Integral representation of martingales and endogenous completeness of financial models

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Abstract

Let $\mathbb{Q}$ and $\mathbb{P}$ be equivalent probability measures and let ψ be a J-dimensional vector of random variables such that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and ψ are defined in terms of a weak solution X to a d-dimensional stochastic differential equation. Motivated by the problem of *endogenous completeness* in financial economics we present conditions which guarantee that any local martingale under $\mathbb{Q}$ is a stochastic integral with respect to the J-dimensional martingale $S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t]$. While the drift $b = b(t,x)$ and the volatility $\sigma = \sigma(t,x)$ coefficients for X need to have only minimal regularity properties with respect to x, they are assumed to be analytic functions with respect to t. We provide a counter-example showing that this t-analyticity assumption for σ cannot be removed.

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1 Introduction

Let $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ be a complete filtered probability space, $\mathbb{Q}$ be an equivalent probability measure, and $S = (S_t^j)$ be a J-dimensional martingale under $\mathbb{Q}$. It is often important to know whether any local martingale $M = (M_t)$ under $\mathbb{Q}$ admits an integral representation with respect to S, that is,

$$M_t = M_0 + \int_0^t H_u dS_u, \quad t \in [0,1], \tag{1.1}$$

for some predictable S-integrable process $H = (H_t^j)$. For instance, in mathematical finance, which is the topic of a particular interest to us, the existence of such a martingale representation corresponds to the *completeness* of the market model driven by stock prices S, see Harrison and Pliska [5].

A general answer is given in Jacod [7, Section XI.1(a)]. Jacod's theorem states that the integral representation property holds if and only if $\mathbb{Q}$ is the *only* equivalent martingale measure for S. In mathematical finance this result is sometimes referred to as the 2nd fundamental theorem of asset pricing.

In many applications, including those in finance, the process S is defined in terms of its predictable characteristics under $\mathbb{P}$. The construction of a martingale measure $\mathbb{Q}$ for S is then accomplished through the use of the Girsanov theorem and its generalizations, see Jacod and Shiryaev [8]. The verification of the existence of integral representations for all $\mathbb{Q}$-martingales under S is often straightforward. For example, if S is a diffusion process under $\mathbb{P}$ with the drift vector-process $b = (b_t)$ and the volatility matrix-process $\sigma = (\sigma_t)$, then such a representation exists if and only if the volatility matrix-process σ has full rank $d\mathbb{P} \times dt$ almost surely.

In this paper we assume that the inputs are the random variables $\xi > 0$ and $\psi = (\psi^j)_{j=1,\dots,J}$, while $\mathbb{Q}$ and S are defined as

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} &\triangleq \frac{\xi}{\mathbb{E}[\xi]}, \\ S_t &\triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad t \in [0,1]. \end{aligned}$$

We are looking for (easily verifiable) conditions on ξ and ψ guaranteeing the integral representation of all $\mathbb{Q}$-martingales with respect to S.

Our study is motivated by the problem of *endogenous completeness* in financial economics, see Anderson and Raimondo [1]. Here ξ is an equilibrium state price density, usually defined implicitly by a fixed point argument, and

$\psi = (\psi^j)$ is the random vector of the cumulative discounted dividends for traded stocks. The term "endogenous" is used because the stock prices S are now *computed* as an output of equilibrium.

We focus on the case when ξ and ψ are defined in terms of a weak solution X to a d-dimensional stochastic differential equation. With respect to x the coefficients of this equation satisfy classical conditions guaranteeing weak existence and uniqueness: the drift vector $b(t, \cdot)$ is measurable and bounded and the volatility matrix $\sigma(t, \cdot)$ is uniformly continuous and bounded and has a bounded inverse. With respect to t our assumptions are more stringent: $b(\cdot, x)$ and $\sigma(\cdot, x)$ are required to be analytic functions. We give an example showing that the t-analyticity assumption on the volatility matrix σ cannot be removed.

Our results complement and generalize those in Anderson and Raimondo [1], Hugonnier, Malamud, and Trubowitz [6], and Riedel and Herzberg [18]. In the pioneering paper [1], X is a Brownian motion. The proofs in this paper rely on non-standard analysis. In [6] and [18], among other conditions, the diffusion coefficients $b = b(t, x)$ and $\sigma = \sigma(t, x)$ are assumed to be analytic functions with respect to (t, x). In one important aspect, however, the assumptions in [1], [6], and [18] are less restrictive. If $\psi = F(X_1)$, where $F = F^j(x)$ is a J-dimensional vector-function on $\mathbb{R}^d$, then these papers require the Jacobian matrix of F to have rank d only on some open set. In our framework, this property needs to hold almost everywhere on $\mathbb{R}^d$. We provide an example showing that in the absence of the x-analyticity assumption on b and σ this stronger condition cannot be relaxed.

2 Main results

Let $\mathbf{X}$ be a Banach space and D be a subset of either the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. We remind the reader that a map $f : D \to \mathbf{X}$ is called *analytic* if for any $x \in D$ there exist a number $\epsilon > 0$ and a sequence $A = (A_n)_{n \geq 0}$ in $\mathbf{X}$ (both ϵ and A depend on x) such that

$$f(y) = \sum_{n=0}^{\infty} A_n (y - x)^n, \quad y \in D, |y - x| < \epsilon,$$

where the series converges in the norm $\|\cdot\|_{\mathbf{X}}$ of $\mathbf{X}$.

In the statements of our results, $D = [0, 1]$ and $\mathbf{X}$ will be one of the following spaces:

$\mathbf{L}_\infty = \mathbf{L}_\infty(\mathbb{R}^d, dx)$: the Lebesgue space of bounded measurable real-valued functions f on an Euclidean space $\mathbb{R}^d$ with the norm $\|f\|_{\mathbf{L}_\infty} \triangleq \operatorname{ess\,sup}_{x \in \mathbb{R}^d} |f(x)|$.

As usual, the term *a measurable function* is used for an equivalence class of Borel measurable functions indistinguishable with respect to the Lebesgue measure.

$\mathbf{C} = \mathbf{C}(\mathbb{R}^d)$: the Banach space of bounded and continuous real-valued functions f on $\mathbb{R}^d$ with the norm $\|f\|_{\mathbf{C}} \triangleq \sup_{x\in\mathbb{R}^d}|f(x)|$.

We shall use standard notations of linear algebra. If x and y are vectors in an Euclidean space $\mathbb{R}^n$, then xy denotes the scalar product and $|x| \triangleq \sqrt{xx}$. If $a \in \mathbb{R}^{m\times n}$ is a matrix with m rows and n columns, then ax denotes its product on the (column-)vector x, a^* stands for the transpose, and $|a| \triangleq \sqrt{\mathrm{trace}(aa^*)}$.

Let $\mathbb{R}^d$ be an Euclidean space and the functions $b = b(t,x) : [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma = \sigma(t,x) : [0,1] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be such that for all $i,j = 1,\dots,d$:

(A1) $t \mapsto b^i(t,\cdot)$ is an analytic map of $[0,1]$ to $\mathbf{L}_\infty$.

(A2) $t \mapsto \sigma^{ij}(t,\cdot)$ is an analytic map of $[0,1]$ to $\mathbf{C}$. For $t \in [0,1]$ and $x \in \mathbb{R}^d$ the matrix $\sigma(t,x)$ has the inverse $\sigma^{-1}(t,x)$ and there exists a constant $N > 0$, same for all t and x, such that

$$|\sigma^{-1}(t,x)| \le N. \tag{2.1}$$

Moreover, there exists a strictly increasing function $\omega = (\omega(\epsilon))_{\epsilon>0}$ such that $\omega(\epsilon) \to 0$ as $\epsilon \downarrow 0$ and, for all $t \in [0,1]$ and all $x, y \in \mathbb{R}^d$,

$$|\sigma(t,x) - \sigma(t,y)| \le \omega(|x-y|).$$

Note that (2.1) is equivalent to the uniform ellipticity of the matrix-function $a \triangleq \sigma\sigma^*$: for all $y \in \mathbb{R}^d$ and $(t,x) \in [0,1]\times\mathbb{R}^d$,

$$ya(t,x)y = |\sigma(t,x)y|^2 \ge \frac{1}{N^2}|y|^2.$$

Let $X_0 \in \mathbb{R}^d$. The classical results of Stroock and Varadhan [19, Theorem 7.2.1] and Krylov [16, 13] imply that under (A1) and (A2) there exist a complete filtered probability space $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$, a Brownian motion W, and a stochastic process X, both with values in $\mathbb{R}^d$, such that

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s, \quad t \in [0,1], \tag{2.2}$$

and, moreover, all finite dimensional distributions of X are defined uniquely. In view of (2.1), we can (and will) assume that the filtration $\mathbf{F}$ is generated by X:

$$\mathbf{F} = \mathbf{F}^X. \tag{2.3}$$

In this case, $\mathbb{P}$ is defined uniquely in the sense that if $\mathbb{Q} \sim \mathbb{P}$ is an equivalent probability measure on $(\Omega, \mathcal{F}_1) = (\Omega, \mathcal{F}_1^X)$ such that

$$W_t = \int_0^t \sigma^{-1}(s, X_s)(dX_s - b(s, X_s)ds), \quad t \in [0, 1],$$

is a Brownian motion under $\mathbb{Q}$, then $\mathbb{Q} = \mathbb{P}$.

Remark 2.1. With respect to x, (A1) and (A2) are, essentially, the minimal classical assumptions guaranteeing the existence and the uniqueness of the weak solution to (2.2). This weak solution is also well-defined when b and σ are only measurable functions with respect to t. As we shall see in Example 2.6, the requirement on σ to be t-analytic is, however, essential for the validity of our main results, Theorems 2.3 and 2.5.

Remark 2.2. It is well-known that any local martingale M adapted to the filtration $\mathbf{F}^W$, generated by the Brownian motion W, is a stochastic integral with respect to W, that is, there exists an $\mathbf{F}^W$-predictable process H with values in $\mathbb{R}^d$ such that

$$M_t = M_0 + \int_0^t H_u dW_u \triangleq M_0 + \sum_{i=1}^d \int_0^t H_u^i dW_u^i, \quad t \in [0, 1]. \tag{2.4}$$

The example in Barlow [2] shows that under (A1) and (A2) the filtration $\mathbf{F}^W$ may be strictly smaller than $\mathbf{F} = \mathbf{F}^X$. Nevertheless, for every local martingale M adapted to $\mathbf{F}$ the integral representation (2.4) still holds with some $\mathbf{F}$-predictable H. This follows from the mentioned fact that any $\mathbb{Q} \sim \mathbb{P}$ such that W is a $\mathbb{Q}$-local martingale (equivalently, a $\mathbb{Q}$-Brownian motion) coincides with $\mathbb{P}$ and the integral representation theorems in Jacod [7, Section XI.1(a)].

Recall that a locally integrable function f on $(\mathbb{R}^d, dx)$ is *weakly differentiable* if for any index $i = 1, \dots, d$ there is a locally integrable function g^i such that the identity

$$\int_{\mathbb{R}^d} g^i(x)h(x)dx = -\int_{\mathbb{R}^d} f(x)\frac{\partial h}{\partial x^i}(x)dx$$

holds for any function $h \in \mathbf{C}^\infty$ with compact support, where $\mathbf{C}^\infty$ is the space of infinitely many times differentiable functions. In this case, we set $\frac{\partial f}{\partial x^i} \triangleq g^i$. The weak derivatives of higher orders are defined recursively.

