continuity, $g(u, 0) \triangleq 0$. Then for $y \geq 0$ we define, by lower semi-continuity,

$$g(u, y) \triangleq \lim_{\varepsilon\to 0} \inf_{z\in B(u,\varepsilon)} g(z, y), \quad u \in \partial\,(-\infty, 0]^M, \tag{6.14}$$

where $B(u, \varepsilon)$ is the ball of the radius ε centered at u:

$$B(u, \varepsilon) \triangleq \{z \in (-\infty, 0)^M : |u - z| \leq \varepsilon\}.$$

Note that in (6.14) the value of the limit may be infinite.

As in the proof of Lemma 6.1 we deduce the existence of a saddle function $f = f(v,x)$ defined on $(C \times \operatorname{int} D) \cup (D \times \operatorname{int} C)$, where

$$\begin{aligned} C &\triangleq \{v \in \mathbf{R}^M : \ f(v,x) < \infty \text{ for all } x \in \mathbf{R}\} \\ &= \{v \in \mathbf{R}^M : \ \sup_{u \in (-\infty,0]^M} [\langle v,u\rangle - g(u,y)] < \infty \text{ for some } y \in \mathbf{R}_+\}, \\ D &\triangleq \{x \in \mathbf{R} : \ f(v,x) > -\infty \text{ for all } v \in \mathbf{R}^M\} \\ &= \{x \in \mathbf{R} : \ \inf_{y \in \mathbf{R}_+} [xy - g(u,y)] > -\infty \text{ for some } u \in (-\infty,0]^M\}, \end{aligned}$$

such that, for any $(v,x) \in (C \times \operatorname{int} D) \cup (D \times \operatorname{int} C)$,

$$\begin{aligned} f(v,x) &= \sup_{u \in (-\infty,0]^M} \inf_{y \in \mathbf{R}_+} [\langle v,u\rangle + xy - g(u,y)], \\ &= \inf_{y \in \mathbf{R}_+} \sup_{u \in (-\infty,0]^M} [\langle v,u\rangle + xy - g(u,y)], \end{aligned} \tag{6.15}$$

and, for any $(u,y) \in (-\infty,0)^M \times (0,\infty)$,

$$\begin{aligned} g(u,y) &= \sup_{v \in C} \inf_{x \in \operatorname{int} D} [\langle v,u\rangle + xy - f(v,x)] \\ &= \inf_{x \in D} \sup_{v \in \operatorname{int} C} [\langle v,u\rangle + xy - f(v,x)]. \end{aligned} \tag{6.16}$$

As $g(u,y) = yg(u,1)$, and, by (G2)(c), for any $x \in \mathbf{R}$ there is $u \in (-\infty,0)^M$ such that $x \geq g(u,1)$, we have

$$D = \mathbf{R}.$$

Choosing $y = 0$ in the second description of C above, we obtain

$$\sup_{u \in (-\infty,0]^M} [\langle u,v\rangle - g(u,0)] = \sup_{u \in (-\infty,0]^M} [\langle u,v\rangle] < \infty \text{ iff } v \in \mathbf{R}^M_+.$$

If $v \notin \mathbf{R}^M_+$, then there is $u_0 \in (-\infty,0)^M$ such that $\langle u_0,v\rangle > 0$. By (G2)(c), for any $y > 0$,

$$\lim_{n \to \infty} g(nu_0,y) = -\infty,$$

and, therefore,

$$\sup_{u \in (-\infty,0]^M} [\langle u,v\rangle - g(u,y)] \geq \limsup_{n \to \infty} [\langle nu_0,v\rangle - g(nu_0,y)] = \infty.$$

It follows that

$$C = \mathbf{R}_+^M.$$

For $u \in (-\infty, 0)^M$ and $y > 0$ we have $\nabla g(u, y) \in (0, \infty)^M \times \mathbf{R}$, implying that $(v, x) \triangleq \nabla g(u, y)$ is the unique saddle point of (6.16). In particular, we deduce that the minimax identities (6.1) and (6.16) have the same unique saddle points.

