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ANALYSIS OF A LINEAR FLUID-STRUCTURE INTERACTION PROBLEM

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Abstract. A time-dependent system modeling the interaction between a Stokes fluid and an elastic structure is studied. A divergence-free weak formulation is introduced which does not involve the fluid pressure field. The existence and uniqueness of a weak solution is proved. Strong energy estimates are derived under additional assumptions on the data. The existence of an L^2 integrable pressure field is established after the verification of an inf-sup condition.

1. **Introduction.**

Fluid-structure interaction problems have been extensively studied in the past and continue to be the focus of much attention today. We classify a number of different types of mathematical models for fluid-structure interactions into the following three categories.

Elementary fluid. The fluid motion is governed by equations for a potential function, e.g., the Laplace equation or the wave equation. In [25], a coupled system of a potential equation and a wave equation was considered. Elementary fluids interacting with a rigid cavity or a moving wall were studied in [15] and with an elastic solid in [3].

Inviscid fluid. The fluid motion is governed by inviscid fluid models, e.g., the Euler equations. Interactions between linearized inviscid fluids and elastic solids were analyzed in [1, 27]. An algorithm applicable to an inviscid nonlinear fluid coupled with rigid walls was given in [2].

Viscous fluid. The fluid motion is governed by viscous, incompressible or compressible fluid models, e.g., the Stokes or Navier-Stokes equations. There is an extensive literature on linearized viscous fluids coupled with solids. Solids modeled by plate equations or shell equations were treated in [11, 12, 14, 23]. The Stokes

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equations coupled with a beam equation was analyzed in [18]. In [6, 22], interactions between a linearized viscous fluid and elastic solids were studied; [4] discussed interactions with rigid walls. There also is a vast literature on fluid-structure interactions for which the fluid is modeled by nonlinear viscous fluid models. Rigid body motions of solids in a nonlinear viscous fluid were studied in [5, 7, 20, 19, 21]. In [13], the Navier-Stokes equations coupled to the plate equations were studied. The work of [5, 8, 10, 28, 29] treated interactions between nonlinear viscous fluids and elastic solids.

It should be noted that the majority of the references cited use solid models of lower spatial dimensions, e.g., one-dimensional beams interacting with two-dimensional fluids or two-dimensional plates interacting with three-dimensional fluids. Rigorous mathematical results are rare for fluid-solid interaction problems in which both the fluid and the solid occupy true spatial domains. Eigenmodes of the coupled system (1.1)–(1.2) were studied in [29]. In [6], the homogenization of a mathematical model for the Stokes equation coupled to the eqations of linear elasticity was considered. Both existence of a solution and numerical experiments for a problem in which a nonlinear viscous fluids is coupled to elastic solids in one dimension were discussed in [8]. For a numerical algorithm for solving interaction problems of elastic body motions in a fluid flow, see [10].

In this paper, we consider the interaction of a linear, viscous fluid with elastic body motions in a two or three dimensional bounded domain. The specific fluid-structure interaction model is described as follows. We assume the fluid and solid occupy the open, Lipschitz domains $\Omega_1 \subset \mathbb{R}^d$ and $\Omega_2 \subset \mathbb{R}^d$, respectively, where $d = 2$ or 3 is the space dimensions. We denote by Ω the entire fluid-solid region under consideration, i.e., Ω is the interior of $\overline{\Omega}_1 \cup \overline{\Omega}_2$. Let $\Gamma_0 = \partial\Omega_1 \cap \partial\Omega_2$ denote the interface between the fluid and solid and let $\Gamma_1 = \partial\Omega_1 \setminus \Gamma_0$ and $\Gamma_2 = \partial\Omega_2 \setminus \Gamma_0$ respectively denote the parts of the fluid and solid boundaries excluding the interface Γ_0. For obvious reasons, we assume that $\text{meas}(\Gamma_1 \cup \Gamma_2) \neq 0$.

In the fluid region Ω_1, we consider the Stokes system

$$\begin{cases} \rho_1 \mathbf{v}_t + \nabla p - \mu_1 \nabla \cdot (\nabla \mathbf{v} + \nabla \mathbf{v}^T) = \rho_1 \mathbf{f}_1 \qquad \text{in } (0,T) \times \Omega_1 \\ \nabla \cdot \mathbf{v} = 0 \qquad \text{in } (0,T) \times \Omega_1 \\ \mathbf{v} = 0 \quad \text{on } (0,T) \times \Gamma_1\,, \qquad \mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega_1, \end{cases} \tag{1.1}$$

where $\mathbf{v}$ denotes the fluid velocity, p the fluid pressure, $\mathbf{f}_1$ the given body force per unit mass, ρ_1 and μ_1 the constant fluid density and viscosity, $\mathbf{v}_0$ the given initial velocity, and $T > 0$ the terminal time.

In the solid region, we consider the equations of linear elasticity

$$\begin{cases} \rho_2 \mathbf{u}_{tt} - \mu_2 \nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \lambda_2 \nabla(\nabla \cdot \mathbf{u}) = \rho_2 \mathbf{f}_2 \qquad \text{in } (0,T) \times \Omega_2 \\ \mathbf{u} = 0 \quad \text{on } (0,T) \times \Gamma_2\,, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega_2\,, \quad \mathbf{u}_t|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega_2, \end{cases} \tag{1.2}$$

where $\mathbf{u}$ denotes the displacement of the solid, $\mathbf{f}_2$ the given loading force per unit mass, μ_2 and λ_2 the Lamé constants, ρ_2 the constant solid density and $\mathbf{u}_0$ and $\mathbf{u}_1$ the given initial data.

