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NECESSARY AND SUFFICIENT CONDITIONS IN THE PROBLEM OF OPTIMAL INVESTMENT IN INCOMPLETE MARKETS

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Following *Ann. Appl. Probab.* **9** (1999) 904–950 we continue the study of the problem of expected utility maximization in incomplete markets. Our goal is to find *minimal* conditions on a model and a utility function for the validity of several key assertions of the theory to hold true. In the previous paper we proved that a minimal condition on the utility function *alone*, that is, a minimal *market independent* condition, is that the asymptotic elasticity of the utility function is strictly less than 1. In this paper we show that a *necessary and sufficient* condition on *both*, the utility function and the model, is that the value function of the dual problem is finite.

1. Introduction and main results. We study the same financial framework as in [10] and refer to this paper for more details and references. We consider a model of a security market which consists of $d+1$ assets, one bond and d stocks. We work in discounted terms, that is, we suppose that the price of the bond is constant, and denote by $S=(S^i)_{1\le i\le d}$ the price process of the d stocks. The process S is assumed to be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. Here T is a finite time horizon. To simplify notation we assume that $\mathcal{F}=\mathcal{F}_T$.

A (self-financing) portfolio Π is defined as a pair (x, H), where the constant x is the initial value of the portfolio, and $H=(H^i)_{1\le i\le d}$ is a predictable S-integrable process, where H^i_t specifies how many units of asset i are held in the portfolio at time t. The value process $X=(X_t)_{0\le t\le T}$ of such a portfolio Π is given by

$$X_t = X_0 + \int_0^t H_u\, dS_u, \qquad 0\le t\le T. \tag{1}$$

We denote by $\mathcal{X}(x)$ the family of wealth processes with nonnegative capital at any instant, that is, $X_t\ge 0$ for all $t\in[0,T]$, and with initial value equal to x. In other words

$$\mathcal{X}(x) = \{X\ge 0 : X \text{ is defined by (1) with } X_0 = x\}.$$

We shall use the shorter notation $\mathcal{X}$ for $\mathcal{X}(1)$. Clearly,

$$\mathcal{X}(x) = x\mathcal{X} = \{xX : X\in\mathcal{X}\} \qquad \text{for } x\ge 0.$$

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A probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an *equivalent local martingale measure* if any $X \in \mathcal{X}$ is a local martingale under $\mathbb{Q}$. The family of equivalent local martingale measures will be denoted by $\mathcal{M}$. We assume throughout that

$$\mathcal{M} \neq \varnothing. \tag{2}$$

This condition is intimately related to the absence of arbitrage opportunities on the security market. See [4, 5] for precise statements and references.

We also consider an economic agent in our model, whose preferences are modeled by a utility function $U:(0,\infty) \to \mathbf{R}$ for wealth at maturity time T. Hereafter we will assume that the function U is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions

$$\begin{aligned} U'(0) &= \lim_{x\to 0} U'(x) = \infty, \\ U'(\infty) &= \lim_{x\to\infty} U'(x) = 0. \end{aligned} \tag{3}$$

For a given initial capital $x > 0$, the goal of the agent is *to maximize the expected value of terminal utility*. The value function of this problem is denoted by

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)]. \tag{4}$$

Intuitively speaking, the value function u plays the role of the utility function of the investor at time 0, if she subsequently invests in an optimal way. A well-known tool in studying the optimization problem (4) is the use of duality relationships in the spaces of convex functions and semimartingales; see, for example, [1–3, 6–11, 13].