Let now $v \in (0, \infty)^M$ and $x \in \mathbf{R}$. As (v, x) belongs to the *interior* of the effective domain of f, the minimax values in (6.15) are attained on a closed convex set of saddle points belonging to the subdifferential of f evaluated at (v, x), see Corollary 37.5.3 in [24]. We are going to show that this set is a singleton in $(-\infty, 0)^M \times (0, \infty)$.

Let $(\widehat{u}, \widehat{y})$ be a saddle point. Then

$$(\widehat{u}, \widehat{y}) \in \operatorname{dom} g \subset (-\infty, 0]^M \times \mathbf{R}_+,$$

and

$$\begin{aligned} f(v, x) &= x\widehat{y} + \langle \widehat{u}, v \rangle - g(\widehat{u}, \widehat{y}) = x\widehat{y} + \sup_{u \in (-\infty, 0]^M} [\langle u, v \rangle - g(u, \widehat{y})] \\ &= \langle \widehat{u}, v \rangle + \inf_{y \in \mathbf{R}_+} [xy - g(\widehat{u}, y)]. \end{aligned}$$

As $g(u, y) = yg(u, 1)$, we deduce that

$$f(v, x) = \langle \widehat{u}, v \rangle \,, \tag{6.17}$$

$$x = g(\widehat{u}, 1), \tag{6.18}$$

$$\langle \widehat{u}, v \rangle - x\widehat{y} = \sup_{u \in (-\infty, 0]^M} [\langle u, v \rangle - \widehat{y} g(u, 1)]. \tag{6.19}$$

By (6.18) and (G2)(a), $\widehat{u} \neq 0$, and then, by (6.19), $\widehat{y} > 0$. The attainability of the upper bound in (6.19) at $\widehat{u}$ implies that the subdifferential $\partial g(\widehat{u}, 1)$ is well-defined and $v \in \widehat{y} \partial g(\widehat{u}, 1)$. From (G2)(b) we deduce that $g(\cdot, 1)$ is not subdifferentiable on the boundary of $(-\infty, 0]^M$ and, therefore, $\widehat{u} \in (-\infty, 0)^M$. As $g(\cdot, 1)$ is strictly convex on $(-\infty, 0)^M$, (6.18) defines $\widehat{u}$ uniquely, and, as $g(\cdot, 1)$ is differentiable on $(-\infty, 0)^M$, $\widehat{y}$ is uniquely determined by the equality

$$v = \widehat{y} \frac{\partial g}{\partial u} (\widehat{u}, 1).$$

The uniqueness of the saddle points $(\widehat{u}, \widehat{y})$ implies the continuous differentiability of f on $(0, \infty)^M \times \mathbf{R}$. Finally, from (6.17) we deduce that $f < 0$ on $(0, \infty)^M \times \mathbf{R}$. □

Lemma 6.5. *Let g and f be as in Lemma 6.4. Then f satisfies the positive homogeneity condition* (3.2) *and, for $(v,x) \in (0,\infty)^M \times \mathbf{R}$ and $(u,y) \in (-\infty,0)^M \times (0,\infty)$, the assertions of Lemma 6.2 hold.*

Proof. The positive homogeneity property (3.2) for f is a consequence of the corresponding feature (3.13) for g. The remaining assertions follow by the same arguments as in the proof of Lemma 6.2. □

Lemma 6.6. *Let g and f be as in Lemma 6.4. Then f satisfies* (F2).

Proof. Fix $x \in \mathbf{R}$. The positive homogeneity with respect to v was already established in Lemma 6.5. By item 4 of Lemma 6.2, $\frac{\partial f}{\partial v} < 0$, implying that the function $f(\cdot, x)$ is strictly decreasing.