Across the *fixed* interface Γ_0 between the fluid and solid, the velocity and stress vector are continuous. Thus, we have

$$\mathbf{u}_t = \mathbf{v} \qquad \text{on } \Gamma_0 \tag{1.3}$$

and

$$\mu_2(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \cdot \mathbf{n}_2 + \lambda_2(\nabla \cdot \mathbf{u})\mathbf{n}_2 = p\mathbf{n}_1 - \mu_1(\nabla \mathbf{v} + \nabla \mathbf{v}^T) \cdot \mathbf{n}_1 \qquad \text{on } \Gamma_0, \quad (1.4)$$

where $\mathbf{n}_i$ is the outward-pointing unit normal vector along $\partial\Omega_i$, $i = 1, 2$.

Some remarks about the use of a fixed fluid-solid interface Γ_0 are in order. The general motion of a solid body immersed in a fluid involves rigid body motions superimposed with displacements caused by the stresses and strains induced in the solid by the loads resulting from the interaction with the fluid motion. Even if the the latter are infinitesimal in size, the rigid motion of the solid by itself would result in a moving fluid-solid interface. Certainly, if the solid motion involves large stress-induced displacements, the fluid-solid interface is not stationary. Thus, the general case is that of a *moving fluid-solid interface* which, in fact, is generally not known and must be found as part of the solution process.

At the other extreme is the case for which, first, there is no rigid motion or that motion is purely translational with constant speed and one attaches the coordinate system to the solid, and, second, the solid undergoes only infinitesimal elastic displacements. Then, since the motion of the fluid-solid interface is wholly determined by the elastic displacement, one may assume that that interface is stationary. Moreover, it is usually the case that the velocity in the solid is also infinitesimal so that that the interface condition $\mathbf{v} = \mathbf{u}_t$ simply reduces to $\mathbf{v} = \mathbf{0}$, i.e., the classical no-slip condition, and the fluid motion uncouples from that of the solid. Once the fluid motion is determined, the other interface condition provides an inhomogeneous traction boundary condition which may be used to determine the elastic motion of the solid.

In between the two extremes, i.e., between having a moving interface and a fixed interface with uncoupled fluid motion, is the case considered here. We begin as in the second case: the motion of the solid is wholly due to infinitesimal displacements. Again, we may then assume that the fluid-solid interface is stationary. However, we operate in the case that although the displacement $\mathbf{u}$ is small, the velocity $\mathbf{u}_t$ is not. Thus, we cannot impose the no-slip condition on the fluid velocity and must retain the interface condition $\mathbf{v} = \mathbf{u}_t$, but the latter may be imposed along a fixed boundary. We note that the case of having $O(1)$ solid velocities even though the solid displacements are of infinitesimal size is of practical interest. For example, this setting arises in the high frequency, small displacement oscillation of elastic structures. Our model may be justified by standard asymptotic analysis techniques (see, e.g., [16]) and the details are omitted.

The plan of the paper is as follows. In Section 2, working in a divergence-free setting, we first introduce some notations and define a weak formulation for the model fluid-structure interaction problem. We next formulate an auxiliary, equivalent parabolic type problem, define its Galerkin approximations and derive a priori estimates for the Galerkin sequence. By passing to the limit in the Galerkin approximations, we prove the existence and uniqueness of a solution for the auxiliary problem. The existence and uniqueness of the velocity $\mathbf{v}$ and displacement $\mathbf{u}$ for the fluid-structure interaction problem follows directly from the existence of a unique solution for the auxiliary problem. In Section 3, we prove a regularity result for the weak solution under additional assumptions. We then verify an inf-sup condition and establish the existence of an L^2 integrable pressure.

2. **The existence of a weak solution.**

2.1. **Notations.** Throughout this paper, C denotes a positive constant, depending on the domains Ω, Ω_1 and Ω_2, whose meaning and value changes with context. $H^s(\mathcal{D})$, $s \in \mathbb{R}$, denotes the standard Sobolev space of order s with respect to the set $\mathcal{D}$ equipped with the standard norm $\|\cdot\|_{s,\mathcal{D}}$. Vector-valued Sobolev spaces are denoted by $\mathbf{H}^s(\mathcal{D})$, with norms still denoted by $\|\cdot\|_{s,\mathcal{D}}$. $H^1_0(\mathcal{D})$ denotes the space of functions belonging to $H^1(\mathcal{D})$ that vanish on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$; $\mathbf{H}^1_0(\mathcal{D})$ denotes the vector-valued counterpart.

We will use the following L^2 inner product notations on scalar and vector-valued L^2 spaces:

$$[p,q]_{\mathcal{D}} = \int_{\mathcal{D}} pq\, d\mathcal{D} \quad \forall\, p,q \in L^2(\mathcal{D}), \qquad [\mathbf{u},\mathbf{v}]_{\mathcal{D}} = \int_{\mathcal{D}} \mathbf{u}\cdot\mathbf{v}\, d\mathcal{D} \quad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{L}^2(\mathcal{D}),$$

where the spatial set $\mathcal{D}$ is Ω or Γ_0 or Ω_i, for $i = 1,2$.