Let $J \geq d$ be an integer and the functions $F^j, G: \mathbb{R}^d \to \mathbb{R}$ and $f^j, \alpha^j, \beta:$ $[0,1] \times \mathbb{R}^d \to \mathbb{R}$, $j = 1, \dots, J$, be such that for some $N \geq 0$

(A3) The functions F^j are weakly differentiable, $e^{-N|\cdot|}\frac{\partial F^j}{\partial x^i} \triangleq \left(e^{-N|x|}\frac{\partial F^j}{\partial x^i}(x)\right)_{x \in \mathbf{R}^d} \in \mathbf{L}_\infty$, $i = 1, \dots, d$, and the Jacobian matrix $\left(\frac{\partial F^j}{\partial x^i}\right)_{i=1,\dots,d,\ j=1,\dots,J}$ has rank d almost surely under the Lebesgue measure on $\mathbb{R}^d$.

(A4) The function G is strictly positive and weakly differentiable and $e^{-N|\cdot|}\frac{\partial G}{\partial x^i} \in \mathbf{L}_\infty$, $i = 1, \dots, d$.

(A5) The maps $t \mapsto e^{-N|\cdot|}f^j(t,\cdot) \triangleq \left(e^{-N|x|}f^j(t,x)\right)_{x \in \mathbf{R}^d}$, $t \mapsto \alpha^j(t,\cdot)$, and $t \mapsto \beta(t,\cdot)$ of $[0,1]$ to $\mathbf{L}_\infty$ are analytic.

We now define the random variables

(2.5)
$$\psi^j \triangleq F^j(X_1)e^{\int_0^1 \alpha^j(t,X_t)dt} + \int_0^1 e^{\int_0^t \alpha^j(s,X_s)ds} f^j(t,X_t)dt, \quad j = 1, \dots, J,$$

(2.6) $\xi \triangleq G(X_1)e^{\int_0^1 \beta(t,X_t)dt}$,

and state the main results of the paper.

Theorem 2.3. *Suppose that* (2.3) *and (A1)–(A5) hold. Then the equivalent probability measure* $\mathbb{Q}$ *with the density*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{\xi}{\mathbb{E}[\xi]},$$

and the $\mathbb{Q}$*-martingale*

$$S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad t \in [0,1],$$

with values in $\mathbb{R}^J$ *are well-defined and any local martingale* M *under* $\mathbb{Q}$ *is a stochastic integral with respect to* S*, that is,* (1.1) *holds.*

Remark 2.4. The t-analyticity condition on f^j in (A5) cannot be relaxed even if X is a one-dimensional Brownian motion, see Example 2.7 below. By contrast, the x-regularity assumptions on the functions F^j, G, and f^j in (A3), (A4), and (A5) admit weaker formulations with the $\mathbf{L}_\infty$ space being replaced by appropriate $\mathbf{L}_p$ spaces (with the power $p > 1$ different for each of these functions). This generalization leads, however, to more delicate and longer proofs and will be dealt with elsewhere.

The proof of Theorem 2.3 will be given in Section 5 and will rely on the study of parabolic equations in Section 4. In Section 3.2 we shall apply Theorem 2.3 to the problem of endogenous completeness in an economy with terminal consumption.

The following result, which, in fact, is an easy corollary of Theorem 2.3, will be used in Section 3.3 to study the endogenous completeness in an economy with intermediate consumption. For $i = 1, \dots, d$ let the functions $\gamma^i = \gamma^i(t,x)$ on $[0,1] \times \mathbb{R}^d$ be such that

(A6) the maps $t \mapsto \gamma^i(t,\cdot)$ of $[0,1]$ to $\mathbf{L}_\infty$ are analytic.

Theorem 2.5. *Suppose that* (2.3)*, (A1)–(A3), and (A5)–(A6) hold. Then the equivalent probability measure $\mathbb{Q}$ with the density*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^1 \gamma(s,X_s)dW_s - \frac{1}{2}\int_0^1 |\gamma(s,X_s)|^2 ds\right)$$

and the $\mathbb{Q}$-martingale

$$S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad t \in [0,1],$$

with values in $\mathbb{R}^J$ are well-defined and any local martingale under $\mathbb{Q}$ is a stochastic integral with respect to S.

Proof. By Girsanov's theorem,

$$W_t^{\mathbb{Q}} = W_t - \int_0^t \gamma(s,X_s)ds$$

is a Brownian motion under $\mathbb{Q}$. After this substitution the equation (2.2) becomes

$$dX_t = \big(b(t,X_t) + \sigma(t,X_t)\gamma(t,X_t)\big)dt + \sigma(t,X_t)dW_t^{\mathbb{Q}}, \quad X_0 = x.$$

The result now follows from Theorem 2.3, where we can assume $\xi = 1$, if we observe that, similarly with b, each component of $\widetilde{b} \triangleq b + \sigma\gamma$ defines an analytic map of $[0,1]$ to $\mathbf{L}_\infty$. $\square$

We conclude with a few counter-examples illustrating the sharpness of the conditions of the theorems. Our first two examples show that the time analyticity assumptions on the volatility coefficient $\sigma = \sigma(t,x)$ and on the functions $f^j = f^j(t,x)$ in Theorems 2.3 and 2.5 cannot be relaxed. In both cases, we take $b(t,x) = \alpha(t,x) = \beta(t,x) = \gamma(t,x) = 0$ and $G(x) = 1$; in particular, $\mathbb{Q} = \mathbb{P}$.

Example 2.6. We show that the assertions of Theorems 2.3 and 2.5 can fail to hold when all their conditions are satisfied except the t-analyticity of the volatility matrix σ. In our construction, $d = J = 2$ and both σ and its inverse σ^{-1} are $\mathbf{C}^\infty$-matrices on $[0,1] \times \mathbb{R}^2$ which are bounded with all their derivatives and have analytic restrictions to $[0, \frac{1}{2}) \times \mathbb{R}^2$ and $(\frac{1}{2}, 1] \times \mathbb{R}^2$.

Let $g = g(t)$ be a $\mathbf{C}^\infty$-function on $[0,1]$ which equals 0 on $[0, \frac{1}{2}]$ and is analytic and strictly positive on $(\frac{1}{2}, 1]$. Let $h = h(t,y)$ be a non-constant analytic function on $[0,1] \times \mathbb{R}$ such that $0 \le h \le 1$ and

$$\frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial y^2} = 0.$$

For instance, we can take

$$h(t,y) = \frac{1}{2}(1 + e^{\frac{t-1}{2}} \sin y).$$

Define a 2-dimensional diffusion (X,Y) on $[0,1]$ by

$$\begin{aligned} X_t &= \int_0^t \sqrt{1 + g(s)h(s,Y_s)}dB_s, \\ Y_t &= W_t, \end{aligned}$$

where B and W are independent Brownian motions. Clearly, the volatility matrix

$$\sigma(t,x,y) = \begin{pmatrix} \sqrt{1 + g(t)h(t,y)} & 0 \\ 0 & 1 \end{pmatrix}$$

has the announced properties and coincides with the identity matrix for $t \in [0, \frac{1}{2}]$.

Define the functions $F = F(x,y)$ and $H = H(x,y)$ on $\mathbb{R}^2$ as

$$\begin{aligned} F(x,y) &= x, \\ H(x,y) &= x^2 - 1 - h(1,y)\int_0^1 g(t)dt. \end{aligned}$$

As $h(1,\cdot)$ is non-constant and analytic, the set of zeros for $\frac{\partial h}{\partial y}(1,\cdot)$ is at most countable. Since the determinant of the Jacobian matrix for (F,H) is given by

$$\frac{\partial F}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial H}{\partial x} = -\frac{\partial h}{\partial y}(1,y)\int_0^1 g(t)dt,$$

it follows that this Jacobian matrix has full rank almost surely.

Observe now that

$$
\begin{aligned}
S_t &\triangleq \mathbb{E}[F(X_1, Y_1)|\mathcal{F}_t] = X_t, \\
R_t &\triangleq \mathbb{E}[H(X_1, Y_1)|\mathcal{F}_t] = X_t^2 - t - h(t, Y_t) \int_0^t g(s)ds,
\end{aligned}
$$

which can be verified by Ito's formula. As $g(t) = 0$ for $t \in [0, \frac{1}{2}]$, it follows that $S_t = B_t$ and $R_t = B_t^2 - t$ on $[0, \frac{1}{2}]$. Hence, the Brownian motion $Y = W$ cannot be written as a stochastic integral with respect to (S, R).

Example 2.7. This counter-example shows the necessity of the t-analyticity assumption on $f^j = f^j(t, x)$ in (A5). Let $g = g(t)$ be a $\mathbf{C}^\infty$-function on $[0, 1]$ which equals 0 on $[0, \frac{1}{2}]$, is analytic on $(\frac{1}{2}, 1]$, and is such that $g(1) \neq 0$. For the functions

$$
\begin{aligned}
f(t, x) &= -\big(g'(t)x + \frac{1}{2}g^2(t)\big)e^{g(t)x}, \\
F(x) &= e^{g(1)x},
\end{aligned}
$$

the conditions (A3) and (A5) hold except the time analyticity of the map $t \to e^{-N|\cdot|}f(t, \cdot)$ of $[0, 1]$ to $\mathbf{L}_\infty$. This map belongs instead to $\mathbf{C}^\infty$ and has analytic restrictions to $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$.

Take X to be a one-dimensional Brownian motion:

$$
X_t = W_t, \quad t \in [0, 1],
$$

and observe that, by Ito's formula,

$$
S_t \triangleq \mathbb{E}[\psi|\mathcal{F}_t] = e^{g(t)W_t} - \int_0^t \big(g'(s)W_s + \frac{1}{2}g^2(s)\big)e^{g(s)W_s}ds,
$$

where

$$
\psi = F(X_1) + \int_0^1 f(t, X_t)dt.
$$

For $t \in [0, \frac{1}{2}]$ we have $g(t) = 0$ and, therefore, $S_t = 1$. Hence, any local martingale M which is non-constant on $[0, \frac{1}{2}]$ cannot be a stochastic integral with respect to S.

When the diffusion coefficients $\sigma^{ij} = \sigma^{ij}(t, x)$ and $b^i = b^i(t, x)$ and the functions $f^j = f^j(t, x)$ in (A5) are also analytic with respect to the state variable x, the results in [6] and [18] show that in (A3) it is sufficient to require the Jacobian matrix of $F = F(x)$ to have rank d only on an open set. The following example shows that in the case of $\mathbf{C}^\infty$ functions this simplification is not possible anymore.

Example 2.8. Let $d = J = 2$ and let $g : \mathbb{R} \to \mathbb{R}$ be a $\mathbf{C}^\infty$ function such that $g(x) = 0$ for $x \leq 0$, while $g'(x) > 0$ and $g''(x)$ is bounded for $x > 0$.

Define the diffusion processes X and Y on $[0, 1]$ by

$$
\begin{aligned}
X_t &= B_t, \\
Y_t &= \int_0^t g''(X_s)ds + W_t,
\end{aligned}
$$

where B and W are independent Brownian motions. Clearly, the diffusion coefficients of (X, Y) satisfy (A1) and (A2).