Let $(w_i)_{i=1,2}$ be distinct points in $\mathbf{S}^M$, w_3 be their midpoint, and, for $i = 1,2,3$, denote $u_i \triangleq \frac{\partial f}{\partial v}(w_i, x)$ and $y_i \triangleq \frac{\partial f}{\partial x}(w_i, x)$. By the characterizations of saddle points in Lemma 6.2, for $i = 1,2,3$, we have

$$f(w_i, x) = \langle u_i, w_i \rangle, \quad x = g(u_i, 1), \text{ and } w_i = y_i \frac{\partial g}{\partial u}(u_i, 1).$$

From the last equality we deduce that the points $(u_i)_{i=1,2,3}$ are distinct. The uniqueness of saddle points for (6.2) then implies $\langle u_3, w_i \rangle < \langle u_i, w_i \rangle$, for $i = 1,2$, and, therefore,

$$\begin{aligned} f(w_3, x) = \langle u_3, w_3 \rangle &= \frac{1}{2}(\langle u_3, w_1 \rangle + \langle u_3, w_2 \rangle) \\ &< \frac{1}{2}(\langle u_1, w_1 \rangle + \langle u_2, w_2 \rangle) = \frac{1}{2}(f(w_1, x) + f(w_2, x)), \end{aligned}$$

proving the strict convexity of $f(\cdot, x)$ on $\mathbf{S}^M$.

Let now $(w_n)_{n\geq 1}$ be a sequence in $\mathbf{S}^M$ converging to $w \in \partial\, \mathbf{S}^M$. From (G2)(c) for any $\varepsilon > 0$ we deduce the existence of $u(\varepsilon) \in (-\infty, 0)^M$ such that $g(u(\varepsilon), 1) \leq x$ and

$$-\varepsilon \leq \langle u(\varepsilon), w \rangle = \lim_{n\to\infty} \langle u(\varepsilon), w_n \rangle .$$

From the construction of f in (6.2) we deduce $f(v,x) \geq \langle u(\varepsilon), v \rangle$ for any $v \in (0,\infty)^M$. It follows that

$$\liminf_{n\to\infty} f(w_n, x) \geq \lim_{n\to\infty} \langle u(\varepsilon), w_n \rangle \geq -\varepsilon,$$

proving (3.3). □

Lemma 6.7. *Let g and f be as in Lemma 6.4. Then f satisfies* (F4).

Proof. Fix $v \in (0,\infty)^M$. As, by item 4 of Lemma 6.2, $\frac{\partial f}{\partial x} > 0$, the function $f(v,\cdot)$ is strictly increasing.

Let $(x_i)_{i=1,2}$ be distinct elements of $\mathbf{R}$, $x_3 \triangleq \frac{1}{2}(x_1+x_2)$, and $u_i \triangleq \frac{\partial f}{\partial v}(v,x_i)$, $i=1,2,3$. From Lemma 6.5 we deduce

$$g(u_i,1) = x_i, \quad f(v,x_i) = \langle u_i, v\rangle, \quad i=1,2,3.$$

It follows that $(u_i)_{i=1,2,3}$ are distinct, and, hence, by the strict convexity of $g(\cdot,1)$,

$$g(\frac{1}{2}(u_1+u_2),1) < \frac{1}{2}(g(u_1,1)+g(u_2,1)) = \frac{1}{2}(x_1+x_2) = x_3.$$

From the uniqueness of saddle points in (6.2) we deduce that if $g(u,1) < x$ then $f(v,x) > \langle u,v\rangle$. It follows that

$$f(v,x_3) > \left\langle \frac{1}{2}(u_1+u_2), v \right\rangle = \frac{1}{2}(f(v,x_1)+f(v,x_2)),$$

proving the strict concavity of $f(v,\cdot)$.