We introduce the function spaces

$$X_i = [\mathbf{H}^1_0(\Omega)]|_{\Omega_i} \text{ with the norm } \|\cdot\|_{X_i} = \|\cdot\|_{1,\Omega_i}, \quad i = 1,2,$$

$$V_1 = \{\mathbf{v} \in \mathbf{H}^1_0(\Omega_1) \ : \ \operatorname{div}\mathbf{v} = 0\} \text{ with the norm } \|\cdot\|_{V_1} = \|\cdot\|_{1,\Omega_1},$$

$$\mathbf{\Psi} = \{\boldsymbol{\eta} \in \mathbf{H}^1_0(\Omega) : \operatorname{div}\boldsymbol{\eta} = 0 \text{ in } \Omega_1\} \text{ with the norm } \|\cdot\|_{1,\Omega},$$

$$\mathbf{\Phi} = \text{the closure of } \mathbf{\Psi} \text{ for the norm induced by the inner product } [[\cdot,\cdot]],$$

where $[[\cdot,\cdot]]$ denotes the weighted L^2 inner product

$$[[\boldsymbol{\xi},\boldsymbol{\eta}]] = [\rho_1\boldsymbol{\xi},\boldsymbol{\eta}]_{\Omega_1} + [\rho_2\boldsymbol{\xi},\boldsymbol{\eta}]_{\Omega_2} \qquad \forall\, \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbf{L}^2(\Omega). \tag{2.1}$$

Clearly, the induced norm on $\mathbf{\Phi}$ is equivalent to $\|\cdot\|_{0,\Omega}$ and $\mathbf{\Phi}$ preserves the divergence-free property in Ω_1.

We denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality pairing between $\mathbf{\Psi}^*$ and $\mathbf{\Psi}$ that is generated from the weighted L^2 inner product $[[\cdot,\cdot]]$. The norm on the dual space $\mathbf{\Psi}^*$ is defined in the conventional manner:

$$\|\mathbf{g}\|_{\mathbf{\Psi}^*} = \sup_{\boldsymbol{\eta}\in\mathbf{\Psi},\, \|\boldsymbol{\eta}\|_{1,\Omega}\le 1} |\langle\langle \mathbf{g},\boldsymbol{\eta}\rangle\rangle| \qquad \forall\, \mathbf{g} \in \mathbf{\Psi}^*.$$

We define the bilinear forms

$$a_1[\mathbf{u},\mathbf{v}] = \frac{1}{2}\int_{\Omega_1} \mu_1(\nabla\mathbf{u} + \nabla\mathbf{u}^T) : (\nabla\mathbf{v} + \nabla\mathbf{v}^T)\, d\Omega \qquad \forall\, \mathbf{u},\mathbf{v} \in X_1,$$

$$b[\mathbf{v},q] = -\int_{\Omega_1} q\nabla\cdot\mathbf{v}\, d\Omega \qquad \forall\, \mathbf{v} \in X_1,\ \forall\, q \in L^2(\Omega_1)$$

and

$$a_2[\mathbf{u},\mathbf{v}] = \int_{\Omega_2} \Big\{ \frac{\mu_2}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) : (\nabla\mathbf{v} + \nabla\mathbf{v}^T) + \lambda_2(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v}) \Big\}\, d\Omega \quad \forall\, \mathbf{u},\mathbf{v} \in X_2.$$

It is evident that the forms $a_1[\cdot,\cdot]$, $a_2[\cdot,\cdot]$ and $b[\cdot,\cdot]$ are continuous, i.e., there exist positive constants K_1, K_2 and K_b such that

$$|a_1[\mathbf{u},\mathbf{v}]| \le K_1\|\mathbf{u}\|_{1,\Omega_1}\|\mathbf{v}\|_{1,\Omega_1} \qquad \forall\, \mathbf{u},\mathbf{v} \in X_1,$$

$$|a_2[\mathbf{u},\mathbf{v}]| \le K_2\|\mathbf{u}\|_{1,\Omega_2}\|\mathbf{v}\|_{1,\Omega_2} \qquad \forall\, \mathbf{u},\mathbf{v} \in X_2,$$

$$|b[\mathbf{v},q]| \le K_b\|\mathbf{v}\|_{1,\Omega_1}\|q\|_{0,\Omega_1} \qquad \forall\, \mathbf{v} \in X_1, \quad \forall\, q \in L^2(\Omega_1).$$

Also, it can be verified with the help of Korn's inequalities [26, p.31, p.120] that for $i = 1,2$,

$$a_i[\boldsymbol{\eta},\boldsymbol{\eta}] \ge k_i\|\boldsymbol{\eta}\|^2_{1,\Omega_i} \quad \forall\, \boldsymbol{\eta} \in X_i, \qquad \text{if } meas(\Gamma_i) \ne 0 \tag{2.2}$$

and

$$[\boldsymbol{\eta},\boldsymbol{\eta}]_{\Omega_i} + a_i[\boldsymbol{\eta},\boldsymbol{\eta}] \geq k_i \|\boldsymbol{\eta}\|^2_{1,\Omega_i} \quad \forall\, \boldsymbol{\eta} \in X_i, \qquad \text{if } meas(\Gamma_i) = 0. \tag{2.3}$$