Define the functions $F = F(x, y)$ and $H = H(x, y)$ on $\mathbb{R}^2$ as

$$
\begin{aligned}
F(x, y) &= y, \\
H(x, y) &= y - 2g(x),
\end{aligned}
$$

and the function $f = f(t, x, y)$ on $[0, 1] \times \mathbb{R}^2$ as

$$
f(t, x, y) = -g''(x).
$$

Observe that the determinant of the Jacobian matrix for (F, H) is given by

$$
\frac{\partial F}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial H}{\partial x} = 2g'(x),
$$

and, hence, this Jacobian matrix has full rank on the set $(0, \infty) \times \mathbb{R}$.

A simple application of Ito's formula yields

$$
\begin{aligned}
S_t &\triangleq \mathbb{E}[F(X_1, Y_1) + \int_0^1 f(s, X_s, Y_s)ds|\mathcal{F}_t] = W_t, \\
R_t &\triangleq \mathbb{E}[H(X_1, Y_1)|\mathcal{F}_t] = W_t - 2\int_0^t g'(X_s)dB_s.
\end{aligned}
$$

Hence, any martingale in the form

$$
M_t = \int_0^t h(X_s)dB_s,
$$

where the function $h = h(x)$ is different from zero for $x \leq 0$, cannot be written as a stochastic integral with respect to (S, R).

3 Endogenous completeness

In this section, Theorems 2.3 and 2.5 will be applied to the problem of *endogenous completeness* in financial economics.

As before, the uncertainty and the information flow are modeled by the filtered probability space $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ with the filtration $\mathbf{F}$ generated by the solution X to (2.2).

3.1 Financial market with exogenous prices

Recall first the "standard" model of mathematical finance, where the prices of traded securities are given as model inputs or, in more economic terms, *exogenously*.

Consider a financial market with $J+1$ traded assets: a bank account and J stocks. The bank account pays the continuous interest rate $r = (r_t)$ and the stocks pay the continuous dividends $\theta = (\theta_t^j)$ and have the prices $P = (P_t^j)$, where $t \in [0,1]$ and $j = 1, \ldots, J$. We assume that P is a continuous semimartingale with values in $\mathbb{R}^J$ and

$$\int_0^1 (|r_t| + |\theta_t|)dt < \infty.$$

We shall use the abbreviation (r, θ, P) for such a model.

The wealth of a (self-financing) strategy evolves as

$$V_t = v + \int_0^t \zeta_u(dP_u + \theta_u du) + \int_0^t (V_u - \zeta_u P_u) r_u du, \quad t \in [0,1], \tag{3.1}$$

where $v \in \mathbb{R}$ is the initial wealth and $\zeta = (\zeta_t^j)$ is the predictable process with values in $\mathbb{R}^J$ of the number of stocks such that the integrals in (3.1) are well-defined. This balance equation can be written more compactly in terms of discounted values:

$$V_t e^{-\int_0^t r_u du} = v + \int_0^t \zeta_u dS_u, \quad t \in [0,1],$$

where, for $j = 1, \ldots, J$,

$$S_t^j \triangleq P_t^j e^{-\int_0^t r_u du} + \int_0^t \theta_s^j e^{-\int_0^s r_u du} ds, \quad t \in [0,1],$$

denotes the discounted wealth of the "buy and hold" strategy for jth stock, that is, the strategy where we start with one unit of such a stock and reinvest the continuous dividends $\theta = (\theta_t)$ in the bank account.

It is common to assume that the family $\mathcal{Q}$ of the equivalent martingale measures for S is not empty:

$$\mathcal{Q} = \mathcal{Q}(r, \theta, P) \triangleq \{\mathbb{Q} \sim \mathbb{P} : S \text{ is a } \mathbb{Q}\text{-martingale}\} \neq \emptyset.$$

This is equivalent to the absence of arbitrage if one is allowed to sell short both the bank account and the stock until the maturity; see [4].

The following property is the primary focus of our study.

Definition 3.1. The model (r, θ, P) is called *complete* if for any random variable μ such that $|\mu| \leq 1$ there is a self-financing strategy such that $|V_t e^{-\int_0^t r_u du}| \leq 1$, $t \in [0,1]$, and $V_1 e^{-\int_0^1 r_u du} = \mu$.

Recall, see Harrison and Pliska [5] and Jacod [7, Section XI.1(a)], that for a (r, θ, P)-model with $\mathcal{Q} \neq \emptyset$ the completeness is equivalent to any of the following conditions:

1. there exists only one $\mathbb{Q} \in \mathcal{Q}$;
2. if $\mathbb{Q} \in \mathcal{Q}$ then any $\mathbb{Q}$-local martingale is a discounted wealth process or, equivalently, is a stochastic integral with respect to S.

3.2 Economy with terminal consumption

Consider an economy with a single (representative) agent. We assume that the agent consumes only at maturity 1 and denote by $U = \big(U(x)\big)_{x>0}$ his utility function for terminal wealth.

(B1) The utility function $U = U(x)$ is twice weakly differentiable on $(0, \infty)$ and $U' > 0$. Moreover, it has a bounded relative risk aversion, that is, for some constant $N > 0$,

$$\frac{1}{N} \leq A(x) \triangleq -\frac{xU''(x)}{U'(x)} \leq N, \quad x \in (0, \infty).$$

Note that (B1) implies that U is strictly increasing, strictly concave, and continuously differentiable, that it satisfies the Inada conditions:

$$\lim_{x \downarrow 0} U'(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} U'(x) = 0,$$

and that its asymptotic elasticity is strictly less than 1:

$$\limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1.$$

Given an (r, θ, P)-market, a basic problem of financial economics is to determine an optimal investment strategy $\widehat{V}(v)$ of the agent starting with the initial capital $v > 0$. More formally, if

$$\mathcal{V}(v) \triangleq \{V \geq 0 : \ (3.1) \text{ holds for some } \zeta\}$$

denotes the family of positive wealth processes starting from $v > 0$, then $\widehat{V}(v)$ is defined as an element of $\mathcal{V}(v)$ such that

$$\infty > \mathbb{E}[U(\widehat{V}_1(v))] \geq \mathbb{E}[U(V_1)] \quad \text{for all} \quad V \in \mathcal{V}(v), \tag{3.2}$$

where we used the convention:

$$\mathbb{E}[U(V_1)] \triangleq -\infty \quad \text{if} \quad \mathbb{E}[\min(U(V_1), 0)] = -\infty.$$

We are interested in an inverse problem: given a terminal wealth Λ for the agent and final dividends $\Theta = (\Theta^j)$ for the stocks find a price process $P = (P^j_t)$ such that $P_1 = \Theta$ and, in the (r, θ, P)-market, $\widehat{V}_1(v) = \Lambda$ for some initial wealth $v > 0$. We particularly want to know whether the family $\mathcal{Q}(r, \theta, P)$ is a singleton and, hence, the (r, θ, P)-model is complete. Since the price process P is now an outcome, rather than an input, the latter property is referred to as an *endogenous completeness*.

We make the following assumptions:

(B2) The interest rate $r_t = \beta(t, X_t)$, $t \in [0, 1]$, where the function $\beta = \beta(t, x)$ satisfies (A5).

(B3) The continuous dividends $\theta = (\theta^j_t)$ and the terminal dividends $\Theta = (\Theta^j)$ are such that, for $t \in [0, 1]$ and $j = 1, \ldots, J$,

$$\theta^j_t = f^j(t, X_t) e^{\int_0^t \alpha^j(s, X_s) ds},$$
$$\Theta^j = F^j(X_1) e^{\int_0^1 \alpha^j(s, X_s) ds},$$

where the functions $F^j = F^j(x)$ satisfy (A3) and the functions f^j and α^j satisfy (A5).

(B4) The terminal wealth $\Lambda = e^{H(X_1)}$, where the function $H = H(x)$ on $\mathbb{R}^d$ is weakly differentiable and $\frac{\partial H}{\partial x^i} \in \mathbf{L}^\infty$, $i = 1, \ldots, d$.

Note that a function $H = H(x)$ on $\mathbb{R}^d$ satisfies (B4) if and only if it is Lipschitz continuous, that is, there is $N \geq 0$ such that

$$|H(x) - H(y)| \leq N|x - y|, \quad x, y \in \mathbb{R}^d.$$

For $j = 1, \ldots, J$ denote

$$\psi^j \triangleq \Theta^j e^{-\int_0^1 r_u du} + \int_0^1 \theta^j_u e^{-\int_0^u r_s ds} du, \tag{3.3}$$

the cumulative values of the discounted cash flows generated by the stocks.

Theorem 3.2. *Let* (2.3), *(A1)–(A2), and (B1)–(B4) hold. Then there exists a continuous process $P = (P^j_t)$ with the terminal value $P_1 = \Theta$ such that, in the (r, θ, P)-market, for some initial capital $v_0 > 0$ the optimal terminal wealth $\widehat{V}_1(v_0)$ in* (3.2) *equals Λ and such that the set of martingale measures $\mathcal{Q} = \mathcal{Q}(r, \theta, P)$ is a singleton; in particular, the (r, θ, P)-market is complete.*

Further, $P = (P^j_t)$, $\mathbb{Q} \in \mathcal{Q}$, and v_0 are unique and given by

$$P_t = S_t e^{\int_0^t r_u du} - \int_0^t e^{\int_s^t r_u du} \theta_s ds, \quad t \in [0, 1], \tag{3.4}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{U'(\Lambda) e^{\int_0^1 r_u du}}{\mathbb{E}[U'(\Lambda) e^{\int_0^1 r_u du}]}, \tag{3.5}$$

$$v_0 = \mathbb{E}^{\mathbb{Q}}[\Lambda e^{-\int_0^1 r_u du}], \tag{3.6}$$

where, for $\psi = (\psi^j)$ from (3.3),

$$S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t], \quad t \in [0, 1]. \tag{3.7}$$

Proof. It is well-known, see [9, Theorem 3.7.6] and [11, Theorem 2.0], that for the utility function $U = U(x)$ as in (B1) and a complete market with unique $\mathbb{Q} \in \mathcal{Q}$ the optimal terminal wealth equals Λ if and only if (3.5) holds. Clearly, the martingale property of the discounted wealth process of an optimal strategy yields (3.6). Hence, it remains only to verify the completeness of the (r, θ, P)-market with $P = (P^j_t)$ given by (3.4).

Define the function

$$G(x) \triangleq U'(e^{H(x)}), \quad x \in \mathbb{R}^d,$$

and observe that

$$\frac{\partial \ln G}{\partial x^i} = \frac{U''}{U'}(e^H) e^H \frac{\partial H}{\partial x^i} = -A(e^H) \frac{\partial H}{\partial x^i} \in \mathbf{L}_\infty,$$

by the boundedness of A and $\frac{\partial H}{\partial x^i}$. This implies the existence of $N \geq 0$ such that

$$e^{-N|\cdot|} \left(G + \sum_{i=1}^d |\frac{\partial G}{\partial x^i}| \right) \in \mathbf{L}_\infty,$$

which, in particular, yields (A4).