The conjugate function V to the utility function U is defined as

$$V(y) = \sup_{x>0}[U(x) - xy], \qquad y > 0. \tag{5}$$

It is well known (see, e.g., [12]) that if U satisfies the hypotheses stated above, then V is a continuously differentiable, decreasing, strictly convex function satisfying $V'(0) = -\infty$ and $V'(\infty) = 0$, $V(0) = U(\infty)$, $V(\infty) = U(0)$, and the following relation holds true:

$$U(x) = \inf_{y>0}[V(y) + xy], \qquad x > 0.$$

In addition the derivative of U is the inverse function of the negative of the derivative of V, that is,

$$U'(x) = y \quad \Longleftrightarrow \quad x = -V'(y).$$

Further, we define the family $\mathcal{Y}$ of nonnegative semimartingales, which is dual to $\mathcal{X}$ in the following sense:

$$\mathcal{Y} = \{Y \geq 0 : Y_0 = 1 \text{ and } XY \text{ is a supermartingale for all } X \in \mathcal{X}\}.$$

Note that, as $1 \in \mathcal{X}$, any $Y \in \mathcal{Y}$ is a supermartingale. Note also that the set $\mathcal{Y}$ contains the density processes of all $\mathbb{Q} \in \mathcal{M}$. For $y > 0$, we define

$$\mathcal{Y}(y) = y\mathcal{Y} = \{yY : Y \in \mathcal{Y}\}$$

and consider the following optimization problem:

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)]. \tag{6}$$

The next result from [10] shows that the value functions u and v to the optimization problems (4) and (6) are conjugate.

THEOREM 1 ([10], Theorem 2.1). *Assume that* (2) *and* (3) *hold true and*

$$u(x) < \infty \qquad \textit{for some } x > 0. \tag{7}$$

Then:

1. $u(x) < \infty$ *for all* $x > 0$, *and there exists* $y_0 \geq 0$ *such that* $v(y)$ *is finitely valued for* $y > y_0$. *The value functions* u *and* v *are conjugate*

$$\begin{aligned} v(y) &= \sup_{x>0}[u(x) - xy], \qquad y > 0, \\ u(x) &= \inf_{y>0}[v(y) + xy], \qquad x > 0. \end{aligned} \tag{8}$$

The function u *is continuously differentiable on* $(0, \infty)$ *and the function* v *is strictly convex on* $\{v < \infty\}$.

The functions u' *and* v' *satisfy*

$$u'(0) = \lim_{x \to 0} u'(x) = \infty,$$

$$v'(\infty) = \lim_{y \to \infty} v'(y) = 0.$$

2. *The optimal solution* $\widehat{Y}(y) \in \mathcal{Y}(y)$ *to* (6) *exists and is unique provided that* $v(y) < \infty$.

As in [10] we are interested in the following questions related to the optimization problems (4) and (6):

1. Does the optimal solution $\widehat{X} \in \mathcal{X}(x)$ to (4) exist?
2. Does the value function $u(x)$ satisfy the usual properties of a utility function, that is, is it increasing, strictly concave, continuously differentiable and such that $u'(0) = \infty$, $u'(\infty) = 0$?
3. Does the dual value function v have the representation

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \tag{9}$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ denotes the Radon–Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}) = (\Omega, \mathcal{F}_T)$?

In [10] (see Theorem 2.2 and the counterexamples in Section 5) we proved that a minimal assumption on the utility function U, which implies positive answers to these questions for an *arbitrary* financial model, is the condition on the asymptotic behavior of the elasticity of U,

$$AE(U) \stackrel{\triangle}{=} \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1.$$

The subsequent theorem, which is the main result of the present paper, and Note 1 below imply that a necessary and sufficient condition for all three assertions to have positive answers in the framework of a *particular* financial model is the finiteness of the dual value function.

