Qualification Exam Syllabus

Major: Set Theory

21-602: Set Theory I

- The axioms of ZFC
- Ordinal and cardinal arithmetic
- König's lemma
- Transfinite induction and recursion scheme
- The rank hierarchy and rank function
- The Mostowski collapse theorem scheme
- Foundation $\iff V = WF$
- $H_\lambda \models$ZFC-P for regular uncountable λ
- $H_\lambda \prec_{\Sigma_1} V$ for uncountable λ
- Δ_1-absoluteness theorem
- The absoluteness of well-foundedness
- The reflection theorem scheme of hierarchy of sets: ZFC is not finite axiomatizable
- Δ-system lemma
- Internal model theory and proof theory
- HOD$\models$ZFC; HOD$(X) \models$ZF
- $L \models$ZFC+GCH
- ${}^\omega\omega \cap L$ is Σ^1_2 set of reals
- $L \models <_L \cap {}^\omega\omega \times {}^\omega\omega$ is a Δ^1_2 wellordering of ordertype ω_1
- $<_L \cap {}^\omega\omega \times {}^\omega\omega$ is not Lebesgue measurable in L
- Suslin representation for $\mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$ sets of reals
- Shoenfield's absoluteness theorem
- $\mathbf{\Sigma}^1_2 \cap \mathcal{P}({}^\omega\omega) = \mathbf{\Sigma}^{HC}_1 \cap \mathcal{P}({}^\omega\omega)$
- Suslin's problem

- Trees and tree property
- Construction of various uncountable trees: Aronszajn, Suslin, Kurepa, Special
- Diagonal intersection and normality of filters
- Fodor's lemma
- Solovay's splitting theorem
- Elementary substructures and internal approaching sequences
- Ultrapowers and Łoś's theorem
- Clubs and stationary sets of $[X]^\omega$
- The diamond principle $\diamondsuit_\kappa(E)$ and applications
- $L \models \diamondsuit_\kappa(E)$
- The square principle $\square_\kappa$ and applications
- Large cardinals: Inaccessible, Mahlo, weakly compact, measurable, strong, superstrong, supercompact, huge cardinals
- Equivalent definitions for weakly compact cardinals
- Equivalent definitions for measurable cardinals
- Scott's theorem of $L \models$"There is no measurable cardinals"
- Kunen's theorem that the only elementary embedding from V to V is the identity
- Silver's theorem on singular cardinals

21-702: Set Theory II

- Generic model theorem
- Generic filter existence lemma
- Theorem on forcing and truth
- Theorem on the definability of forcing
- Eliminating the countable transitive model assumption
- Existential completeness of forcing
- A model of ZFC + V $\neq$ L
- Properties that some posets have, such as:

  - Splitting
  - Chain condition
  - Distributive
  - Closed
  - Weakly homogeneous
  - Partial ordering in the strict sense
  - Separative

  together with corresponding forcing facts, examples and applications

- Cohen's consistency proof for $\mathrm{ZFC} + \neg\,\mathrm{CH}$
- Nice names lemma
- Properties and applications of the poset $\mathrm{Add}(\kappa, \lambda)$
- Products of posets with various supports
- Product lemmas
- Properties and applications of the posets $\mathrm{Coll}(\kappa, \lambda)$ and $\mathrm{Coll}(\kappa, < \mu)$.
- Easton's theorem and Easton's lemma
- Various models in which AC fails, including:
  - ZF+ There is an infinite set A for which there is no injection $f : \omega \to A$
  - ZF+ ω_1 is singular
  - ZF + DC+ There is no wellordering of $\mathbb{R}$
- Boolean completion of a poset
- Intermediate models are forcing extensions by complete subalgebras
- Quotient Boolean algebras and two step iterations
- Characterization of $\mathrm{ro}(\mathrm{Coll}(\omega, \kappa))$
- $\mathrm{ro}(\mathrm{Coll}(\omega, \kappa))$ is countably generated
- Classical results on Baire category, Lebesgue measure and perfect sets
- If ω_1 is inaccessible to reals, then every $\mathbf{\Sigma}^1_2$ set has the Baire property
- Borel/meager $\simeq$ $\mathrm{ro}(\mathrm{Cohen})$
- Solovay's model for ZF + DC + BP + LM + PSP

- Mansfield perfect set theorem
- Sacks-Guasparie-Kechris theorem
- Martin's theorem that if there is a measurable cardinal, then $\mathbf{\Pi}^1_1$ sets are homogeneously Suslin, hence determined
- Homogeneous and weakly homogeneous systems
- Martin-Solovay Σ^1_3-absoluteness theorem

21-800: Advanced Topics in Logic(Inner Model Theory)

The fine structure theory of L:

- Jensen's hierarchy
- Σ_1-condensation theorem
- Theorems about soundness, acceptability, and solidity
- Equivalent definitions of projecta and standard parameters
- The decoding process
- Downward and upward extension theorem
- $\mathfrak{C} = (J_\rho, \in, A) = \mathfrak{C}_1^{J_\alpha} \implies \forall n < \omega[\mathcal{P}(J_\rho) \cap \Sigma_{2+n}^{J_\alpha} = \Sigma_{1+n}^{\mathfrak{C}}]$
- Amenability of the coding structure
- Σ_2-condensation theorem of the coding structure
- Higher-order coding structures and corresponding theorems
- Examples of the projecta sequence

The covering lemma of L: The proof and some applications

Sharps:

- The factor lemma of ultrapowers
- Indiscernibles of L is a club class
- Equivalence of non-rigidity of L: There is an active baby mice; There is a club class of L-indiscernibles; There is a club class of L-indiscernibles that generates L
- Iterability of $(L, \in, U_j)$ with $j : L \to L$
- Realization lemma and fixed points lemma of L-iterates

- $\mathcal{M}$ is iterable iff it is $|\mathcal{M}|^+$-iterable
- $\{\mathcal{M} : \mathcal{M}$ is a countable active baby mouse$\}$ is Π_1 over HC
- The singleton containing the minimal active baby mouse is Σ^1_3
- Uniqueness lemma of the minimal active baby mouse
- Σ_1-condensation of active baby premouse
- Copying construction
- Uniqueness lemma of every active baby mouse with respect to the critical point
- Uniqueness of L-indiscernibles and its generation of L
- $\exists 0^\#$ is absolute between forcing extensions
- $\exists\theta$-iterable,countable active baby premouse is absolute between L and V for all $\theta < \omega_1^L$
- $\exists 0^\# \implies \Sigma^1_1$-Det

$L[U]$ *premice:*

- Copying construction beween coarse premice
- ω_1-iterability $\iff$ ω_1-completeness $\iff$ iterability
- Realization lemma of coarse premice
- Comparison lemma between mice
- Silver's theorem: All weasels satisfy GCH
- All weasels have the same reals and the same wellordering of their reals
- ${}^\omega\omega \cap W$ is a Σ^1_3 set of reals
- The order of construction of every weasel W restricts to a Σ^1_3 wellordering of ${}^\omega\omega \cap W$
- A proof of Martin-Solovay using mice
- In iteration $i_{0,\theta} M \to M_\theta$ with $crit(i_{0,\theta}) = \kappa$, all cardinal $\mu > \theta$ of M such that $M \models cf(\mu) > \kappa \wedge \forall\lambda < \mu(\lambda^\kappa < \mu)$ is fixed by $i_{0,\theta}$
- The class of ordinals fixed by the iteration map is thick
- Hull property and definability property of weasels
- The weasel W is uniquely determined by $\dot{\kappa}^W$
- For any two weasels W and W', $\dot{\kappa}^W < \dot{\kappa}^{W'} \implies W'$ is an iterate of W

- $L[U] \models$ ZFC+"there is only one measurable cardinal with only one normal measure"
- $L[U] \models \forall \bar{U}[$"$\bar{U}$ is a measure over $\dot{\kappa}$" $\implies \exists n < \omega[\bar{U} \simeq \dot{U}^n]$"
- Mathias theorem about Prikry forcing

Minor: Topology

- Topological space. (Sequentially) continuity. Subspace topology. Basis and subbasis. Interior and closure. Limit points and boundary. Neighborhood basis.
- Initial and final topology. Initial and final topology exists and unique. Product topology. Topological sum.
- (Pathwise/locally) connectedness. (Pathwise) connected components. Properties of (pathwise/locally) connectedness.
- Compactness and variations (paracompact, sequentially compact, locally compact, Lindelöf). Properties of compactness.
- First and second countablility. Separable space.
- Metric space. Isometry. Compact metric spaces. Totally boundedness. Separable metric spaces. Completion of metric spaces. Arzela-Ascoli theorem.
- Axiom of Choice. Zorn's lemma. Baire category theorem. Banach-Steinhaus theorem.
- Filters. Ultrafilters. Acummulation points. Push-forward filters. Tychonoff's theorem.
- Separation axioms($T_0, T_1, T_2, T_3, T_{3a}, T_4$). Regular, completely regular and normal spaces. Urysohn's lemma. Tietze's extension theorem.
- Stone-Čech compactification of completely regular spaces and discrete spaces. Universal property. One-point compactification of locally compact, non-compact, Hausdorff spaces.
- Quotient spaces. Gluing one space to another. Cone and suspensions. Configuration spaces. Projective spaces.
- Simplical complexes. PL category. Orientations. $\mathbb{R}P^2$ does not embed into $\mathbb{R}^3$.
- Brouwer's fixed point theorem. KKM theorem. Appications. Intersecting paths in the unit square.
- Borsuk-Ulam theorem. Jordan curve theorem.
- Antipodally-labelled triangulations and alternating simplices. Topological Radon theorem. $K_{3,3}$ is not planar. Linked curves. K_6 is intrinsically linked.
- Homotopy. Homotopy equivalence. Contractible spaces. B^n is contractible. S^n is not contractible. Simply connected spaces.