Since $e^{-N|\cdot|}(G + e^H + |F|) \in \mathbf{L}_\infty$ for some $N \geq 0$, we deduce the existence of $N \geq 0$ such that

$$U'(\Lambda)(1 + \Lambda + |\psi|) \leq e^{N(1 + \sup_{t \in [0,1]} |X_t|)}.$$

As the diffusion coefficients $b = b(t,x)$ and $\sigma = \sigma(t,x)$ are bounded, the random variable $\sup_{t\in[0,1]}|X_t|$ has all exponential moments. It follows that

$$\mathbb{E}[U'(\Lambda)(1+\Lambda+|\psi|)] < \infty,$$

and, in particular, P, $\mathbb{Q}$, v_0, and S are well-defined by (3.4)–(3.7).

By construction, $\mathbb{Q} \in \mathcal{Q}(r,\theta,P)$. With (A4) verified above, the assumptions of Theorem 2.3 for $\mathbb{Q}$ and S hold trivially. The results cited after Definition 3.1 then imply that the (r,θ,P)-market is complete and that $\mathbb{Q}$ is the only martingale measure. $\square$

We conclude this section with an important corollary of Theorem 3.2. Theorem 3.3 below yields dynamic completeness of all Pareto equilibria in an economy where M investors trade in the *exogenous* bank account paying the interest rate r and in the *endogenous* stocks paying the continuous dividends θ and the terminal dividends Θ. The economic agents have utility functions $U_m = U_m(x)$, $m = 1,\dots,M$, and they collectively possess the terminal wealth Λ. A result of this kind plays a crucial role in the proof of the existence of a continuous-time Arrow-Debreu-Radner equilibrium, see [1], [6], and [18].

Theorem 3.3. *Let* (2.3)*, (A1)–(A2), and (B2)–(B4) hold. Suppose each utility function U_m, $m = 1,\dots,M$, satisfies (B1). Fix $w \in (0,\infty)^N$ and define the function*

$$U(x) \triangleq \sup_{x^1+\dots+x^M=x} \sum_{m=1}^{M} w^m U_m(x^m), \quad x \in (0,\infty). \tag{3.8}$$

Let the price process P be defined by (3.4)*,* (3.5)*, and* (3.7)*. Then the (r,θ,P)-market is complete.*

Proof. The result is an immediate consequence of Theorem 3.2 as soon as we verify that U satisfies (B1). This follows from the well-known identity for the relative risk-aversions:

$$\sum_{m=1}^{M} \frac{\widehat{x}_m(x)}{A_m(\widehat{x}_m(x))} = \frac{x}{A(x)}, \quad x \in (0,\infty).$$

Here $\widehat{x}_1(x) > 0, \dots, \widehat{x}_m(x) > 0$ are the arguments of maximum in (3.8) and A_m is the relative risk-aversion of U_m. The arguments leading to this equality will be recalled in the proof of Lemma 3.6. $\square$

3.3 Economy with intermediate consumption

Consider now an economy where a single (representative) agent consumes *continuously* on $[0,1]$. We denote by $u(t,x): [0,1]\times(0,\infty)\to\mathbb{R}$ the agent's utility function for intermediate consumption and assume that

(B5) $u = u(t,x)$ is analytic in t and 3-times weakly differentiable in x. Moreover, $u_x > 0$ and $u_{xx} < 0$ and $t\mapsto a(t,\cdot)$, $t\mapsto \frac{1}{a(t,\cdot)}$, $t\mapsto p(t,\cdot)$, and $t\mapsto q(t,\cdot)$ are analytic maps of $[0,1]$ to $\mathbf{L}_\infty$, where

$$\begin{aligned} a(t,x) &\triangleq -\frac{xu_{xx}(t,x)}{u_x(t,x)}, \\ p(t,x) &\triangleq -\frac{xu_{xxx}(t,x)}{u_{xx}(t,x)}, \\ q(t,x) &\triangleq -\frac{\partial \ln u_x(t,x)}{\partial t} = -\frac{u_{xt}}{u_x}(t,x), \end{aligned}$$

are, respectively, the relative risk aversion, the relative prudence, and an "impatience" rate of the utility function u.

Note that (B5) implies that $u(t,\cdot)$ is twice continuously differentiable, strictly increasing, and strictly concave and that there is a constant $N>0$ such that

$$a(t,x) + \frac{1}{a(t,x)} + |p(t,x)| + |q(t,x)| \leq N, \quad (t,x)\in[0,1]\times\mathbb{R}. \tag{3.9}$$

Recall the formulation of the investment problem with continuous consumption in a *given* (r,θ,P)-market. Let $\eta = (\eta_t)$ be a non-negative adapted process such that $\int_0^1 \eta_t dt < \infty$. The wealth process of a strategy with the consumption process η is defined as

$$V_t = v + \int_0^t \zeta_u(dP_u + \theta_u du) + \int_0^t (V_u - \zeta_u P_u) r_u du - \int_0^t \eta_u du, \tag{3.10}$$

or, in discounted terms,

$$V_t e^{-\int_0^t r_s ds} = v + \int_0^t \zeta_u dS_u - \int_0^t \eta_u e^{-\int_0^u r_s ds} du, \quad t\in[0,1].$$

Here, as before, v and $\zeta = (\zeta_t^j)$ stand, respectively, for the initial wealth and the process of the number of stocks. We consider the optimization problem

$$\mathbb{E}[\int_0^1 u(t,\eta_t)dt] \to \max, \quad \eta\in\mathcal{W}(v), \tag{3.11}$$

where $\mathcal{W}(v)$ denotes the family of consumption processes obtained from the initial wealth v, that is,

$$\mathcal{W}(v) \triangleq \{\eta \geq 0 : \text{ (3.10) holds for some } V \geq 0 \text{ and } \zeta\},$$

and we have used the convention:

$$\mathbb{E}[\int_0^1 u(t, \eta_t)dt] \triangleq -\infty \quad \text{if} \quad \mathbb{E}[\int_0^1 \min(u(t, \eta_t), 0)dt] = -\infty.$$

As in the previous section, we study an inverse problem to (3.11): given a consumption process $\lambda = (\lambda_t)$ for the agent and final dividends $\Theta = (\Theta^j)$ for the stocks, find an interest rate process $r = (r_t)$ and a price process $P = (P_t^j)$ such that $P_1 = \Theta$ and, in the (r, θ, P)-model, the upper bound in (3.11) is attained at $\lambda = (\lambda_t)$ for some initial wealth $v > 0$. We are particularly interested in the completeness of the resulting (r, θ, P)-market.

(B6) The consumption process $\lambda_t = e^{g(t, X_t)}$, $t \in [0, 1]$, where the function $g = g(t, x)$ on $[0, 1] \times \mathbb{R}^d$ is analytic in t and twice weakly differentiable in x. Moreover, $t \mapsto \frac{\partial g}{\partial t}(t, \cdot)$, $t \mapsto \frac{\partial g}{\partial x^i}(t, \cdot)$, and $t \mapsto \frac{\partial^2 g}{\partial x^i \partial x^j}(t, \cdot)$ are analytic maps of $[0, 1]$ to $\mathbf{L}_\infty$ for all $i, j = 1, \dots, J$.

Theorem 3.4. *Suppose that* (2.3), *(A1)–(A2), (B3), and (B5)–(B6) hold. Then there exist a bounded process $r = (r_t)$ and a continuous process $P = (P_t^j)$ with the terminal value $P_1 = \Theta$ such that, in the (r, θ, P)-market, the set of martingale measures $\mathcal{Q}$ is a singleton and, for some initial wealth $v_0 > 0$, the consumption process $\lambda = (\lambda_t)$ solves* (3.11).

The interest rate process $r = (r_t)$ and the density process $Z = (Z_t)$ of $\mathbb{Q} \in \mathcal{Q}$ are uniquely determined from the decomposition

$$u_x(t, \lambda_t) = u_x(0, \lambda_0) Z_t e^{-\int_0^t r_s ds}, \quad t \in [0, 1]. \tag{3.12}$$

The price process $P = (P_t)$ is unique and given, in terms of $r = (r_t)$ and $\mathbb{Q}$, by (3.4), (3.7), *and* (3.3). *Finally, the initial wealth v_0 is unique and given by*

$$v_0 = \mathbb{E}^{\mathbb{Q}} \left[\int_0^1 e^{-\int_0^t r_u du} \lambda_t dt \right] < \infty. \tag{3.13}$$

Proof. The well-known results on optimal consumption in complete markets, see [9, Theorem 3.7.3], imply that for a utility function $u = u(t, x)$ as in (B1) and a complete (r, θ, P)-market with unique $\mathbb{Q} \in \mathcal{Q}$, a non-negative process $\lambda = (\lambda_t)$ solves (3.11) if and only if (3.12) holds. Moreover, the initial wealth of an optimal strategy yielding the consumption process $\lambda = (\lambda_t)$ is given by (3.13).

The function

$$w(t,x) \triangleq u_x(t, e^{g(t,x)}), \quad (t,x) \in [0,1] \times \mathbf{R}^d,$$

is analytic in t and twice weakly differentiable in x. Further, there is $N > 0$ such that the second derivatives $\frac{\partial^2 w}{\partial x^i \partial x^j}$ are bounded by $e^{N|x|}$. Although the second derivatives are not continuous, a version of Ito's formula from Krylov [14, Theorem 2.10.1] can still be applied to

$$Y_t \triangleq u_x(t, \lambda_t) = u_x(t, e^{g(t,X_t)}) = w(t, X_t), \quad t \in [0,1],$$

yielding

$$dY_t = Y_t(-\beta(t,X_t)dt + \gamma(t,X_t)dW_t). \tag{3.14}$$

The functions $\beta = \beta(t,x)$ and $\gamma^i = \gamma^i(t,x)$, $i = 1, \dots, d$, on $[0,1] \times \mathbb{R}^d$ are given by

$$\beta = q(t,e^g) + a(t,e^g) \left(\frac{\partial g}{\partial t} + \sum_{k=1}^d \frac{\partial g}{\partial x^k} b^k + \frac{1}{2} \sum_{k,l,m=1}^d \sigma^{km} \sigma^{lm} c^{kl} \right),$$

$$\gamma^i = -a(t,e^g) \sum_{k=1}^d \frac{\partial g}{\partial x^k} \sigma^{ki},$$

where we omitted the common argument (t,x) and

$$c^{kl} = (1 - p(t,e^g)) \frac{\partial g}{\partial x^k} \frac{\partial g}{\partial x^l} + \frac{\partial^2 g}{\partial x^k \partial x^l}.$$

The assumptions of the theorem imply that $\beta = \beta(t,x)$ and $\gamma^i = \gamma^i(t,x)$, $i = 1, \dots, d$, satisfy the conditions (A5) and (A6), respectively.

From (3.14) we deduce that a local martingale Z such that $Z_0 = 1$ and a predictable process $r = (r_t)$ are uniquely determined by (3.12) and are given by

$$Z_t = \exp\left(\int_0^t \gamma(s,X_s)dW_s - \frac{1}{2}\int_0^t |\gamma(s,X_s)|^2 ds \right),$$

$$r_t = \beta(t, X_t).$$

Since $\gamma = \gamma(t,x)$ is bounded on $[0,1] \times \mathbb{R}^d$, we obtain that Z is, in fact, a martingale and, hence, is a density of some $\mathbb{Q} \sim \mathbb{P}$. Given $r = (r_t)$ and $\mathbb{Q}$

we define $P = (P_t^j)$ and $S = (S_t^j)$ by (3.4) and (3.7), respectively. By construction, $\mathbb{Q} \in \mathcal{Q}(r, \theta, P)$. Observe now that the conditions of Theorem 2.5 hold trivially for these $\mathbb{Q}$ and S. Hence the (r, θ, P)-market is complete and $\mathbb{Q}$ is its only martingale measure.