For any $\varepsilon > 0$ we can clearly find $u(\varepsilon) \in (-\infty,0)^M$ such that $\langle u(\varepsilon), v\rangle \geq -\varepsilon$. Denoting $x(\varepsilon) \triangleq g(u(\varepsilon),1)$ we deduce

$$\lim_{x\to\infty} f(v,x) > f(v,x(\varepsilon)) \geq \langle u(\varepsilon), v\rangle \geq -\varepsilon,$$

proving (3.4). □

After these preparations we are ready to complete the proof of Theorem 3.3. From this moment, the functions f and g will depend on the "auxiliary" variable $q \in \mathbf{R}^J$.

Proof of Theorem 3.3. If $f = f(v,x,q) \in \mathbf{F}^1$, then, by Lemmas 6.1 and 6.3, the function

$$g(u,y,q) \triangleq \sup_{v\in(0,\infty)^M} \inf_{x\in\mathbf{R}} [\langle v,u\rangle + xy - f(v,x,q)]$$

satisfies (G2) and (G4) and is differentiable with respect to u and y. Moreover, the concavity of $f(v,\cdot,\cdot)$ implies the convexity of $g(\cdot,y,\cdot)$. Conversely, if $g = g(u,y,q) \in \mathbf{G}^1$, then, by Lemmas 6.4–6.7, the function

$$f(v,x,q) \triangleq \sup_{u\in(-\infty,0)^M} \inf_{y\in(0,\infty)} [\langle u,v\rangle + xy - g(u,y,q)]$$

satisfies (F2) and (F4) and is differentiable with respect to v and x. Moreover, as g is convex with respect to (u, q), f is concave with respect to (x, q).

The rest of the proof, namely, the equivalence of the differentiability of f and g with respect to q and the relation (3.19), follows from the *envelope theorem* for saddle functions, Theorem A.1, given in Appendix A. Finally, we recall that for saddle functions the existence of derivatives implies the continuity of derivatives, see Theorem 35.8 and Corollary 35.7.1 in [24]. $\square$

6.2 Proof of Theorem 3.4

As in the proof of Theorem 3.3 we begin with several lemmas, where we omit the dependence on q.

Lemma 6.8. *Let $f = f(v, x)$ and $g = g(u, y)$ be as in Lemma 6.1. Then the following assertions are equivalent:*

1. *f is twice continuously differentiable and for all $v \in (0, \infty)^M$ and $x \in \mathbf{R}$ its Hessian matrix $K(v, x) = (K^{kl}(v, x))_{1\leq k,l\leq M+1}$ has full rank.*

2. *g is twice continuously differentiable and for all $u \in (-\infty, 0)^M$ and $y \in (0, \infty)$ its Hessian matrix $L(u, y) = (L^{kl}(u, y))_{1\leq k,l\leq M+1}$ has full rank.*

Moreover, if $(v, x) \in (0, \infty)^M \times \mathbf{R}$ and $(u, y) \in (-\infty, 0)^M \times (0, \infty)$ are conjugate saddle points in the sense of Lemma 6.2, then $L(u, y)$ is inverse to $K(v, x)$.

Proof. The asserted equivalence is a well-known fact in the theory of saddle functions and is a direct consequence of the characterization of the gradients of the conjugate functions f and g given in Lemma 6.2 and the Implicit Function Theorem. $\square$

In the following statement we shall make the relationship between the Hessian matrices of f and g more explicit by taking into account the positive homogeneity property (3.13) of g.

Lemma 6.9. *Let f and g be as in Lemma 6.1. Then the following assertions are equivalent:*

1. f is twice continuously differentiable and for all $v \in (0,\infty)^M$ and $x \in \mathbf{R}$

$$\frac{\partial^2 f}{\partial x^2}(v,x) < 0 \tag{6.20}$$

and the Hessian matrix $K(v,x)$ of f has full rank.