THEOREM 2. *Assume that* (2) *and* (3) *hold true and*

$$v(y) < \infty \qquad \forall\, y > 0. \tag{10}$$

Then in addition to the assertions of Theorem 1 *we have the following*:

1. *The value functions u and $-v$ are continuously differentiable, increasing and strictly concave on* $(0, \infty)$ *and satisfy*

$$u'(\infty) = \lim_{x\to\infty} u'(x) = 0,$$

$$-v'(0) = \lim_{y\to 0} -v'(y) = \infty.$$

2. *The optimal solution $\widehat{X}(x) \in \mathcal{X}(x)$ to* (4) *exists, for any $x > 0$, and is unique. In addition, if $y = u'(x)$ then*

$$U'(\widehat{X}_T(x)) = \widehat{Y}_T(y),$$

where $\widehat{Y}(y) \in \mathcal{Y}(y)$ is the optimal solution to (6). *Moreover, the process $\widehat{X}(x)\widehat{Y}(y)$ is a martingale.*

3. *The dual value function v satisfies* (9).

PROOF. Theorem 2 is a rather straightforward consequence of its "abstract version," Theorem 4. Admitting Theorem 4 as well as Proposition 1, the proof of Theorem 2 proceeds as follows.

For $x > 0$ and $y > 0$, let

$$\mathcal{C}(x) = \{g \in \mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P}) : 0 \le g \le X_T, \text{ for some } X \in \mathcal{X}(x)\}, \tag{11}$$

$$\mathcal{D}(y) = \{h \in \mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P}) : 0 \le h \le Y_T, \text{ for some } Y \in \mathcal{Y}(y)\}. \tag{12}$$

In other words, $\mathcal{C}(x)$ and $\mathcal{D}(y)$ are the sets of random variables dominated by the final values of elements from $\mathcal{X}(x)$ and $\mathcal{Y}(y)$, respectively. With this notation, the value functions u and v take the form

$$u(x) = \sup_{g\in\mathcal{C}(x)} \mathbb{E}[U(g)],$$

$$v(y) = \inf_{h\in\mathcal{D}(y)} \mathbb{E}[V(h)].$$

According to Proposition 3.1 in [10] the sets $\mathcal{C}(x)$, $x > 0$, and $\mathcal{D}(y)$, $y > 0$, satisfy the conditions (16)–(18). Hence Theorem 4 implies assertions 1 and 2 of Theorem 2, except for the claim that the product $\widehat{X}(x)\widehat{Y}(y)$ is a martingale. To prove the martingale property, note that $\widehat{X}(x)\widehat{Y}(y)$ is a positive supermartingale [by the construction of the set $\mathcal{Y}(y)$] and that we obtain the following equality from item 2 of Theorem 4:

$$\mathbb{E}[\widehat{X}_T(x)\widehat{Y}_T(y)] = xy = \widehat{X}_0(x)\widehat{Y}_0(y).$$

This readily implies the martingale property of $\widehat{X}(x)\widehat{Y}(y)$.

To prove the final assertion 3, we use Proposition 1. We denote by $\widetilde{\mathcal{D}}$ the set of Radon–Nikodym derivatives of equivalent martingale measures

$$\widetilde{\mathcal{D}} = \left\{h = \frac{d\mathbb{Q}}{d\mathbb{P}},\ \mathbb{Q} \in \mathcal{M}\right\}.$$

The set $\widetilde{\mathcal{D}}$ is closed under countable convex combinations. In addition,

$$g \in \mathcal{C} \Leftrightarrow g \geq 0 \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}[g] \leq 1 \qquad \forall\, \mathbb{Q} \in \mathcal{M}$$

by the general duality relationships between the terminal values of strategies and the densities of equivalent martingale measures (see [4] and [5]). Hence the set $\widetilde{\mathcal{D}}$ satisfies the assumptions of Proposition 1 and the result follows. □

NOTE 1. In view of the duality relation (8), condition (10) is equivalent to

$$u'(\infty) = \lim_{x\to\infty} u'(x) = 0,$$

which may equivalently be restated as

$$\lim_{x\to\infty} \frac{u(x)}{x} = 0.$$

In particular, this shows the necessity of (10) for Theorem 2 to hold true.

NOTE 2. In [10], Theorem 2.2, we proved that the assertions of Theorem 2 follow from the assumptions of Theorem 1 and the condition $AE(U) < 1$ on the asymptotic elasticity of U. Let us now deduce this result as an easy consequence of Theorem 2.