Finally, from (B6) we deduce the existence of $N \geq 0$ such that

$$\lambda_t = e^{g(t,X_t)} \leq e^{N(1+|X_t|)},$$

which, in view of the boundedness of the functions β and γ^i and of the diffusion coefficients b^i and σ^{ij}, easily yields the finiteness of v_0 in (3.13). $\square$

We conclude with a criteria for dynamic completeness of Pareto equilibria in the case of intermediate consumption. Consider an economy populated by M investors who trade in the bank account and the stocks; both are defined *endogenously*. The stocks pay the continuous dividends θ and the terminal dividends Θ. The economic agents jointly consume with the rate $\lambda = (\lambda_t)$ and have the utility functions $u^m = u^m(t, x)$, $m = 1, \dots, M$.

We are interested in the validity of the assertions of Theorem 3.4 when the function $u = u(t, x)$ is given by

$$u(t,x) \triangleq \sup_{x^1+\dots+x^M=x} \sum_{m=1}^{M} w^m u^m(t, x^m), \quad (t,x) \in [0,1] \times (0,\infty), \tag{3.15}$$

for some $w \in (0,\infty)^M$. The delicacy of the situation is that the t-analyticity of u does not follow automatically from the t-analyticity of u^m, $m = 1, \dots, M$. We consider two special cases:

(B7) For every $m = 1, \dots, M$ the function $u^m = u^m(t, x)$ satisfies (B5) and is jointly analytic in (t, x).

(B8) For every $m = 1, \dots, M$ the function $u^m = u^m(t, x)$ is given by

$$u^m(t,x) = e^{-\nu(t)} U_m(x), \quad (t,x) \in [0,1] \times (0,\infty),$$

where $\nu = \nu(t)$ is an analytic function on $[0, 1]$ and the function $U_m = U_m(x)$ satisfies (B1) and has a bounded relative risk-prudence:

$$-N \leq -\frac{xU'''(x)}{U''(x)} \leq N, \quad x \in (0,\infty),$$

for some $N > 0$.

Theorem 3.5. *Assume* (2.3), *(A1)–(A2), (B3), and (B6). Suppose also that the utility functions* $u^m = u^m(t,x)$ *satisfy either (B7) or (B8). Fix* $w \in (0,\infty)^M$ *and define* $u = u(t,x)$ *by* (3.15). *Then the assertions of Theorem 3.4 hold.*

The result is an immediate corollary of Theorem 3.4 and the following

Lemma 3.6. *Assume the utility functions* $u^m = u^m(t,x)$ *satisfy either (B7) or (B8) and let* $w \in (0,\infty)^M$. *Then for* $u = u(t,x)$ *defined by* (3.15) *condition (B5) holds true.*

Proof. We shall focus on the case when (B7) holds. The proof under (B8) is analogous. Denote by a^m, p^m, and q^m the coefficients for u^m from (B5).

Condition (B5) for u^m implies that

$$\lim_{x\downarrow 0} u^m_x(t,x) = \infty, \quad \lim_{x\to\infty} u^m_x(t,x) = 0.$$

It follows that the upper bound in (3.15) is attained at unique $\widehat{x}(t,x) = (\widehat{x}^m(t,x))_{m=1,\dots,M}$ determined by

$$\sum_{m=1}^{M} \widehat{x}^m(t,x) = x, \tag{3.16}$$

$$w^m u^m_x(t,\widehat{x}^m(t,x)) = w^M u^M_x(t,\widehat{x}^M(t,x)), \quad m = 1,\dots,M-1. \tag{3.17}$$

On $[0,1]\times(0,\infty)\times(0,\infty)^M$ define the functions

$$h^m(t,x,y) = w^m u^m_x(t,y^m) - w^M u^M_x(t,y^M), \quad m = 1,\dots,M-1,$$

$$h^M(t,x,y) = \sum_{m=1}^{M} y^m - x.$$

Clearly,

$$h^m(t,x,\widehat{x}(t,x)) = 0, \quad m = 1,\dots,M,$$

and

$$\frac{\partial h^m}{\partial y^l}(t,x,\widehat{x}(t,x)) = w^m u^m_{xx}(t,\widehat{x}^m(t,x)) 1_{\{l=m\}}, \quad m,l = 1,\dots,M-1,$$

$$\frac{\partial h^m}{\partial y^M}(t,x,\widehat{x}(t,x)) = -w^M u^M_{xx}(t,\widehat{x}^M(t,x)) \quad m = 1,\dots,M-1,$$

$$\frac{\partial h^M}{\partial y^m}(t,x,\widehat{x}(t,x)) = 1, \quad m = 1,\dots,M.$$

As $u^m_{xx} < 0$ the Jacobian matrix of $h^1(t,x,\cdot),\dots,h^m(t,x,\cdot)$ at $\widehat{x}(t,x)$ has a full rank. Since the functions h^m are analytic in (t,x,y) the implicit function theorem yields that the functions $\widehat{x}^m$ are analytic in (t,x), see Krantz and Parks [12, Theorem 2.3.5]. Moreover, standard computations in the implicit function theorem show that

$$\frac{\partial \widehat{x}^l}{\partial x}(t,x) = \frac{\widehat{x}^l}{a^l(t,\widehat{x}^l)} / \left(\sum_{m=1}^M \frac{\widehat{x}^m}{a^m(t,\widehat{x}^m)} \right). \tag{3.18}$$

Since

$$u(t,x) = \sum_{m=1}^M w^m u^m(t,\widehat{x}^m(t,x)),$$

the function u is analytic in (t,x). Hence, to complete the proof it only remains to verify (3.9) for this function.

Accounting for (3.16) and (3.17) we obtain

$$\begin{aligned} u_t(t,x) &= \sum_{m=1}^M w^m u^m_t(t,\widehat{x}^m), \\ u_x(t,x) &= w^m u^m_x(t,\widehat{x}^m), \quad m = 1,\dots,M. \end{aligned}$$

By differentiating these equalities a necessary number of times with respect to x and accounting for (3.18) we arrive to the identities:

$$\begin{aligned} \frac{1}{a(t,x)} &= \sum_{m=1}^M \frac{1}{a^m(t,\widehat{x}^m)} \frac{\widehat{x}^m}{x}, \\ p(t,x) &= \sum_{m=1}^M p^m(t,\widehat{x}^m) \left(\frac{a(t,x)}{a^m(t,\widehat{x}^m)} \right)^2 \frac{\widehat{x}^m}{x}, \\ q(t,x) &= \sum_{m=1}^M q^m(t,\widehat{x}^m) \frac{a(t,x)}{a^m(t,\widehat{x}^m)} \frac{\widehat{x}^m}{x}, \end{aligned}$$

which readily imply (3.9). □

4 A time analytic solution of a parabolic equation

The proof of Theorem 2.3 will rely on the study of a parabolic equation in Theorem 4.4 below.

For reader's convenience, recall the definition of the classical Sobolev spaces $\mathbf{W}_p^m$ on $\mathbb{R}^d$ where $m \in \{0, 1, \dots\}$ and $p \geq 1$. When $m = 0$ we get the classical Lebesgue spaces $\mathbf{L}_p = \mathbf{L}_p(\mathbb{R}^d, dx)$ with the norm

$$\|f\|_{\mathbf{L}_p} \triangleq \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

When $m \in \{1, \dots\}$ the Sobolev space $\mathbf{W}_p^m$ consists of all m-times weakly differentiable functions f such that

$$\|f\|_{\mathbf{W}_p^m} \triangleq \|f\|_{\mathbf{L}_p} + \sum_{1 \leq |\alpha| \leq m} \|D^\alpha f\|_{\mathbf{L}_p} < \infty$$

and is a Banach space with such a norm. The summation is taken with respect to multi-indexes $\alpha = (\alpha_1, \dots, \alpha_d)$ of non-negative integers, $|\alpha| \triangleq \sum_{i=1}^d \alpha_i$ and

$$D^\alpha \triangleq \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

Recall also that a function $h = h(t): \ [0, 1] \to \mathbf{X}$ with values in a Banach space $\mathbf{X}$ is called Hölder continuous if there is $0 < \gamma < 1$ such that

$$\sup_{t \in [0,1]} \|h(t)\|_{\mathbf{X}} + \sup_{0 \leq s < t \leq 1} \frac{\|h(t) - h(s)\|_{\mathbf{X}}}{|t - s|^\gamma} < \infty.$$

For $t \in [0, 1]$ and $x \in \mathbb{R}^d$ consider an elliptic operator

$$A(t) \triangleq \sum_{i,j=1}^d a^{ij}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i(t, x) \frac{\partial}{\partial x^i} + c(t, x),$$

where a^{ij}, b^i, and c are measurable functions on $[0, 1] \times \mathbb{R}^d$ such that

(C1) $t \mapsto a^{ij}(t, \cdot)$ is an analytic map of $[0, 1]$ to $\mathbf{C}$, $t \mapsto b^i(t, \cdot)$ and $t \mapsto c(t, \cdot)$ are analytic maps of $[0, 1]$ to $\mathbf{L}_\infty$. The matrix a is symmetric: $a^{ij} = a^{ji}$, uniformly elliptic: there exists $N > 0$ such that

$$y a(t, x) y \geq \frac{1}{N^2} |y|^2, \quad (t, x) \in [0, 1] \times \mathbb{R}^d, \quad y \in \mathbb{R}^d,$$

and is uniformly continuous with respect to x: there exists a decreasing function $\omega = (\omega(\epsilon))_{\epsilon > 0}$ such that $\omega(\epsilon) \to 0$ as $\epsilon \downarrow 0$ and for all $t \in [0, 1]$ and $y, z \in \mathbb{R}^d$

$$|a^{ij}(t, y) - a^{ij}(t, z)| \leq \omega(|y - z|).$$

Let $g = g(x) : \mathbb{R}^d \to \mathbb{R}$ and $f = f(t,x) : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ be measurable functions such that for some $p > 1$

(C2) the function g belongs to $\mathbf{W}^1_p$ and $t \mapsto f(t,\cdot)$ is a Hölder continuous map from $[0,1]$ to $\mathbf{L}_p$ whose restriction to $(0,1]$ is analytic.

Theorem 4.1. *Let $p > 1$ and suppose the conditions (C1) and (C2) hold. Then there exists a unique measurable function $u = u(t,x)$ on $[0,1] \times \mathbb{R}^d$ such that*

1. *$t \mapsto u(t,\cdot)$ is a Hölder continuous map of $[0,1]$ to $\mathbf{L}_p$,*
2. *$t \mapsto u(t,\cdot)$ is a continuous map of $[0,1]$ to $\mathbf{W}^1_p$,*
3. *$t \mapsto u(t,\cdot)$ is an analytic map of $(0,1]$ to $\mathbf{W}^2_p$,*

and such that $u = u(t,x)$ solves the parabolic equation:

$$\frac{\partial u}{\partial t} = A(t)u + f, \quad t \in (0,1], \tag{4.1}$$

$$u(0,\cdot) = g. \tag{4.2}$$

The proof is essentially a compilation of references to known results. We first introduce some notations and state a few lemmas.