2. f is twice continuously differentiable and for all $v \in (0,\infty)^M$ and $x \in \mathbf{R}$ the inequality (6.20) holds and the matrix
(6.21)

$$\widetilde{A}^{kl}(v,x) \triangleq \left(\frac{\partial^2 f}{\partial v_k \partial v^l} - \frac{1}{\frac{\partial^2 f}{\partial x^2}} \frac{\partial^2 f}{\partial v^k \partial x} \frac{\partial^2 f}{\partial v^l \partial x} \right)(v,x), \quad 1 \le k,l \le M,$$

has full rank.

3. g is twice continuously differentiable and for all $u \in (-\infty,0)^M$ and $y \in (0,\infty)$ the matrix

$$\widetilde{B}^{kl}(u,y) \triangleq \frac{\partial^2 g}{\partial u_k \partial u^l}(u,y), \quad 1 \le k,l \le M, \tag{6.22}$$

has full rank.

Moreover, if $(v,x) \in (0,\infty)^M \times \mathbf{R}$ and $(u,y) \in (-\infty,0)^M \times (0,\infty)$ are conjugate saddle points in the sense of Lemma 6.2, then $\widetilde{B}(u,y)$ is the inverse of $\widetilde{A}(v,x)$.

Proof. 1 $\iff$ 2. From (6.20) and the construction of the matrix $\widetilde{A}(v,x)$ in (6.21) we deduce that for $a \in \mathbf{R}^M$ and $b \in \mathbf{R}$ the equation

$$K(v,x)\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

is equivalent to

$$b = -\frac{1}{\frac{\partial^2 f}{\partial x^2}(v,x)} \sum_{m=1}^{M} \frac{\partial^2 f}{\partial v^m \partial x}(v,x) a^m$$

and

$$\widetilde{A}(v,x)a = 0.$$

It follows that under (6.22) the matrices $K(v,x)$ and $\widetilde{A}(v,x)$ can have full rank only simultaneously.

$1 \iff 3$. We fix arguments (u,y) and (v,x) satisfying the conjugacy relations of Lemma 6.2. From the definition of $\widetilde{B} = \widetilde{B}(u,y)$ in (6.22) we deduce that the Hessian matrix of g at (u,y) has the representation

$$L(u,y) = \begin{pmatrix} \widetilde{B}(u,y) & \frac{\partial g}{\partial u}(u,1) \\ (\frac{\partial g}{\partial u}(u,1))^T & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{B} & \frac{1}{y}v \\ \frac{1}{y}v^T & 0 \end{pmatrix}.$$

To simplify notations we shall also represent the Hessian matrix of f at (v,x) as

$$K(v,x) = \begin{pmatrix} M & p \\ p^T & z \end{pmatrix},$$

where M is the Hessian matrix of $f(\cdot,x)$ at v, $p \triangleq (\frac{\partial^2 f}{\partial v^m \partial x}(v,x))_{m=1,\dots,M}$ is the vector-column of mixed derivatives, and $z \triangleq \frac{\partial^2 f}{\partial x^2}(v,x)$. Observe that the matrix $\widetilde{A} = \widetilde{A}(v,x)$ defined in (6.21) is given by

$$\widetilde{A} = M - \frac{1}{z}pp^T. \tag{6.23}$$

As $\widetilde{B}$ is a symmetric positive semi-definite matrix and $\frac{1}{y}v \neq 0$, the full rank of $\widetilde{B}$ implies the full rank of $L(u,y)$. Hence, by Lemma 6.8, under the conditions of either item 1 or item 3, the Hessian matrices $K(v,x)$ and $L(u,y)$ have full rank and are inverse to each other. Denoting by I the $M \times M$ identity matrix we deduce

$$\begin{aligned} \widetilde{B}M + \frac{1}{y}vp^T &= I, \\ \widetilde{B}p + \frac{1}{y}vz &= 0. \end{aligned} \tag{6.24}$$

If $z < 0$, that is, (6.20) holds, then, by (6.23) and (6.24), $\widetilde{B}\widetilde{A} = I$. Hence, $\widetilde{B}$ is the inverse of $\widetilde{A}$ and, in particular, it has full rank, proving $1 \implies 3$.