We need to show that $AE(U) < 1$ implies that $v(y) < \infty$ for all $y > 0$. By Theorem 1 there is $y_0 > 0$ such that

$$v(y) < \infty, \qquad y > y_0. \tag{13}$$

Further, the condition $AE(U) < 1$ is equivalent to the following property of V (see Lemma 6.3 in [10]): there are positive constants c_1 and c_2 such that

$$V\left(\frac{y}{2}\right) \leq c_1 V(y) + c_2, \qquad y > 0. \tag{14}$$

The finiteness of v now follows from (13) and (14).

NOTE 3. Condition (10) may also be stated in the following equivalent form:

$$\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] < \infty \qquad \forall\, y > 0. \tag{15}$$

Indeed, the implication (15) $\Rightarrow$ (10) is trivial, as the density processes of martingale measures belong to $\mathcal{Y}$. The more difficult reverse implication follows from Theorem 2.

2. The abstract version of the theorem. Let $\mathcal{C}$ and $\mathcal{D}$ be nonempty sets of positive random variables such that

1. The set $\mathcal{C}$ is bounded in $\mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ and contains the constant function $g = 1$,

$$\lim_{n\to\infty} \sup_{g\in\mathcal{C}} \mathbb{P}[|g| \geq n] = 0, \tag{16}$$

$$1 \in \mathcal{C}. \tag{17}$$

2. The sets $\mathcal{C}$ and $\mathcal{D}$ satisfy the bipolar relations

$$\begin{aligned} g \in \mathcal{C} &\Leftrightarrow g \geq 0 \quad \text{and} \quad \mathbb{E}[gh] \leq 1 \qquad \forall\, h \in \mathcal{D},\\ h \in \mathcal{D} &\Leftrightarrow h \geq 0 \quad \text{and} \quad \mathbb{E}[gh] \leq 1 \qquad \forall\, g \in \mathcal{C}. \end{aligned} \tag{18}$$

For $x > 0$ and $y > 0$, we define the sets

$$\mathcal{C}(x) = x\mathcal{C} = \{xg : g \in \mathcal{C}\},$$
$$\mathcal{D}(y) = y\mathcal{D} = \{yh : h \in \mathcal{D}\},$$

and the optimization problems

$$u(x) = \sup_{g\in\mathcal{C}(x)} \mathbb{E}[U(g)], \tag{19}$$

$$v(y) = \inf_{h\in\mathcal{D}(y)} \mathbb{E}[V(h)]. \tag{20}$$

Here $U = U(x)$ and $V = V(y)$ are the functions defined in Section 1. If $\mathcal{C}(x)$ and $\mathcal{D}(y)$ are defined by (11) and (12), these value functions coincide with the value functions defined in (4) and (6).

Let us recall the following result from [10], which is the abstract version of Theorem 1.

THEOREM 3 (Theorem 3.1 in [10]). *Assume that the sets $\mathcal{C}$ and $\mathcal{D}$ satisfy* (16)–(18). *Assume also that the utility function U satisfies* (3) *and that*

$$u(x) < \infty \qquad \textit{for some } x > 0. \tag{21}$$

Then:

1. $u(x) < \infty$ *for all* $x > 0$, *and there exists* $y_0 \geq 0$ *such that* $v(y)$ *is finitely valued for* $y > y_0$. *The value functions* u *and* v *are conjugate*:

$$
\begin{aligned}
v(y) &= \sup_{x>0}[u(x) - xy], \qquad y > 0, \\
u(x) &= \inf_{y>0}[v(y) + xy], \qquad x > 0.
\end{aligned}
\tag{22}
$$

The function u *is continuously differentiable on* $(0, \infty)$, *and the function* v *is strictly convex on* $\{v < \infty\}$.
The functions u' *and* $-v'$ *satisfy*

$$
\begin{aligned}
u'(0) &= \lim_{x \to 0} u'(x) = \infty, \\
v'(\infty) &= \lim_{y \to \infty} v'(y) = 0.
\end{aligned}
$$

2. *If* $v(y) < \infty$, *then the optimal solution* $\widehat{h}(y) \in \mathcal{D}(y)$ *to* (19) *exists and is unique*.