Let $\mathbf{X}$ and $\mathbf{D}$ be Banach spaces. By $\mathcal{L}(\mathbf{X},\mathbf{D})$ we denote the Banach space of bounded linear operators $T : \mathbf{X} \to \mathbf{D}$ endowed with the operator norm. A shorter notation $\mathcal{L}(\mathbf{X})$ is used for $\mathcal{L}(\mathbf{X},\mathbf{X})$. We shall write $\mathbf{D} \subset \mathbf{X}$ if $\mathbf{D}$ is *continuously embedded* into $\mathbf{X}$, that is, the elements of $\mathbf{D}$ form a subset of $\mathbf{X}$ and there is a constant $N > 0$ such that $\|x\|_{\mathbf{X}} \le N\|x\|_{\mathbf{D}}$, $x \in \mathbf{D}$. We shall write $\mathbf{D} = \mathbf{X}$ if $\mathbf{D} \subset \mathbf{X}$ and $\mathbf{X} \subset \mathbf{D}$.

Let $\mathbf{D} \subset \mathbf{X}$. A Banach space $\mathbf{E}$ is called an *interpolation space* between $\mathbf{D}$ and $\mathbf{X}$ if $\mathbf{D} \subset \mathbf{E} \subset \mathbf{X}$ and any linear operator $T \in \mathcal{L}(\mathbf{X})$ whose restriction to $\mathbf{D}$ belongs to $\mathcal{L}(\mathbf{D})$ also has its restriction to $\mathbf{E}$ in $\mathcal{L}(\mathbf{E})$; see Bergh and Löfström [3, Section 2.4].

The following lemma will be used in the proof of item 2 of the theorem.

Lemma 4.2. *Let $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{X}$ be Banach spaces such that $\mathbf{D} \subset \mathbf{X}$, $\mathbf{E}$ is an interpolation space between $\mathbf{D}$ and $\mathbf{X}$, and $\mathbf{D}$ is dense in $\mathbf{E}$. Let $(T_n)_{n\ge 1}$ be a sequence of linear operators in $\mathcal{L}(\mathbf{X})$ such that $\lim_{n\to\infty}\|T_n x\|_{\mathbf{X}} = 0$ for any $x \in \mathbf{X}$ and $\lim_{n\to\infty}\|T_n x\|_{\mathbf{D}} = 0$ for any $x \in \mathbf{D}$. Then $\lim_{n\to\infty}\|T_n x\|_{\mathbf{E}} = 0$ for any $x \in \mathbf{E}$.*

Proof. The uniform boundedness theorem implies that the sequence $(T_n)_{n\geq 1}$ is bounded both in $\mathcal{L}(\mathbf{X})$ and $\mathcal{L}(\mathbf{D})$. Due to the Banach property, $\mathbf{E}$ is a uniform interpolation space between $\mathbf{D}$ and $\mathbf{X}$, that is, there is a constant $M > 0$ such that

$$\|T\|_{\mathcal{L}(\mathbf{E})} \leq M \max(\|T\|_{\mathcal{L}(\mathbf{C})}, \|T\|_{\mathcal{L}(\mathbf{D})}) \text{ for any } T \in \mathcal{L}(\mathbf{X}) \cap \mathcal{L}(\mathbf{D});$$

see Theorem 2.4.2 in [3]. Hence, $(T_n)_{n\geq 1}$ is also bounded in $\mathcal{L}(\mathbf{E})$. The density of $\mathbf{D}$ in $\mathbf{E}$ then yields the result. $\square$

Let A be an (unbounded) closed linear operator on $\mathbf{X}$. We denote by $\mathbf{D}(A)$ the domain of A and assume that it is endowed with the graph norm of A:

$$\|x\|_{\mathbf{D}(A)} \triangleq \|Ax\|_{\mathbf{X}} + \|x\|_{\mathbf{X}}.$$

Then $\mathbf{D}(A)$ is a Banach space. Recall that the resolvent set $\rho(A)$ of A is defined as the set of complex numbers λ for which the operator $\lambda I - A : \mathbf{D}(A) \to \mathbf{X}$, where I is the identity operator, is invertible; the inverse operator is called the resolvent and is denoted by $R(\lambda, A)$. The bounded inverse theorem implies that $R(\lambda, A) \in \mathcal{L}(\mathbf{X}, \mathbf{D}(A))$ and, in particular, $R(\lambda, A) \in \mathcal{L}(\mathbf{X})$.

The operator A is called *sectorial* if there are constants $M > 0$, $r \in \mathbb{R}$, and $\theta \in \left(0, \frac{\pi}{2}\right)$ such that the sector

$$S_{r,\theta} \triangleq \{\lambda \in \mathbb{C} : \ \lambda \neq r \text{ and } |\arg(\lambda - r)| \leq \pi - \theta\} \tag{4.3}$$

of the complex plane $\mathbb{C}$ is a subset of $\rho(A)$ and

$$\|R(\lambda, A)\|_{\mathcal{L}(\mathbf{X})} \leq \frac{M}{1 + |\lambda|}, \quad \lambda \in S_{r,\theta}. \tag{4.4}$$

The set of such sectorial operators will be denoted by $\mathcal{S}(M, r, \theta)$. Sectorial operators are important, because when their domains are dense in $\mathbf{X}$ they coincide with generators of analytic semi-groups, see Pazy [17, Section 2.5].

The following lemma will enable us to use the results from Kato and Tanabe [10] to verify item 3 of the theorem.

Lemma 4.3. *Let $\mathbf{X}$ and $\mathbf{D}$ be Banach spaces such that $\mathbf{D} \subset \mathbf{X}$ and let $A = (A(t))_{t\in[0,1]}$ be closed linear operators on $\mathbf{X}$ such that $\mathbf{D}(A(t)) = \mathbf{D}$ for all $t \in [0, 1]$. Suppose $A : [0, 1] \to \mathcal{L}(\mathbf{D}, \mathbf{X})$ is an analytic map, and there are $M > 0$, $r < 0$, and $\theta \in \left(0, \frac{\pi}{2}\right)$ such that $A(t) \in \mathcal{S}(M, r, \theta)$ for all $t \in [0, 1]$.*

Then there exist a convex open set U in $\mathbb{C}$ containing $[0, 1]$ and an analytic extension of A to U such that $A(z) \in \mathcal{S}(2M, r, \theta)$ for all $z \in U$ and the function $A^{-1} : [0, 1] \to \mathcal{L}(\mathbf{X}, \mathbf{D})$ is analytic.

Proof. If $A \in \mathcal{S}(M, r, \theta)$, then for $\lambda \in S_{r,\theta}$

$$\|R(\lambda, A)\|_{\mathcal{L}(\mathbf{X},\mathbf{D}(A))} = \|R(\lambda, A)\|_{\mathcal{L}(\mathbf{X})} + \|AR(\lambda, A)\|_{\mathcal{L}(\mathbf{X})} \leq M + 1,$$

where we used (4.4) and the identity $AR(\lambda, A) = \lambda R(\lambda, A) - I$. As $A : [0, 1] \to \mathcal{L}(\mathbf{D}, \mathbf{X})$ is a continuous function, the Banach spaces $\mathbf{D}$ and $\mathbf{D}(A(t))$, $t \in [0, 1]$, are uniformly equivalent, that is, there is $L > 0$ such that $\|x\|_{\mathbf{D}(A(t))} \leq L\|x\|_{\mathbf{D}}$ and $\|x\|_{\mathbf{D}} \leq L\|x\|_{\mathbf{D}(A(t))}$ for every $t \in [0, 1]$ and every $x \in \mathbf{D}$. It follows that one can find $N > 0$ such that

$$\|R(\lambda, A(t))\|_{\mathcal{L}(\mathbf{X},\mathbf{D})} \leq N, \quad \lambda \in S_{r,\theta}, t \in [0, 1]. \tag{4.5}$$

Since $r < 0$, the operator $A(t)$ is invertible for every $t \in [0, 1]$. As $A : [0, 1] \to \mathcal{L}(\mathbf{D}, \mathbf{X})$ is analytic, the inverse function $B = A^{-1} : [0, 1] \to \mathcal{L}(\mathbf{X}, \mathbf{D})$ is well-defined and analytic. Clearly, there is an open convex set U in $\mathbb{C}$ containing $[0, 1]$ on which both A and B can be analytically extended. Then $B = A^{-1}$ on U, as AB is an analytic function on U with values in $\mathcal{L}(\mathbf{X})$ which on $[0, 1]$ equals the identity operator. Of course, we can choose U so that for any $z \in U$ there is $t \in [0, 1]$ such that

$$\|A(z) - A(t)\|_{\mathcal{L}(\mathbf{D},\mathbf{X})} \leq \frac{1}{2N}, \tag{4.6}$$

where the constant $N > 0$ is taken from (4.5).

Fix $\lambda \in S_{r,\theta}$ and take $t \in [0, 1]$ and $z \in U$ satisfying (4.6). By (4.5) and (4.6)

$$\|(A(z) - A(t))R(t, A(t))\|_{\mathcal{L}(\mathbf{X})} \leq \frac{1}{2}.$$

Hence the operator $I - (A(z) - A(t))R(t, A(t))$ in $\mathcal{L}(\mathbf{X})$ is invertible and its inverse has norm less than 2. Since

$$\lambda I - A(z) = (I - (A(z) - A(t))R(t, A(t)))(\lambda I - A(t)),$$

we obtain that the resolvent $R(\lambda, A(z))$ is well-defined and

$$\|R(\lambda, A(z))\|_{\mathcal{L}(\mathbf{X})} \leq \frac{2M}{1 + |\lambda|}.$$

This completes the proof. $\square$

Proof of Theorem 4.1. It is well-known that under (C1) for every $t \in [0, 1]$ the operator $A(t)$ is closed in $\mathbf{L}_p$ and has $\mathbf{W}_p^2$ as its domain:

$$\mathbf{D}(A(t)) = \mathbf{W}_p^2. \tag{4.7}$$

Moreover, the operators $(A(t))_{t\in[0,1]}$ are *sectorial* with the same constants $M > 0$, $r \in \mathbb{R}$, and $\theta \in \left(0, \frac{\pi}{2}\right)$:

$$A(t) \in \mathcal{S}(M, r, \theta), \quad t \in [0, 1]. \tag{4.8}$$

These results can found, for example, in Krylov [15], see Section 13.4 and Exercise 13.5.1.

It will be convenient for us to assume that that the sector $S_{r,\theta}$ defined in (4.3) contains 0 or, equivalently, that $r < 0$. This does not restrict any generality as for $s \in \mathbb{R}$ the substitution $u(t, x) \to e^{st}u(t, x)$ in (4.1) corresponds to the shift $A(t) \to A(t)+s$ in the operators $A(t)$. Among other benefits, this assumption implies the existence of inverses and fractional powers for the operators $-A(t)$; see Section 2.6 in [17] on fractional powers of sectorial operators.