Conversely, if $\widetilde{B}$ has full rank, then $z = \frac{\partial^2 f}{\partial x^2}(v,x) \neq 0$. Indeed, otherwise from the second equality in (6.24) we obtain $p = 0$ contradicting the full rank of $K(v,x)$. Since, $f(v,\cdot)$ is concave, we deduce $z < 0$, proving $3 \implies 1$. □

Proof of Theorem 3.4. If $(v,x,q) \in \mathbf{A}$ and $(u,y,q) \in \mathbf{B}$ satisfy the equivalent relations of items 1–4 of Theorem 3.3, then the matrices A and B defined in

(3.5) and (3.14) and the matrices $\widetilde{A}$ and $\widetilde{B}$ defined in (6.21) and (6.22) are related by

$$A^{kl} = \frac{v^k v^l}{y} \widetilde{A}^{kl}, \quad B^{kl} = \frac{y}{v^k v^l} \widetilde{B}^{kl}, \quad 1 \leq k, l \leq M.$$

Lemma 6.9 then implies (3.24) as well as the other assertions of the theorem except those involving the second derivatives with respect to q.

Assume first that $f \in \mathbf{F}^2$. We have to show that g is two-times continuously differentiable and (3.25) and (3.26) hold. For $a \in \mathbf{R}^M$ define the function $h: (0,\infty)^M \times (-\infty, 0)^M \times \mathbf{R}^J \to \mathbf{R}^M$ by

$$h(v, u, q) \triangleq (\frac{\partial f}{\partial v}(v, g(u, 1, q), q) - u) + a(\frac{\partial f}{\partial x}(v, g(u, 1, q), q) - 1).$$

From Theorem 3.3 we deduce, for any $(u, y, q) \in \mathbf{B}$,

$$h(\frac{\partial g}{\partial u}(u, y, q), u, q) = 0.$$

Fix $(u_0, y_0, q_0) \in \mathbf{B}$, denote $v_0 \triangleq \frac{\partial g}{\partial u}(u_0, y_0, q_0)$, $x_0 \triangleq g(u_0, 1, q_0)$, and choose

$$a^m \triangleq - \left(\frac{\partial^2 f}{\partial v^m \partial x} / \frac{\partial^2 f}{\partial x^2} \right) (v_0, x_0, q_0), \quad m = 1, \ldots, M.$$

Direct computations show that, for $m, l = 1, \ldots, M$ and $j = 1, \ldots, J$,

$$\frac{\partial h^m}{\partial v^l}(v_0, u_0, q_0) = \widetilde{A}^{ml}(v_0, x_0, q_0) = \frac{y_0}{v_0^m v_0^l} A^{ml}(v_0, x_0, q_0),$$
$$\frac{\partial h^m}{\partial q^j}(v_0, u_0, q_0) = \frac{y_0}{v_0^m} C^{mj}(v_0, x_0, q_0).$$

By the Implicit Function Theorem the function $\frac{\partial g}{\partial u} = \frac{\partial g}{\partial u}(u, y, q)$ is continuously differentiable with respect to q in a neighborhood of (u_0, y_0, q_0) and the relation (3.25) holds at this point.

To prove the existence of the continuous second derivatives of g with respect to q and the remaining identity (3.26) we denote

$$b^j \triangleq (\frac{\partial^2 f}{\partial x \partial q^j} / \frac{\partial^2 f}{\partial x^2})(v_0, x_0, q_0), \quad j = 1, \ldots, J.$$

From Theorem 3.3 we deduce, for any $(u, y, q) \in \mathbf{B}$,

$$\begin{aligned} \frac{\partial g}{\partial q}(u, y, q) + by = &- \frac{\partial f}{\partial q}(\frac{\partial g}{\partial u}(u, y, q), g(u, 1, q), q) \\ &+ b\frac{\partial f}{\partial x}(\frac{\partial g}{\partial u}(u, y, q), g(u, 1, q), q). \end{aligned}$$

This implies the two-times continuous differentiability of g with respect to q. Moreover, direct computations show that the differentiation of the above identity with respect to q at (u_0, y_0, q_0) yields (3.26) at this point.