We now state the abstract version of Theorem 2. This theorem refines Theorem 3.2 in [10] in the sense that the condition $AE(U) < 1$ is replaced by the weaker condition (23) requiring the finiteness of the function $v(y)$, for all $y > 0$.

THEOREM 4. *Assume that the utility function* U *satisfies* (3), *the sets* $\mathcal{C}$ *and* $\mathcal{D}$ *satisfy* (16)–(18), *and that the value function* v *defined in* (20) *is finite*

$$
v(y) < \infty \qquad \forall\, y > 0. \tag{23}
$$

Then, *in addition to the assertions of Theorem* 3, *we have the following*:

1. *The value functions* u *and* $-v$ *are continuously differentiable*, *increasing and strictly concave on* $(0, \infty)$ *and satisfy*

$$
\begin{aligned}
u'(\infty) &= \lim_{x \to \infty} u'(x) = 0, \\
-v'(0) &= \lim_{y \to 0} -v'(y) = \infty.
\end{aligned}
$$

2. *The optimal solution* $\widehat{g}(x) \in \mathcal{C}(x)$ *to* (19) *exists*, *for all* $x > 0$, *and is unique*. *In addition*, *if* $y = u'(x)$, *then*

$$
U'(\widehat{g}(x)) = \widehat{h}(y),
$$

$$
\text{and } \mathbb{E}[\widehat{g}(x)\widehat{h}(y)] = xy,
$$

where $\widehat{h}(y) \in \mathcal{D}(y)$ *is the optimal solution to* (20).

The proof of Theorem 4 is based on the following lemma.

LEMMA 1. *Assume that the set* $\mathcal{C}$ *satisfies* (16)–(18) *and the value function* $u(x)$ *defined in* (19) *is finite* (*for some or, equivalently, for all* $x > 0$) *and satisfies*

$$\lim_{x\to\infty} \frac{u(x)}{x} = 0. \tag{24}$$

Then the optimal solution $\widehat{g}(x) \in \mathcal{C}(x)$ *exists for all* $x > 0$.

PROOF. The assertion that $u(x) < \infty$, for some $x > 0$, iff $u(x) < \infty$, for all $x > 0$, is a straightforward consequence of the concavity and monotonicity of u and the fact that $u \geq U$. Also observe that, as remarked in Note 1, assertion (24) is equivalent to (23).

Fix $x > 0$. Let $(f^n)_{n\geq 1}$ be a sequence in $\mathcal{C}(x)$ such that

$$\lim_{n\to\infty} \mathbb{E}[U(f^n)] = u(x).$$

We can find a sequence of convex combinations $g^n \in \operatorname{conv}(f^n, f^{n+1}, \dots)$ which converges almost surely to a random variable $\widehat{g}$ with values in $[0, \infty]$; see, for example, [4], Lemma A1.1. Since the set $\mathcal{C}(x)$ is bounded in $\mathbf{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, we deduce that $\widehat{g}$ is almost surely finitely valued. By (18) and Fatou's lemma, $\widehat{g}$ belongs to $\mathcal{C}(x)$. We claim that $\widehat{g}$ is the optimal solution to (19), that is,

$$\mathbb{E}[U(\widehat{g})] = u(x).$$

Let us denote by U^+ and U^- the positive and negative parts of the function U. From the concavity of U we deduce that

$$\lim_{n\to\infty} \mathbb{E}[U(g^n)] = u(x)$$

and from Fatou's lemma that

$$\liminf_{n\to\infty} \mathbb{E}[U^-(g^n)] \geq \mathbb{E}[U^-(\widehat{g})].$$

The optimality of $\widehat{g}$ will follow if we show that

$$\lim_{n\to\infty} \mathbb{E}[U^+(g^n)] = \mathbb{E}[U^+(\widehat{g})]. \tag{25}$$

If $U(\infty) \leq 0$, then there is nothing to prove. So we assume that $U(\infty) > 0$.