2.2. **A divergence-free weak formulation.** We are now prepared to introduce a weak formulation for (1.1)-(1.4). We assume that the fluid force $\mathbf{f}_1$, the elastic force $\mathbf{f}_2$ and the initial data $\mathbf{v}_0$, $\mathbf{u}_0$ and $\mathbf{u}_1$ satisfy

$$\begin{cases} \mathbf{f}_1 \in L^2(0,T;\mathbf{L}^2(\Omega_1)), \quad \mathbf{f}_2 \in L^2(0,T;\mathbf{L}^2(\Omega_2)), \quad \mathbf{u}_0 \in X_2, \\ \mathbf{v}_0 \in X_1, \quad \operatorname{div}\mathbf{v}_0 = 0 \text{ in } \Omega_1, \quad \mathbf{u}_1 \in X_2, \quad \mathbf{v}_0|_{\Gamma_0} = \mathbf{u}_1|_{\Gamma_0}. \end{cases} \tag{2.4}$$

A desired weak formulation for (1.1) and (1.2) can be derived by multiplying the Stokes and elasticity equations by an $\boldsymbol{\eta} \in \mathbf{H}^1_0(\Omega)$ and performing integration by parts; this yields

$$\begin{aligned} &\rho_1[\mathbf{v}_t,\boldsymbol{\eta}]_{\Omega_1} + b[\boldsymbol{\eta},p] + a_1[\mathbf{v},\boldsymbol{\eta}] + \rho_2[\mathbf{u}_{tt},\boldsymbol{\eta}]_{\Omega_2} + a_2[\mathbf{u},\boldsymbol{\eta}] \\ &\qquad = \rho_1[\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2} \qquad \forall\, \boldsymbol{\eta} \in \mathbf{H}^1_0(\Omega), \text{ a.e. } t \in [0,T]. \end{aligned} \tag{2.5}$$

The choice of a global test function $\boldsymbol{\eta} \in \mathbf{H}^1_0(\Omega)$ instead of two independent test functions on the two subdomains allows us to incorporate the stress interface condition (1.4) into the weak formulation.

The weak form (2.5) requires

$$\begin{aligned} &\mathbf{v}_t \in L^2(0,T;\mathbf{L}^2(\Omega_1)), \quad \mathbf{u}_t \in L^2(0,T;X_2), \\ &\mathbf{u}_{tt} \in L^2(0,T;\mathbf{L}^2(\Omega_2)), \quad p \in L^2(0,T;L^2(\Omega_1)). \end{aligned}$$

In general, such regularity conditions are too strong to assume for a weak solution. Analogous to the standard practice [30] of defining a weak formulation for the Stokes equation with respect to divergence-free function spaces, we introduce the following divergence-free weak formulation for (1.1)-(1.4): seek a pair $(\mathbf{v},\mathbf{u}) \in L^2(0,T;X_1) \times L^2(0,T;X_2)$ satisfying $\operatorname{div}\mathbf{v} = 0$ in Ω_1,

$$\begin{aligned} &\frac{d}{dt}\Big(\rho_1[\mathbf{v},\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\partial_t\mathbf{u},\boldsymbol{\eta}]_{\Omega_2}\Big) + a_1[\mathbf{v},\boldsymbol{\eta}] + a_2[\mathbf{u},\boldsymbol{\eta}] \\ &\qquad\qquad = \rho_1[\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2} \quad \forall\, \boldsymbol{\eta} \in \boldsymbol{\Psi}, \end{aligned} \tag{2.6}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \qquad \mathbf{u}|_{t=0} = \mathbf{u}_0, \qquad \mathbf{u}_t|_{t=0} = \mathbf{u}_1 \tag{2.7}$$

and

$$\int_0^t \mathbf{v}(s)|_{\Gamma_0}\, ds = \mathbf{u}(t)|_{\Gamma_0} - \mathbf{u}_0|_{\Gamma_0} \qquad \text{a.e. } t, \tag{2.8}$$

where (2.6) holds in the sense of distributions on $(0,T)$ (see [30].) Equations (2.6)–(2.8) will be the primary weak formulation we study in this paper. Note that the pressure does not appear in (2.6)–(2.8). A weak formulation involving the pressure will be considered in Section 3.2.

The "natural" interface condition (1.4) is built into the weak form (2.5) and (2.6). In the divergence-free weak formulation, we enforce the "essential" interface condition (1.3) in the weak sense (2.8).

2.3. **An auxiliary weak formulation and its Galerkin approximations.** Introducing auxiliary functions

$$\boldsymbol{\xi} = \begin{cases} \mathbf{v} & \text{in } \Omega_1 \\ \mathbf{u}_t & \text{in } \Omega_2 \end{cases} \quad \text{and} \quad \boldsymbol{\xi}_0 = \begin{cases} \mathbf{v}_0 & \text{in } \Omega_1 \\ \mathbf{u}_1 & \text{in } \Omega_2, \end{cases} \tag{2.9}$$

we see that (2.6)–(2.8) is equivalent to the auxiliary problem

$$\begin{aligned} &\langle\langle \partial_t \boldsymbol{\xi}, \boldsymbol{\eta} \rangle\rangle + a_1[\boldsymbol{\xi}, \boldsymbol{\eta}] + a_2[\textstyle\int_0^t \boldsymbol{\xi}(s)\, ds, \boldsymbol{\eta}] \\ &\qquad = \rho_1 [\mathbf{f}_1, \boldsymbol{\eta}]_{\Omega_1} + \rho_2 [\mathbf{f}_2, \boldsymbol{\eta}]_{\Omega_2} - a_2[\mathbf{u}_0, \boldsymbol{\eta}] \qquad \forall \boldsymbol{\eta} \in \boldsymbol{\Psi}, \text{ a.e. } t \in [0,T], \end{aligned} \tag{2.10}$$

$$\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \qquad \text{in } \boldsymbol{\Psi}^* \tag{2.11}$$

and

$$\int_0^t \Big(\boldsymbol{\xi}(s)|_{\Omega_1} \Big)\Big|_{\Gamma_0} ds = \int_0^t \Big(\boldsymbol{\xi}(s)|_{\Omega_2} \Big)\Big|_{\Gamma_0} ds \qquad \text{a.e. } t. \tag{2.12}$$