Assume now that $g \in \mathbf{G}^2$. To complete the proof we have to show that f has continuous second derivatives involving q. By Theorem 3.3, for any $(v, x, q) \in \mathbf{A}$ we have the equalities

$$\begin{aligned} \frac{\partial g}{\partial u}(\frac{\partial f}{\partial v}(v, x, q), \frac{\partial f}{\partial x}(v, x, q), q) - v = 0, \\ \frac{\partial g}{\partial y}(\frac{\partial f}{\partial v}(v, x, q), \frac{\partial f}{\partial x}(v, x, q), q) - x = g(\frac{\partial f}{\partial v}(v, x, q), 1, q) - x = 0, \\ \frac{\partial f}{\partial q}(v, x, q) + \frac{\partial g}{\partial q}(\frac{\partial f}{\partial v}(v, x, q), \frac{\partial f}{\partial x}(v, x, q), q) = 0. \end{aligned}$$

By Lemmas 6.8 and 6.9, the full rank of the matrix $B(u, y, q)$ implies the full rank of the Hessian matrix of $g(\cdot, \cdot, q)$ at (u, y). An application of the Implicit Function Theorem to the first two equalities above then leads to the continuous differentiability of $\frac{\partial f}{\partial v}$ and $\frac{\partial f}{\partial x}$ with respect to q. By the third identity, this implies the existence and the continuity of $\frac{\partial^2 f}{\partial q^i \partial q^j}$. $\square$

A An envelope theorem for saddle functions

In the proof of Theorem 3.3 we used the following version of the folklore "envelope" theorem for saddle functions. As usual, $\operatorname{ri} C$ denotes the *relative interior* of a convex set C.

Theorem A.1. *Let C be a convex set in $\mathbf{R}^n$, D be a convex set in $\mathbf{R}^m$, E be a convex open set in $\mathbf{R}^l$, $f = f(x, y, z) : C \times D \times E \to \mathbf{R}$ be a function convex with respect to x and concave with respect to (y, z), and let $z_0 \in E$. Denote*

$$g(z) \triangleq \sup_{y \in D} \inf_{x \in C} f(x, y, z), \quad z \in E,$$

and assume that the maximin value $g(z_0)$ is attained at a unique $x_0 \in \operatorname{ri} C$ and some (not necessarily unique) $y_0 \in D$ and the function $f(x_0, y_0, \cdot)$ is differentiable at z_0.

Then the function $g: E \to \mathbf{R} \cup \{-\infty\}$ is concave, differentiable at z_0 (in particular, finite in a neighborhood of z_0) and

$$\nabla g(z_0) = \frac{\partial f}{\partial z}(x_0, y_0, z_0).$$

Remark A.2. Theorem 5 in Milgrom and Segal [20] is the closest result to ours we could find in the literature. There, the convexity assumptions on f are replaced by compactness requirements on C and D.

The proof of Theorem A.1 relies on two lemmas of independent interest, which were used in the proof of Theorem 4.1. The first lemma is essentially known, see, for example, Corollary 3 in [20].

Lemma A.3. *Let $f = f(x, y) : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R} \cup \{-\infty\}$ be a concave function and let $y_0 \in \mathbf{R}^m$. Denote*

$$g(y) \triangleq \sup_{x \in \mathbf{R}^n} f(x, y), \quad y \in \mathbf{R}^m,$$

and assume that the upper bound $g(y_0)$ is attained at some (not necessarily unique) $x_0 \in \mathbf{R}^n$ and the function $f(x_0, \cdot)$ is differentiable at y_0.