From (C1) we clearly deduce the existence of $M > 0$ such that for any $v \in \mathbf{W}_p^2$

$$\|(A(t) - A(s))v\|_{\mathbf{L}_p} \le M|t - s|\|v\|_{\mathbf{W}_p^2}, \quad s, t \in [0, 1]. \tag{4.9}$$

Conditions (4.7), (4.8), and (4.9) for the operators $A = A(t)$ and condition (C2) for f and g imply the existence and uniqueness of the classical solution $u = u(t, x)$ to the initial value problem (4.1)–(4.2) in $\mathbf{L}_p$; see Theorem 7.1 in Section 5 of [17]. Recall that $u = u(t, x)$ is the *classical solution* to (4.1) and (4.2) if $u(t, \cdot) \in \mathbf{W}_p^2$ for $t \in (0, 1]$, the map $t \mapsto u(t, \cdot)$ of $[0, 1]$ to $\mathbf{L}_p$ is continuous, the restriction of this map to $(0, 1]$ is continuously differentiable, and the equations (4.1) and (4.2) hold.

To verify item 1 we use Theorem 3.10 in Yagi [20] dealing with maximal regularity properties of solutions to evolution equations. This theorem implies the existence of constants $\delta > 0$ and $M > 0$ such that

$$\|\frac{\partial u}{\partial t}(t, \cdot)\|_{\mathbf{L}_p} \le Mt^{\delta-1}, \quad t \in (0, 1], \tag{4.10}$$

provided that the operators $A = A(t)$ satisfy (4.7)–(4.9), the function f is Hölder continuous as in (C2), and for some $0 < \gamma < 1$

$$g \in \mathbf{D}((-A(0))^\gamma), \tag{4.11}$$

where $\mathbf{D}((-A(0))^\gamma)$ is the domain of the fractional power γ of the operator $-A(0)$ acting in $\mathbf{L}_p$. The inequality (4.10) clearly implies the Hölder continuity of $u(t, \cdot) :\ [0, 1] \to \mathbf{L}_p$ and, hence, to complete the proof of item 1 we only need to verify (4.11).

Since $g \in \mathbf{W}_p^1$, we obtain (4.11) if

$$\mathbf{W}_p^1 \subset \mathbf{D}((-A(0))^\gamma), \quad \gamma \in (0, \frac{1}{2}).$$

This embedding is an immediate corollary of the classical characterization of Sobolev spaces $\mathbf{W}_p^m$ as the domains of $(1-\Delta)^{m/2}$ in $\mathbf{L}_p$:

$$\mathbf{W}_p^m = \mathbf{D}((1-\Delta)^{m/2}), \quad m \in \{0, 1, \ldots\},$$

where $\Delta \triangleq \sum_i \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and the fact that for $0 < \alpha < \beta < 1$ and sectorial operators A and B such that $\mathbf{D}(B) \subset \mathbf{D}(A)$ and such that the fractional powers $(-A)^\alpha$ and $(-B)^\beta$ are well-defined we have $\mathbf{D}((-B)^\beta) \subset \mathbf{D}((-A)^\alpha)$. These results can be found, respectively, in [15, Theorem 13.3.12] and [20, Theorem 2.25]. This finishes the proof of item 1.

Another consequence of the maximal regularity properties of u given in [20, Theorem 3.10] is that the map $u(t,\cdot) : \ [0,1] \to \mathbf{W}_p^2$ is continuous if $g \in \mathbf{W}_p^2 = \mathbf{D}(A(0))$. We shall apply this result shortly to prove item 2.

For $t \in [0,1]$ define a linear operator $T(t)$ on $\mathbf{L}_p$ such that for $h \in \mathbf{L}_p$ the function $v = v(t,x)$ given by $v(t,\cdot) = T(t)h$ is the unique classical solution in $\mathbf{L}_p$ of the homogeneous problem:

$$\frac{\partial v}{\partial t} = A(t)v, \quad v(0,\cdot) = h. \tag{4.12}$$

Actually, $T(t) = U(t,0)$, where $U = (U(t,s))_{0 \le s \le t \le 1}$ is the evolution system for $A = A(t)$; see Pazy [17, Chapter 5]. However, we shall not use this relation. Of course, the properties established above for $u = u(t,x)$ will also hold for the solution $v = v(t,x)$ to (4.12). It follows that for any $h \in \mathbf{L}_p$ the map $t \mapsto T(t)h$ is well-defined and continuous in $\mathbf{L}_p$ and if $h \in \mathbf{W}_p^2$ then the same map is also continuous in $\mathbf{W}_p^2$. Recall now that $\mathbf{W}_p^1$ is an interpolation space between $\mathbf{L}^p$ and $\mathbf{W}_p^2$, more precisely, a midpoint in complex interpolation, see, for example, Bergh and Löfström [3, Theorem 6.4.5]. Since $\mathbf{W}_p^2$ is dense in $\mathbf{W}_p^1$, Lemma 4.2 yields the continuity of the map $t \mapsto T(t)h$ in $\mathbf{W}_p^1$.

Observe now that $u = u(t,x)$ can be decomposed as

$$u(t,\cdot) = T(t)g + w(t,\cdot),$$

where $w(t,\cdot)$ is the unique classical solution in $\mathbf{L}_p$ of the inhomogeneous problem:

$$\frac{\partial w}{\partial t} = A(t)w + f, \quad w(0,\cdot) = 0.$$

Since w coincides with u in the special case $g = 0$, the map $t \mapsto w(t,\cdot)$ is continuous in $\mathbf{W}_p^2$ and, hence, also continuous in $\mathbf{W}_p^1$. This completes the proof of item 2.

Finally, let us prove item 3. To simplify notations suppose that the map $f = f(t,\cdot) : [0,1] \to \mathbf{L}_p$ is actually analytic; otherwise, we repeat the same arguments on $[\epsilon, 1]$ for $0 < \epsilon < 1$. The condition (C1) implies the analyticity of the function $A = A(t) : [0,1] \to \mathcal{L}(\mathbf{W}_p^2, \mathbf{L}_p)$. Let U be an open convex set in $\mathbb{C}$ containing $[0,1]$ on which there is an analytic extension of A satisfying the assertions of Lemma 4.3. We choose U so that $f = f(t,\cdot) : [0,1] \to \mathbf{L}_p$ can also be analytically extended on U. Theorem 2 in Kato and Tanabe [10] now implies the analyticity of the map $t \mapsto u(t,\cdot)$ in $\mathbf{L}_p$. However, as

$$u(t,\cdot) = \big(A(t)\big)^{-1}\Big(\frac{\partial u}{\partial t} - f(t,\cdot)\Big),$$

and since, by Lemma 4.3, the $\mathcal{L}(\mathbf{L}_p, \mathbf{W}_p^2)$-valued function $\big(A(t)\big)^{-1}$ on $[0,1]$ is analytic, the map $t \mapsto u(t,\cdot)$ is also analytic in $\mathbf{W}_p^2$.

The proof is completed. □

In the proof of our main Theorem 2.3 we actually need Theorem 4.4 below, which is a corollary of Theorem 4.1. Instead of (C2) we assume that the measurable functions $g = g(x) : \mathbb{R}^d \to \mathbb{R}$ and $f = f(t,x) : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ have the following properties:

(C3) There is a constant $N \geq 0$ such that $e^{-N|\cdot|}\frac{\partial g}{\partial x^i}(\cdot) \in \mathbf{L}_\infty$ and for any $p \geq 1$ we have $t \mapsto e^{-N|\cdot|} f(t,\cdot)$ is a Hölder continuous map from $[0,1]$ to $\mathbf{L}_p$ whose restriction to $(0,1]$ is analytic.

Fix a function $\phi = \phi(x)$ such that

$$\phi \in \mathbf{C}^\infty(\mathbb{R}^d) \text{ and } \phi(x) = |x| \text{ when } |x| \geq 1. \tag{4.13}$$

Theorem 4.4. *Suppose the conditions (C1) and (C3) hold. Let $\phi = \phi(x)$ be as in (4.13). Then there exists a unique continuous function $u = u(t,x)$ on $[0,1] \times \mathbb{R}^d$ and a constant $N \geq 0$ such that for any $p \geq 1$*

1. *$t \mapsto e^{-N\phi}u(t,\cdot)$ is a Hölder continuous map of $[0,1]$ to $\mathbf{L}_p$,*
2. *$t \mapsto e^{-N\phi}u(t,\cdot)$ is a continuous map of $[0,1]$ to $\mathbf{W}_p^1$,*
3. *$t \mapsto e^{-N\phi}u(t,\cdot)$ is an analytic map of $(0,1]$ to $\mathbf{W}_p^2$,*

and such that $u = u(t,x)$ solves the Cauchy problem (4.1) and (4.2).

Proof of Theorem 4.4. From (C3) we deduce the existence of $M > 0$ such that

$$|\frac{\partial g}{\partial x^i}|(x) \le M e^{M|x|}, \quad x \in \mathbb{R}^d,$$

and, therefore, such that

$$|g(x) - g(0)| \le M|x|e^{M|x|}, \quad x \in \mathbb{R}^d.$$

Hence, for any $N > M$ and any function $\phi = \phi(x)$ as in (4.13)

$$e^{-N\phi} g \in \mathbf{W}^1_p, \quad p \ge 1.$$

Hereafter, we choose the constant $N \ge 0$ so that in addition to (C3) it also has the property above.

Define the functions $\widetilde{b}^i = \widetilde{b}^i(t,x)$ and $\widetilde{c} = \widetilde{c}(t,x)$ so that for any $t \in [0,1]$ and any $v \in \mathbf{C}^\infty$

$$\widetilde{A}(t)\big(e^{-N\phi}v\big) = e^{-N\phi} A(t)v,$$

where

$$\widetilde{A}(t) \triangleq \sum_{i,j=1}^d a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d \widetilde{b}^i(t,x)\frac{\partial}{\partial x^i} + \widetilde{c}(t,x).$$

It is easy to see that $\widetilde{b}^i$ and $\widetilde{c}$ satisfy the same conditions as b^i and c in (C1). From Theorem 4.1 we deduce the existence of a measurable function $\widetilde{u} = \widetilde{u}(t,x)$ which for any $p > 1$ complies with the items 1–3 of this theorem and solves the Cauchy problem:

$$\frac{\partial \widetilde{u}}{\partial t} = \widetilde{A}(t)\widetilde{u} + e^{-N\phi} f, \quad \widetilde{u}(0,\cdot) = e^{-N\phi} g.$$

For $p > d$, by the classical Sobolev's embedding, the continuity of the map $t \mapsto \widetilde{u}(t,\cdot)$ in $\mathbf{W}^1_p$ implies its continuity in $\mathbf{C}$. In particular, we obtain that the function $\widetilde{u} = \widetilde{u}(t,x)$ is continuous on $[0,1] \times \mathbb{R}^d$.

To conclude the proof it only remains to observe that $u = u(t,x)$ complies with the assertions of the theorem for $p > 1$ if and only if $\widetilde{u} \triangleq e^{-N\phi}u$ has the properties just established. The case $p = 1$ follows trivially from the case $p > 1$ by taking N slightly larger. $\square$

5 Proof of Theorem 2.3

Throughout this section we assume the conditions and the notations of Theorem 2.3. We fix a function ϕ satisfying (4.13). We also denote by $L(t)$ the infinitesimal generator of X at $t \in [0,1]$:

$$L(t) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x)\frac{\partial}{\partial x^i},$$

where $a \triangleq \sigma\sigma^*$ is the covariation matrix of X. The proof is divided into several lemmas.