The validity of (25) is equivalent to the uniform integrability of the sequence $(U^+(g^n))_{n\geq 1}$. If this sequence is not uniformly integrable then, passing if necessary to a subsequence still denoted by $(g^n)_{n\geq 1}$, we can find a constant $\alpha > 0$ and a disjoint sequence $(A^n)_{n\geq 1}$ of $(\Omega, \mathcal{F})$, that is,

$$A^n \in \mathcal{F}, \qquad A^i \cap A^j = \varnothing \qquad \text{if } i \neq j,$$

such that

$$\mathbb{E}[U^+(g^n) I(A^n)] \geq \alpha \qquad \text{for } n \geq 1.$$

We define the sequence of random variables $(h^n)_{n\geq 1}$

$$h^n = x_0 + \sum_{k=1}^{n} g^k I(A^k),$$

where

$$x_0 = \inf\{x > 0 : U(x) \geq 0\}.$$

For any $f \in \mathcal{D}$,

$$\mathbb{E}[h^n f] \leq x_0 + \sum_{k=1}^{n} \mathbb{E}[g^k f] \leq x_0 + nx.$$

Hence $h^n \in \mathcal{C}(x_0 + nx)$. On the other hand,

$$\mathbb{E}[U(h^n)] \geq \sum_{k=1}^{n} \mathbb{E}[U^+(g^k) I(A^k)] \geq \alpha n,$$

and therefore

$$\limsup_{x\to\infty} \frac{u(x)}{x} \geq \limsup_{n\to\infty} \frac{\mathbb{E}[U(h^n)]}{x_0 + nx} \geq \limsup_{n\to\infty} \frac{\alpha n}{x_0 + nx} = \alpha > 0.$$

This contradicts (24). Therefore (25) holds true. $\square$

PROOF OF THEOREM 4. Since, for $x > 0$ and $y > 0$,

$$U(x) \leq V(y) + xy,$$

and, for $g \in \mathcal{C}(x)$ and $h \in \mathcal{D}(y)$,

$$\mathbb{E}[gh] \leq xy,$$

we have

$$u(x) \leq v(y) + xy.$$

In particular, the finiteness of $v(y)$, for some $y > 0$, implies the finiteness of $u(x)$, for all $x > 0$. It follows that the conditions of Theorem 3 hold true.

From the assumption that $v(y) < \infty$, $y > 0$, and the duality relations (22) between u and v, we deduce that

$$\lim_{x\to\infty} \frac{u(x)}{x} = \lim_{x\to\infty} u'(x) = 0. \tag{26}$$

Lemma 1 now implies that the optimal solution $\widehat{g}(x)$ to (19) exists, for any $x > 0$. The strict concavity of U implies the uniqueness of $\widehat{g}(x)$, as well as the fact that the function u is strictly concave too. The remaining assertions of item 1 related to the function v follow from the established properties of u, because of the duality relations (22) (see, e.g., [12]).