Then the function $g: \mathbf{R}^m \to \mathbf{R} \cup \{-\infty\}$ is concave, differentiable at y_0 and

$$\nabla g(y_0) = \frac{\partial f}{\partial y}(x_0, y_0). \tag{A.1}$$

Proof. The concavity of g follows from the concavity of f with respect to both arguments. As $g(y_0) = f(x_0, y_0) < \infty$, this concavity property implies that $g < \infty$. Since $g \geq f(x_0, \cdot)$, the function g is finite in a neighborhood of y_0. It follows that $\partial g(y_0)$, the subdifferential of g at y_0, is not empty.

If $y^* \in \partial g(y_0)$, then

$$g(y) \leq g(y_0) + \langle y^*, y - y_0 \rangle, \quad y \in \mathbf{R}^m.$$

As $f(x_0, y) \leq g(y)$, $y \in \mathbf{R}^m$, and $f(x_0, y_0) = g(y_0)$, it follows that

$$f(x_0, y) \leq f(x_0, y_0) + \langle y^*, y - y_0 \rangle, \quad y \in \mathbf{R}^m.$$

Hence, y^* belongs to the subdifferential of $f(x_0, \cdot)$ at y_0, and, therefore, $y^* = \frac{\partial f}{\partial y}(x_0, y_0)$. It follows that y^* is the only element of $\partial g(y_0)$, proving the differentiability of g at y_0 and the identity (A.1). □

Lemma A.4. *Let C be a convex set in $\mathbf{R}^n$, D be a convex open set in $\mathbf{R}^m$, $f = f(x, y) : C \times D \to \mathbf{R}$ be a function concave with respect to x and convex with respect to y, and let $y_0 \in D$. Define the function*

$$g(y) \triangleq \sup_{x \in C} f(x, y), \quad y \in D,$$

and assume that the upper bound $g(y_0)$ is attained at a unique $x_0 \in \operatorname{ri} C$ and the function $f(x_0, \cdot)$ is differentiable at y_0.

Then the function $g : D \to \mathbf{R} \cup \{\infty\}$ is convex, differentiable at y_0, and the identity (A.1) *holds.*

Remark A.5. The proof of Lemma A.4 will follow from the well-known analogous result in convex optimization, where the assumption of concavity in x is replaced by the requirement that C is compact, see, for example, Corollary 4.45 in Hiriart-Urruty and Lemaréchal [17].

Proof. The convexity of g is straightforward. Let $\varepsilon > 0$ be such that

$$C(\varepsilon) \triangleq \{x \in C : |x - x_0| \leq \varepsilon\} \subset \operatorname{ri} C.$$

If $(y_n)_{n \geq 1}$ is a sequence in D converging to y_0, then the concave functions $f(\cdot, y_n)$, $n \geq 1$, converge to $f(\cdot, y_0)$ uniformly on compact subsets of C. Since x_0 is the unique point of maximum for $f(\cdot, y_0)$, there is $n_0 > 0$ such that for every $n \geq n_0$ the concave function $f(\cdot, y_n)$ attains its maximum at some point $x_n \in C(\varepsilon)$. This argument implies the existence of $\delta > 0$ such that

$$g(y) = \sup_{x \in C(\varepsilon)} f(x, y) < \infty, \quad y \in D, |y - y_0| < \delta.$$

As $C(\varepsilon)$ is a compact set, the result now follows from the well-known fact in convex optimization mentioned in Remark A.5. □

Proof of Theorem A.1. The function $h = h(y, z) : D \times E \to \mathbf{R} \cup \{-\infty\}$ given by

$$h(y, z) \triangleq \inf_{x \in C} f(x, y, z), \quad y \in D, z \in E,$$

is clearly concave. By Lemma A.4 the function $h(y_0, \cdot)$ is differentiable at z_0 and $\frac{\partial h}{\partial z}(y_0, z_0) = \frac{\partial f}{\partial z}(x_0, y_0, z_0)$. An application of Lemma A.3 completes the proof. □

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