Here we recall that the space $\boldsymbol{\Psi}$ introduced in Section 2.1 is divergence-free in Ω_1 so that the term $b[\boldsymbol{\eta}, p]$ vanishes. Under assumption (2.4), $\boldsymbol{\xi}_0$ defined by (2.9) obviously satisfies $\boldsymbol{\xi}_0 \in \boldsymbol{\Psi}$. The initial condition (2.11) is equivalent to

$$\langle\langle \boldsymbol{\xi}(0), \boldsymbol{\eta} \rangle\rangle = [[\boldsymbol{\xi}_0, \boldsymbol{\eta}]] \qquad \forall \boldsymbol{\eta} \in \boldsymbol{\Psi}.$$

The precise meaning of the initial condition requires certain continuity of $\boldsymbol{\xi}$ in t and will be made clear in subsequent discussions.

To prove the existence of a solution for (2.10)–(2.12), we employ and adapt the widely-used Galerkin approach (see, e.g., [9] and [24].) Let $\{\boldsymbol{\psi}_j\}_{j=1}^\infty$ be a basis for the space $\boldsymbol{\Psi}$. In particular, we choose $\{\boldsymbol{\psi}_j\}_{j=1}^\infty$ to be the complete set of eigenfunctions for the eigenvalue problem

$$\boldsymbol{\psi} \in \boldsymbol{\Psi}, \qquad [\nabla \boldsymbol{\psi}, \nabla \boldsymbol{\eta}]_\Omega = \lambda [[\boldsymbol{\psi}, \boldsymbol{\eta}]] \quad \forall \boldsymbol{\eta} \in \boldsymbol{\Psi}, \tag{2.13}$$

where the weighted $\mathbf{L}^2(\Omega)$ inner product $[[\cdot,\cdot]]$ is defined by (2.1). Moreover, we assume that $\{\boldsymbol{\psi}_j\}_{j=1}^\infty$ is orthonormalized with respect to the $\mathbf{H}_0^1(\Omega)$-inner product $[\nabla \cdot, \nabla \cdot]_\Omega$. Then, from equation (2.13) we also have that $\{\boldsymbol{\psi}_j\}_{j=1}^\infty$ is orthogonal with respect to the weighted $\mathbf{L}^2(\Omega)$ inner product $[[\cdot,\cdot]]$.

Let $\boldsymbol{\Psi}_m = \operatorname{span}\{\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_m\}$. A Galerkin approximation $\boldsymbol{\xi}_m \in C^1([0,T]; \boldsymbol{\Psi}_m)$ of the form

$$\boldsymbol{\xi}_m = \sum_{j=1}^m g_j^{(m)}(t) \boldsymbol{\psi}_j(\mathbf{x}) \tag{2.14}$$

is defined as the solution of

$$\begin{aligned} &\rho_1 [\partial_t \boldsymbol{\xi}_m(t), \boldsymbol{\eta}]_{\Omega_1} + \rho_2 [\partial_t \boldsymbol{\xi}_m(t), \boldsymbol{\eta}]_{\Omega_2} + a_1[\boldsymbol{\xi}_m(t), \boldsymbol{\eta}] + a_2[\textstyle\int_0^t \boldsymbol{\xi}_m(s)\, ds, \boldsymbol{\eta}] \\ &\qquad = [[\mathbf{f}(t), \boldsymbol{\eta}]] - a_2[\mathbf{u}_0, \boldsymbol{\eta}] + a_1[\boldsymbol{\xi}_m(0) - \mathbf{v}_0, \boldsymbol{\eta}] \qquad \forall \boldsymbol{\eta} \in \boldsymbol{\Psi}_m\,, \ t \in [0,T] \end{aligned} \tag{2.15}$$

and

$$[[\boldsymbol{\xi}_m(0), \boldsymbol{\eta}]] = [[\boldsymbol{\xi}_0, \boldsymbol{\eta}]] \qquad \forall \boldsymbol{\eta} \in \boldsymbol{\Psi}_m\,, \tag{2.16}$$

where $\mathbf{f} \in L^2(0,T; \mathbf{L}^2(\Omega))$ is defined by

$$\mathbf{f} = \begin{cases} \mathbf{f}_1 & \text{in } \Omega_1 \\ \mathbf{f}_2 & \text{in } \Omega_2. \end{cases}$$

Note that $\boldsymbol{\xi}_m$ defined as in (2.14) trivially satisfies

$$\int_0^t \Big(\boldsymbol{\xi}_m(s)|_{\Omega_1} \Big)|_{\Gamma_0} ds = \int_0^t \Big(\boldsymbol{\xi}_m(s)|_{\Omega_2} \Big)|_{\Gamma_0} ds \qquad \forall t \in [0,T]. \tag{2.17}$$

Remark: Comparing (2.15) and (2.10), the term $a_1[\boldsymbol{\xi}_m(0)-\mathbf{v}_0,\boldsymbol{\eta}]$ appears to be redundant. Indeed, it will be shown in Lemma 2.2 that $\|\boldsymbol{\xi}_m(0)-\boldsymbol{\xi}_0\|_{1,\Omega}\to 0$ so that $a_1[\boldsymbol{\xi}_m(0)-\mathbf{v}_0,\boldsymbol{\eta}]\to 0$ as $m\to\infty$. The term $a_1[\boldsymbol{\xi}_m(0)-\mathbf{v}_0,\boldsymbol{\eta}]$ is added into (2.15) for technical reasons connected with the derivation of strong a priori estimates in Section 3.1.