Let $x > 0$, $y = u'(x)$, $\widehat{g}(x)$ and $\widehat{h}(y)$ be the optimal solutions to (19) and (20), respectively. We have

$$\begin{aligned}\mathbb{E}\big[\big|V\big(\widehat{h}(y)\big) + \widehat{g}(x)\widehat{h}(y) - U(\widehat{g}(x))\big|\big]\\ = \mathbb{E}\big[V\big(\widehat{h}(y)\big) + \widehat{g}(x)\widehat{h}(y) - U(\widehat{g}(x))\big]\\ \leq v(y) + xy - u(x) = 0,\end{aligned}$$

where, in the last step, we have used the relation $y = u'(x)$. It follows that

$$U(\widehat{g}(x)) = V\big(\widehat{h}(y)\big) + \widehat{g}(x)\widehat{h}(y).$$

This readily implies that

$$U'(\widehat{g}(x)) = \widehat{h}(y) \qquad \text{a.s.}$$

and

$$\mathbb{E}\big[\widehat{g}(x)\widehat{h}(y)\big] = \mathbb{E}[U(\widehat{g}(x))] - \mathbb{E}\big[V\big(\widehat{h}(y)\big)\big] = u(x) - v(y) = xy. \qquad \square$$

We complete the section with Proposition 1, which was used in the proof of item 3 of Theorem 2. This proposition was proved in [10] under the additional assumption $AE(U) < 1$.

Let $\widetilde{\mathcal{D}}$ be a convex subset of $\mathcal{D}$ such that:

1. For any $g \in \mathcal{C}$,

$$\sup_{h\in\widetilde{\mathcal{D}}} \mathbb{E}[gh] = \sup_{h\in\mathcal{D}} \mathbb{E}[gh]. \tag{27}$$

2. The set $\widetilde{\mathcal{D}}$ is closed under countable convex combinations, that is, for any sequence $(h^n)_{n\geq 1}$ of elements of $\widetilde{\mathcal{D}}$ and any sequence of positive numbers $(a^n)_{n\geq 1}$ such that $\sum_{n=1}^{\infty} a^n = 1$ the random variable $\sum_{n=1}^{\infty} a^n h^n$ belongs to $\widetilde{\mathcal{D}}$.

PROPOSITION 1. *Assume that the conditions of Theorem* 4 *hold true and that* $\widetilde{\mathcal{D}}$ *satisfies the above assertions. The value function* $v(y)$ *defined in* (20) *then satisfies*

$$v(y) = \inf_{h\in\widetilde{\mathcal{D}}} \mathbb{E}[V(yh)]. \tag{28}$$

The proof of the proposition will use the following two lemmas.

The first is an easy result, whose proof is analogous to the proof of Proposition 3.1 in [10] and is therefore skipped.

LEMMA 2. *Under the assumptions of Proposition* 1, *let* $\widehat{h}(y)$ *be the optimal solution to* (20). *Then there exists a sequence* $(h^n)_{n\geq 1}$ *in* $\widetilde{\mathcal{D}}$, *that converges almost surely to* $\widehat{h}(y)/y$.

LEMMA 3. *Under the assumptions of Proposition* 1, *we have*, *for each* $y > 0$,

$$\inf_{h \in \widetilde{\mathcal{D}}} \mathbb{E}[V(yh)] < \infty.$$

PROOF. To simplify the notation we shall prove the assertion of the lemma for the case $y = 1$.

Let $(\lambda_n)_{n \geq 1}$ be a sequence of strictly positive numbers such that $\sum_{n=1}^{\infty} \lambda_n = 1$. We denote by $\widehat{h}(\lambda_n)$ the optimal solution to (20) corresponding to the case $y = \lambda_n$. Let $(\delta_n)_{n \geq 2}$ be a sequence of strictly positive numbers, decreasing to 0, such that

$$\sum_{n=1}^{\infty} \mathbb{E}\big[V\big(\widehat{h}(\lambda_n)\big) I(A_n)\big] < \infty \qquad \text{if } A_n \in \mathcal{F}, \mathbb{P}[A_n] \leq \delta_n,\ n \geq 2. \tag{29}$$

From Lemma 2 we deduce the existence of a sequence $(h_n)_{n \geq 1}$ in $\widetilde{D}$ such that

$$\mathbb{P}\big[V(\lambda_n h_n) > V\big(\widehat{h}(\lambda_n)\big) + 1\big] \leq \delta_{n+1}, \qquad n \geq 1.$$