Qualification Exam Transcript*

Examiner: Prof. James Cummings, Prof. Ernest Schimmerling, Prof. Florian Fricks.

Florian: Give a counter-example of sequentially continuity/compactness/closedness not implying continuity/compactness/closedness.

Me: Consider $f : \omega_1 + 1 \to \{0,1\}$, where $\omega_1 + 1$ is equipped with the order topology, $\{0,1\}$ the discrete topology, and $f(\alpha) = 0$ iff $\alpha < \omega_1$. This function is sequentially continuous but not continuous.

Florian: Show that, $f : X \to Y$ is continuous at x iff for every filter $\mathcal{F} \subseteq \mathcal{P}(X)$ and $\mathcal{F} \to x$, $\mathcal{F}^* \to f(x)$, where $\mathcal{F}^*$ is the push-forward filter.

Me: [Omitted response.]†

James: Show why $x = 0^\#$ is $\Pi^1_2(x)$.

Me: [Omitted response.]‡

James: Show why $0^\#$ exists is absolute between V and $V[G]$, where G is a set-generic filter over V.

Me: If V has $0^\#$, then the minimal active baby mouse is still iterable in $V[G]$, thus $V[G] \models 0^\#$ exists. If $\Vdash_{\mathbb{P}} 0^\#$ exists, then we force $\mathbb{P} \times \mathbb{P}$ over V. Let $G \times H$ be the generic filter. Then since $0^\#$ exists in both $V[G]$ and $V[H]$, and by the uniqueness theorem, in $V[G \times H]$ witnesses the fact that $(0^\#)^V[G] = (0^\#)^V[H]$. This implies that $0^\# \in V$, by the product lemma.

Florian: Give the definition of contractible space.

Me: [Omitted response.]

Florian: Show that S^n is not contractible.§

Me: Notice that:

Lemma 1. *S^n is contractible iff for every continuous function $f : S^n \to S^n$, f can be extended to a continuous function $\tilde{f} : B^{n+1} \to S^n$.*

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*I wrote this list after approximately 1hr after the exam, but I won't be sure that these are all of the questions.

†I mistakenly stated the definition of x being an accumulation point of $\mathcal{F}$, and Florian corrected me.

‡I got the correct idea, but I was very sloppy at how I should express that I code some structure in a real and some iteration process can be computable from it. I received several hints and suggestions from James and Ernest.

§I misheard the question and answered that B^n is contractible. Fixed by Florian.

Florian didn't ask me to prove this. I then claimed that I can get a contradiction by using the Borsuk-Ulam theorem, but it turned out that I cannot. Hinted by Florian, I considered the Brower's fixed point theorem, and let $f = id_{S^n}$. Suppose f has a continuous extension $\tilde{f}$, then let $g(x) = -x$ and consider $g \circ \tilde{f} : B^{n+1} \to S^n \subseteq B^{n+1}$. This map has no fixed point, which contradicts the Brower's fixed point theorem.

Ernest: Give some properties hold by L.

Me: $L \models \mathsf{ZFC}$, $L \models 0^\#$ does not exist(I gave a short proof of this, under Ernest's request that I cannot using Kunen's inconsistency), $L \models \Diamond_\kappa(E) \wedge \square_\kappa(E)$, the covering lemma, fine structure...

Ernest: Show some consequence of covering lemma.

Me: If $0^\#$ does not exist, then for any singular uncountable cardinal κ, $\kappa^{+L} = \kappa^+$.

Ernest: Prove this.

Me: [Ernest's hint: Consider $\kappa^{+L} = \alpha < \kappa^+$ and the $cof(\alpha)$.] Otherwise, let $F \subseteq \alpha$ be a cofinal set in V with cardinality $< \kappa$. Now in L, let $F^* \supseteq F$ in L given by the covering lemma. Now F^* witnesses that α is not regular in L. Contradiction.

James: Give an outline of the proof of Solovay splitting theorem.

Me: [Since the time is over we just had a brief conversation about this. He would like to know the proof of the regular case, I basically wrote:

Lemma 2. *Let μ be a weakly inaccessible cardinal and $S \subseteq \mu$ is stationary. Then $T = \{\alpha \in S : \alpha \cap S$ is not stationary in $\alpha\}$ is stationary.*

I said that this is a homework question of 602 and they didn't ask me to prove this. Also I briefly proved that following ladder system lemma:

Lemma 3. *Let $C_\alpha = \{\xi_j^\alpha : j < \alpha\}$ be a club set that $T \cap C_\alpha = \varnothing$. Then there exists $j < \mu$ such that for all $\eta < \mu$, $\{\alpha \in T : \xi_j^\alpha \geqslant \eta\}$ is stationary.*

What I said is just: towards contradiction, we should first take the diagonal intersection. Then Ernest reminds me that we should also take the club set of the closure point of $i \mapsto \eta_i$.]¶

Time span: Approx. 1hr 45min. After the exam, they said that the tradition was to let me wait in my office and they would come to my office after the result was determined.

¶I also remember that Ernest wanted to ask me to give the outline of the proof of the covering lemma and then he changed his mind.