We can write (2.15)–(2.16) as an equivalent system of first-order, linear ordinary differential equations (ODEs) for $\{g_j^{(m)}\}_{j=1}^m$:

$$\begin{aligned}&\sum_{j=1}^m\Big([[\boldsymbol{\psi}_j,\boldsymbol{\psi}_i]]\frac{d}{dt}g_j^{(m)}(t)+a_1[\boldsymbol{\psi}_j,\boldsymbol{\psi}_i]\,g_j^{(m)}(t)+a_2[\boldsymbol{\psi}_j,\boldsymbol{\psi}_i]\int_0^t g_j^{(m)}(s)\,ds\Big)\\&=[[\mathbf{f}(t),\boldsymbol{\psi}_i]]-a_2[\mathbf{u}_0,\boldsymbol{\psi}_i]+a_1[\boldsymbol{\xi}_m(0)-\mathbf{v}_0,\boldsymbol{\psi}_i]\quad i=1,\dots,m,\ t\in[0,T]\end{aligned}\tag{2.18}$$

and

$$\sum_{j=1}^m[[\boldsymbol{\psi}_j,\boldsymbol{\psi}_i]]\,g_j^{(m)}(0)=[[\boldsymbol{\xi}_0,\boldsymbol{\psi}_i]]\qquad i=1,\dots,m.\tag{2.19}$$

The following theorem states the existence of a solution for (2.18)–(2.19), or equivalently, (2.15)–(2.16).

Theorem 2.1. *Assume that $\mathbf{f}_1,\mathbf{v}_0,\mathbf{f}_2,\mathbf{u}_0$ and $\mathbf{u}_1$ satisfy* (2.4)*. Then, for each integer $m>0$, there exists a unique set of functions $\{g_j^{(m)}\}_{j=1}^m\subset C^1[0,T]$ which satisfy* (2.18)*–*(2.19)*.*

Proof: Setting $h_j^{(m)}(t)=\int_0^t g_j^{(m)}(s)\,ds$ for $j=1,\dots,m$, we see that (2.18)–(2.19) is equivalent to the following ODE initial value problem:

$$\begin{cases}\displaystyle\sum_{j=1}^m[[\boldsymbol{\psi}_j,\boldsymbol{\psi}_i]]\frac{d}{dt}g_j^{(m)}(t)+\sum_{j=1}^m a_1[\boldsymbol{\psi}_j,\boldsymbol{\psi}_i]\,g_j^{(m)}(t)+\sum_{j=1}^m a_2[\boldsymbol{\psi}_j,\boldsymbol{\psi}_i]\,h_j^{(m)}(t)\\ \qquad\qquad\qquad\qquad=c_i(t)\qquad i=1,\dots,m,\\ \displaystyle\frac{d}{dt}h_i^{(m)}(t)=g_i^{(m)}(t)\qquad i=1,\dots,m,\\ \displaystyle\sum_{j=1}^m[[\boldsymbol{\psi}_j,\boldsymbol{\psi}_i]]\,g_j^{(m)}(0)=[[\boldsymbol{\xi}_0,\boldsymbol{\psi}_i]]\qquad i=1,\dots,m,\\ h_i^{(m)}(0)=0,\qquad i=1,\dots,m,\end{cases}$$

where

$$c_i(t)=\rho_1[\mathbf{f}_1(t),\boldsymbol{\psi}_i]_{\Omega_1}+\rho_2[\mathbf{f}_2(t),\boldsymbol{\psi}_i]_{\Omega_2}-a_2[\mathbf{u}_0,\boldsymbol{\psi}_i]+\sum_{j=1}^m g_j^{(m)}(0)a_1[\boldsymbol{\psi}_j,\boldsymbol{\psi}_i]-a_1[\mathbf{v}_0,\boldsymbol{\psi}_i].$$

The matrix $\{[[\boldsymbol{\psi}_i,\boldsymbol{\psi}_j]]\}_{i,j=1}^m$ is positive definite as the function set $\{\boldsymbol{\psi}_1,\dots,\boldsymbol{\psi}_m\}$ is linearly independent. Thus, using standard theories for (constant coefficient) systems of linear, first-order ODEs we see that the above ODE system has a unique C^1 solution $(g_1^{(m)},\dots,g_m^{(m)},h_1^{(m)},\dots,h_m^{(m)})$ on $[0,T]$. Upon eliminating $h_j^{(m)}$ in the system, we arrive at the assertion of the theorem. □

Theorem 2.1 immediately yields the existence of the Galerkin approximate solutions $\{\boldsymbol{\xi}_m\}$ satisfying (2.15)–(2.16). We proceed to derive a priori estimates for $\{\boldsymbol{\xi}_m\}$.