We define the sequence of measurable sets $(A_n)_{n \geq 1}$ as follows:

$$A_1 = \big\{V(\lambda_1 h_1) \leq V\big(\widehat{h}(\lambda_1) + 1\big)\big\}$$

$$\vdots$$

$$A_n = \big\{V(\lambda_n h_n) \leq V\big(\widehat{h}(\lambda_n) + 1\big)\big\} \Big\backslash \bigcup_{k=1}^{n-1} A_k.$$

This sequence has the following properties:

$$A_i \cap A_j = \varnothing \qquad \text{if } i \neq j,$$

$$\mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right] = 1,$$

$$\mathbb{P}[A_n] \leq \delta_n, \qquad n \geq 2.$$

We define

$$h = \sum_{n=1}^{\infty} \lambda_n h_n.$$

We have $h \in \widetilde{\mathcal{D}}$ because the set $\widetilde{D}$ is closed under countable convex combinations. The proof now follows from the inequalities

$$\begin{aligned}
\mathbb{E}[V(h)] &= \sum_{n=1}^{\infty} \mathbb{E}[V(h) I(A_n)] \\
&\overset{\text{(i)}}{\leq} \sum_{n=1}^{\infty} \mathbb{E}[V(\lambda_n h_n) I(A_n)]
\end{aligned}$$

$$\overset{\text{(ii)}}{\leq} \sum_{n=1}^{\infty} \mathbb{E}\big[V\big(\widehat{h}(\lambda_n)\big)I(A_n)\big] + 1$$

$$\overset{\text{(iii)}}{<} \infty,$$

where (i) holds true because V is a decreasing function, (ii) follows from the construction of the sequence $(A_n)_{n\geq 1}$, and (iii) is a consequence of (29). $\square$

PROOF OF PROPOSITION 1. Fix $\varepsilon > 0$ and $y > 0$. We have to show that there is $h \in \widetilde{D}$ such that

$$\mathbb{E}\big[V\big((y+\varepsilon)h\big)\big] \leq v(y) + \varepsilon.$$

Let $\widehat{h} = \widehat{h}(y)$ be the optimal solution to the optimization problem (20) and f be an element of $\widetilde{\mathcal{D}}$ such that

$$\mathbb{E}[V(\varepsilon f)] < \infty.$$

The existence of such a function f follows from Lemma 3. Let $\delta > 0$ be a sufficiently small number such that

$$\mathbb{E}\big[\big(\big|V(\widehat{h})\big| + |V(\varepsilon f)|\big)I(A)\big] \leq \frac{\varepsilon}{2} \qquad \text{if } A \in \mathcal{F},\ \mathbb{P}[A] \leq \delta. \tag{30}$$

From Lemma 2 we deduce the existence of $g \in \widetilde{\mathcal{D}}$ such that

$$\mathbb{P}\left[V(yg) > V(\widehat{h}) + \frac{\varepsilon}{2}\right] \leq \delta. \tag{31}$$

Denote

$$A = \left\{V(yg) > V(\widehat{h}) + \frac{\varepsilon}{2}\right\}$$

and define

$$h = \frac{yg + \varepsilon f}{y + \varepsilon}.$$

Since the set $\widetilde{\mathcal{D}}$ is convex, $h \in \widetilde{\mathcal{D}}$. The proof now follows from the inequalities.

$$\begin{aligned}\mathbb{E}\big[V\big((y+\varepsilon)h\big)\big] &= \mathbb{E}[V(yg + \varepsilon f)] \\ &\overset{\text{(i)}}{\leq} \mathbb{E}[V(yg)I(A^c)] + \mathbb{E}[V(\varepsilon f)I(A)] \\ &\overset{\text{(ii)}}{\leq} v(y) + \varepsilon,\end{aligned}$$

where (i) holds true, because V is a decreasing function, and (ii) follows from (30) and (31). $\square$

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