Lemma 5.1. *There exist unique continuous functions $u = u(t,x)$ and $v^j = v^j(t,x)$, $j = 1,\dots,J$, on $[0,1]\times\mathbb{R}^d$ and a constant $N \geq 0$ such that*

1. *For any $p \geq 1$ the maps $t \mapsto e^{-N\phi}u(t,\cdot)$ and $t \mapsto e^{-N\phi}v^j(t,\cdot)$ are*
   - *(a) Hölder continuous maps of $[0,1]$ to $\mathbf{L}_p$;*
   - *(b) continuous maps of $[0,1]$ to $\mathbf{W}^1_p$.*
   - *(c) analytic maps of $[0,1)$ to $\mathbf{W}^2_p$.*
2. *The function $u = u(t,x)$ solves the Cauchy problem:*

$$\frac{\partial u}{\partial t} + (L(t)+\beta)u = 0, \quad t \in [0,1), \tag{5.1}$$

$$u(1,\cdot) = G, \tag{5.2}$$

3. *The function $v^j = v^j(t,x)$ solves the Cauchy problem:*

$$\frac{\partial v^j}{\partial t} + (L(t)+\alpha^j+\beta)v^j + uf^j = 0, \quad t \in [0,1), \tag{5.3}$$

$$v^j(1,\cdot) = F^j G. \tag{5.4}$$

Proof. Observe first that (A2) on $\sigma = \sigma(t,x)$ implies (C1) on the covariation matrix $a = a(t,x)$. The assertions for $u = u(t,x)$ and, then, for $v^j = v^j(t,x)$, $j = 1,\dots,J$, follow now directly from Theorem 4.4, where we need to make the time change $t \to 1-t$. □

Hereafter, we denote by $u = u(t,x)$ and $v^j = v^j(t,x)$, $j = 1,\dots,J$, the functions defined in Lemma 5.1.

Lemma 5.2. *The matrix-function $w = w(t,x)$, with d rows and J columns, given by*

$$w^{ij}(t,x) \triangleq \left(u\frac{\partial v^j}{\partial x^i} - v^j\frac{\partial u}{\partial x^i}\right)(t,x), \quad i = 1,\dots,d,\ j = 1,\dots,J, \tag{5.5}$$

has rank d almost surely with respect to the Lebesgue measure on $[0,1]\times\mathbb{R}^d$.

Proof. Denote

$$g(t,x) \triangleq \det(ww^*)(t,x), \quad (t,x) \in [0,1] \times \mathbb{R}^d,$$

the determinant of the product of w on its transpose, and observe that the result holds if and only if the set

$$A \triangleq \{(t,x) \in [0,1] \times \mathbb{R}^d : \ g(t,x) = 0\}$$

has the Lebesgue measure zero on $[0,1] \times \mathbb{R}^d$ or, equivalently, the set

$$B \triangleq \{x \in \mathbb{R}^d : \ \int_0^1 1_A(t,x)dt > 0\}$$

has the Lebesgue measure zero on $\mathbb{R}^d$.

From Lemma 5.1 we deduce that the existence of a constant $N \geq 0$ such that for any $p \geq 1$ the map $t \mapsto e^{-N\phi}g(t,\cdot)$ from $[0,1)$ to $\mathbf{W}_p^1$ is analytic and the same map of $[0,1]$ to $\mathbf{L}_p$ is continuous. Taking $p \geq d$, we deduce from the classical Sobolev embedding of $\mathbf{W}_p^1$ into $\mathbf{C}$ that this map is also analytic from $[0,1)$ to $\mathbf{C}$. It follows that if $x \in B$ then $g(t,x) = 0$ for all $t \in [0,1)$ and, in particular,

$$\lim_{t \uparrow 1} g(t,x) = 0, \quad x \in B.$$

Since

$$\|g(t,\cdot) - g(1,\cdot)\|_{\mathbf{L}^p} = \|g(t,\cdot) - \det(ww^*)(1,\cdot)\|_{\mathbf{L}^p} \to 0, \quad t \uparrow 1,$$

the Lebesgue measure of B is zero if the matrix-function $w(1,\cdot)$ has rank d almost surely. This follows from the expression for $w(1,\cdot)$:

$$w^{ij}(1,\cdot) = G\frac{\partial(F^jG)}{\partial x^i} - F^jG\frac{\partial G}{\partial x^i} = G^2\frac{\partial F^j}{\partial x^i},$$

and the assumptions (A3) and (A4) on $F = (F^j)$ and G. $\square$

Recall the notations ψ^j, $j = 1, \ldots, J$, and ξ for the random variables defined in (2.5) and (2.6).

Lemma 5.3. *The processes Y and R^j, $j = 1, \ldots, J$, on $[0,1]$ defined by*

$$Y_t \triangleq e^{\int_0^t \beta(s,X_s)ds}u(t,X_t),$$
$$R_t^j \triangleq e^{\int_0^t (\alpha^j+\beta)(s,X_s)ds}v^j(t,X_t) + Y_t\int_0^t e^{\int_0^s \alpha^j(r,X_r)dr}f^j(s,X_s)ds,$$

are continuous uniformly integrable martingales with the terminal values $Y_1 = \xi$ *and* $R_1^j = \xi\psi^j$*. Moreover, for* $t \in [0,1]$*,*

$$Y_t = Y_0 + \sum_{i,k=1}^{d} \int_0^t e^{\int_0^s \beta(r,X_r)dr} \left(\frac{\partial u}{\partial x^i} \sigma^{ik} \right)(s, X_s) dW_s^k, \tag{5.6}$$

$$\begin{aligned} R_t^j = R_0^j &+ \sum_{i,k=1}^{d} \int_0^t e^{\int_0^s (\alpha^j + \beta)(r,X_r)dr} \left(\frac{\partial v^j}{\partial x^i} \sigma^{ik} \right)(s, X_s) dW_s^k \\ &+ \int_0^t \left(\int_0^s e^{\int_0^r \alpha^j(q,X_q)dq} f^j(r, X_r) dr \right) dY_s. \end{aligned} \tag{5.7}$$

Proof. From the continuity of u and v^j on $[0,1] \times \mathbb{R}^d$ we obtain that Y and R^j are continuous processes on $[0,1]$. The expressions (5.2) and (5.4) for $u(1,\cdot)$ and $v^j(1,\cdot)$ imply that $Y_1 = \xi$ and $R_1^j = \xi\psi^j$.

Let $N \geq 0$ be the constant in Lemma 5.1. Choosing $p = d+1$ in Lemma 5.1 we deduce that the maps $t \mapsto e^{-N\phi}u(t,\cdot)$ and $t \mapsto e^{-N\phi}v^j(t,\cdot)$ of $[0,1)$ to $\mathbf{W}_{d+1}^2$ are continuously differentiable. This enables us to use a variant of the Ito formula due to Krylov, see [14, Section 2.10, Theorem 1]. Direct computations, where we account for (5.1) and (5.3), then yield the integral representations (5.6) and (5.7).

In particular, we have shown that Y and R^j are continuous local martingales. It only remains to verify their uniform integrability. By Sobolev's embeddings, since $t \mapsto e^{-N\phi}u(t,\cdot)$ and $t \mapsto e^{-N\phi}v^j(t,\cdot)$ are continuous maps of $[0,1]$ to $\mathbf{W}_{d+1}^1$, they are also continuous maps of $[0,1]$ to $\mathbf{C}$. This implies the existence of $c > 0$ such that

$$\sup_{t\in[0,1]} (|Y_t| + |R_t^j|) \leq e^{c(1+\sup_{t\in[0,1]}|X_t|)}.$$

The result now follows from the well-known fact that, for bounded b^i and σ^{ij}, the random variable $\sup_{t\in[0,1]}|X_t|$ has all exponential moments. $\square$

Proof of Theorem 2.3. Let Y and R be the processes defined in Lemma 5.3. This lemma implies, in particular, that

$$\mathbb{E}[|\xi| + \sum_{j=1}^{J} |\xi\psi^j|] < \infty,$$

and, hence, the probability measure $\mathbb{Q}$ and the $\mathbb{Q}$-martingale $S = (S^j)$ are well-defined. Since $\xi > 0$, the measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and Y is a strictly positive martingale. Observe that

$$S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t] = \frac{\mathbb{E}[\xi\psi|\mathcal{F}_t]}{\mathbb{E}[\xi|\mathcal{F}_t]} = \frac{R_t}{Y_t}, \quad t \in [0,1].$$

From (5.6) and (5.7) we deduce, after some computations, that

$$dS^j_t = d\frac{R^j_t}{Y_t} = e^{\int_0^t \alpha^j(s,X_s)ds} \frac{1}{u^2(t,X_t)} \sum_{i,k=1}^d (w^{ij}\sigma^{ik})(t,X_t)dW^{\mathbb{Q},k}_t, \tag{5.8}$$

where the matrix-function $w = w(t,x)$ is defined in (5.5) and

$$W^{\mathbb{Q},k}_t \triangleq W^k_t - \sum_{l=1}^d \int_0^t \left(\frac{1}{u}\frac{\partial u}{\partial x^l}\sigma^{lk}\right)(t,X_t)dt, \quad k = 1,\dots,d, \ t \in [0,1].$$

By Girsanov's theorem, $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$. Note that the division on $u(t,X_t)$ is safe as the process $u(t,X_t) = Y_t e^{-\int_0^t \beta(s,X_s)ds}$, $t \in [0,1]$, is strictly positive.

As we have already observed in Remark 2.2, any $\mathbb{P}$-local martingale is a stochastic integral with respect to W. This readily implies that any $\mathbb{Q}$-local martingale M is a stochastic integral with respect to $W^{\mathbb{Q}}$. Indeed, since $L \triangleq YM$ is a local martingale under $\mathbb{P}$, there is a predictable process ζ with values in $\mathbb{R}^d$ such that

$$L_t = L_0 + \int_0^t \zeta_u dW_u \triangleq L_0 + \sum_{i=1}^d \int_0^t \zeta^i_u dW^i_u$$

and then

$$dM_t = d\frac{L_t}{Y_t} = \frac{1}{Y_t}\sum_{i=1}^d \left(\zeta^i_t - L_t \sum_{k=1}^d \left(\frac{1}{u}\frac{\partial u}{\partial x^k}\sigma^{ki}\right)(t,X_t)\right) dW^{\mathbb{Q},i}_t.$$

In view of (5.8), to conclude the proof we only have to show that the matrix-process $((w^*\sigma)(t,X_t))_{t\in[0,1]}$ has rank d on $\Omega \times [0,1]$ almost surely under the product measure $dt \times d\mathbb{P}$. Observe first that by (2.1) and Lemma 5.2 the matrix-function $w^*\sigma = (w^*\sigma)(t,x)$ has rank d almost surely under the Lebesgue measure on $[0,1]\times\mathbb{R}^d$. The result now follows from the well-known fact that under (A1) and (A2) the distribution of X_t has a density under the Lebesgue measure on $\mathbb{R}^d$, see [19, Theorem 9.1.9]. □

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