Lemma 2.2. *Let P_m denote the weighted $\mathbf{L}^2(\Omega)$ projection from $\mathbf{L}^2(\Omega)$ onto $\mathbf{\Psi}_m$, i.e., for every $\boldsymbol{\eta} \in \mathbf{L}^2(\Omega)$,*

$$[[P_m\boldsymbol{\eta}, \mathbf{z}]] = [[\boldsymbol{\eta}, \mathbf{z}]] \qquad \forall\, \mathbf{z} \in \mathbf{\Psi}_m. \tag{2.20}$$

Then,

$$\|P_m\boldsymbol{\eta}\|_{0,\Omega} \le \|\boldsymbol{\eta}\|_{0,\Omega} \qquad \forall\, \boldsymbol{\eta} \in \mathbf{L}^2(\Omega), \tag{2.21}$$

$$\|P_m\boldsymbol{\eta}\|_{1,\Omega} \le \|\boldsymbol{\eta}\|_{1,\Omega} \qquad \forall\, \boldsymbol{\eta} \in \mathbf{\Psi}, \tag{2.22}$$

$$\|P_m\boldsymbol{\eta} - \boldsymbol{\eta}\|_{1,\Omega} \to 0 \text{ as } m \to \infty \qquad \forall\, \boldsymbol{\eta} \in \mathbf{\Psi} \tag{2.23}$$

and

$$\|P_m\boldsymbol{\eta} - \boldsymbol{\eta}\|_{0,\Omega} \to 0 \text{ as } m \to \infty \qquad \forall\, \boldsymbol{\eta} \in \mathbf{\Phi}. \tag{2.24}$$

PROOF: Setting $\mathbf{z} = P_m\boldsymbol{\eta}$ in (2.20) and applying the Cauchy-Schwarz inequality we easily obtain (2.21).

Next, we prove (2.22)–(2.23). Let $\boldsymbol{\eta} \in \mathbf{\Psi}$ be given. Since $\{\boldsymbol{\psi}_i\}_{i=1}^\infty$ forms a basis for $\mathbf{\Psi}$, we can write $\boldsymbol{\eta} = \sum_{i=1}^\infty \alpha_i \boldsymbol{\psi}_i$; moreover, we have that

$$\Big\| \sum_{i=1}^m \alpha_i \boldsymbol{\psi}_i \Big\|_{1,\Omega} \le \|\boldsymbol{\eta}\|_{1,\Omega} \qquad \text{and} \qquad \Big\| \sum_{i=1}^m \alpha_i \boldsymbol{\psi}_i - \boldsymbol{\eta} \Big\|_{1,\Omega} \to 0 \text{ as } m \to \infty. \tag{2.25}$$

Using the orthogonality of $\{\boldsymbol{\psi}_i\}$ with respect to the $[[\cdot,\cdot]]$ inner product we find $P_m\boldsymbol{\eta} = \sum_{i=1}^m \alpha_i \boldsymbol{\psi}_i$. Hence, (2.22)–(2.23) follows from (2.25).

It remains to prove (2.24). Let $\boldsymbol{\eta} \in \mathbf{\Phi}$ be given. Using the triangle inequality and (2.21) we have

$$\begin{aligned} \|P_m\boldsymbol{\eta} - \boldsymbol{\eta}\|_{0,\Omega} &\le \|\boldsymbol{\eta} - \boldsymbol{\eta}_\epsilon\|_{0,\Omega} + \|\boldsymbol{\eta}_\epsilon - P_m\boldsymbol{\eta}\|_{0,\Omega} + \|P_m(\boldsymbol{\eta}_\epsilon - \boldsymbol{\eta})\|_{0,\Omega} \\ &\le 2\|\boldsymbol{\eta} - \boldsymbol{\eta}_\epsilon\|_{0,\Omega} + \|\boldsymbol{\eta}_\epsilon - P_m\boldsymbol{\eta}\|_{0,\Omega} \qquad \forall\, \boldsymbol{\eta}_\epsilon \in \mathbf{\Psi}. \end{aligned} \tag{2.26}$$

From relations (2.23) and (2.26) as well as the denseness of $\mathbf{\Psi}$ in $\mathbf{\Phi}$ for the $\mathbf{L}^2(\Omega)$ norm we obtain (2.24). □

Theorem 2.3. *Assume that $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$ and $\mathbf{u}_1$ satisfy (2.4). Then, for each integer $m > 0$, there exists a function $\boldsymbol{\xi}_m \in C^1([0,T]; \mathbf{\Psi}_m)$ of the form (2.14) which satisfies (2.15)–(2.17). Moreover,*

$$\begin{aligned} &\|\boldsymbol{\xi}_m(t)\|_{0,\Omega}^2 + \|\boldsymbol{\xi}_m\|_{L^2(0,T;\mathbf{H}^1(\Omega_1))}^2 + \|\textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds\|_{\mathbf{H}^1(\Omega_2)}^2 \\ &\quad \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{v}_0\|_{1,\Omega_2}^2 + \|\mathbf{u}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2\Big) \end{aligned} \tag{2.27}$$

for all $t \in [0,T]$,

$$\begin{aligned} &\|\boldsymbol{\xi}_m'\|_{L^2(0,T;\mathbf{\Psi}^*)}^2 \\ &\quad \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{v}_0\|_{1,\Omega_2}^2 + \|\mathbf{u}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2\Big), \end{aligned} \tag{2.28}$$

$$\begin{aligned} &\|\partial_t[\boldsymbol{\xi}_m|_{\Omega_1}]\|_{L^2(0,T;V_1^*)}^2 \\ &\quad \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{v}_0\|_{1,\Omega_2}^2 + \|\mathbf{u}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2\Big), \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} &\|\partial_t[\boldsymbol{\xi}_m|_{\Omega_2}]\|_{L^2(0,T;\mathbf{H}^{-1}(\Omega_2))}^2 \\ &\quad \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{v}_0\|_{1,\Omega_2}^2 + \|\mathbf{u}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2\Big). \end{aligned} \tag{2.30}$$

PROOF: The existence of $\boldsymbol{\xi}_m$ follows directly from Theorem 2.1 so we only need to prove the a priori estimates (2.27)–(2.30).

Setting $\boldsymbol{\eta} = \boldsymbol{\xi}_m(t)$ in (2.15) and using the weighted $\mathbf{L}^2(\Omega)$ inner product notation, we have

$$[[\boldsymbol{\xi}'_m(t), \boldsymbol{\xi}_m(t)]] + a_1[\boldsymbol{\xi}_m(t), \boldsymbol{\xi}_m(t)] + a_2[\textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds, \boldsymbol{\xi}_m(t)] \\ = [[\mathbf{f}(t), \boldsymbol{\xi}_m(t)]] - a_2[\mathbf{u}_0, \boldsymbol{\xi}_m(t)] + a_1[\boldsymbol{\xi}_m(0) - \mathbf{v}_0, \boldsymbol{\xi}_m(t)]$$

which can be rewritten as

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}[[\boldsymbol{\xi}_m(t), \boldsymbol{\xi}_m(t)]] &+ a_1[\boldsymbol{\xi}_m(t), \boldsymbol{\xi}_m(t)] + \frac{1}{2}\frac{d}{dt}a_2[\textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds, \int_0^t \boldsymbol{\xi}_m(s)\,ds] \\ &= [[\mathbf{f}(t), \boldsymbol{\xi}_m(t)]] - a_2[\mathbf{u}_0, \boldsymbol{\xi}_m(t)] + a_1[\boldsymbol{\xi}_m(0) - \mathbf{v}_0, \boldsymbol{\xi}_m(t)] \\ &\le C(\|\mathbf{f}(t)\|_{0,\Omega}^2 + \|\boldsymbol{\xi}_m(0) - \mathbf{v}_0\|_{1,\Omega_1}^2) \\ &\qquad + \frac{1}{2}\|\boldsymbol{\xi}_m(t)\|_{0,\Omega}^2 - a_2[\mathbf{u}_0, \boldsymbol{\xi}_m(t)] + \frac{1}{2}a_1[\boldsymbol{\xi}_m(t), \boldsymbol{\xi}_m(t)]\,. \end{aligned}$$

Integrating the last relation in t and noting that (Lemma 2.2)

$$\|\boldsymbol{\xi}_m(0)\|_{1,\Omega}^2 = \|P_m\boldsymbol{\xi}_0\|_{1,\Omega}^2 \le \|\boldsymbol{\xi}_0\|_{1,\Omega}^2 = \|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2$$

we are led to

$$\begin{aligned} [[\boldsymbol{\xi}_m(t), \boldsymbol{\xi}_m(t)]] &+ \int_0^t a_1[\boldsymbol{\xi}_m(s), \boldsymbol{\xi}_m(s)]\,ds + a_2[\textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds, \int_0^t \boldsymbol{\xi}_m(s)\,ds] \\ &\le C\Big(\|\boldsymbol{\xi}_m(0)\|_{0,\Omega}^2 + \|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + T\|\boldsymbol{\xi}_m(0) - \mathbf{v}_0\|_{1,\Omega_1}^2\Big) \\ &\qquad + \int_0^t \|\boldsymbol{\xi}_m(s)\|_{0,\Omega}^2\,ds - a_2[\mathbf{u}_0, \textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds] \\ &\le C\Big(\|\boldsymbol{\xi}_m(0)\|_{0,\Omega}^2 + \|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + T\|\mathbf{v}_0\|_{1,\Omega_1}^2 + T\|\mathbf{u}_1\|_{1,\Omega_2}^2\Big) \\ &\qquad + \int_0^t \|\boldsymbol{\xi}_m(s)\|_{0,\Omega}^2\,ds + \frac{1}{2}a_2[\textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds, \int_0^t \boldsymbol{\xi}_m(s)\,ds] \end{aligned}$$

so that

$$\begin{aligned} \|\boldsymbol{\xi}_m(t)\|_{0,\Omega}^2 &+ \int_0^t a_1[\boldsymbol{\xi}_m(s), \boldsymbol{\xi}_m(s)]\,ds + a_2[\textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds, \int_0^t \boldsymbol{\xi}_m(s)\,ds] \\ &\le \int_0^t \|\boldsymbol{\xi}_m(s)\|_{0,\Omega}^2\,ds \\ &\qquad + C\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + T\|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + T\|\mathbf{u}_1\|_{1,\Omega_2}^2\Big)\,. \end{aligned} \tag{2.31}$$

Dropping the second and third terms on the left side of (2.31) and then applying the following version of Gronwall's inequality:

$$\text{if } h(t) \le C_1 + C_2 \textstyle\int_0^t h(s)\,ds, \text{ then } h(t) \le \dfrac{C_1}{C_2}e^{C_2 t}, \tag{2.32}$$

we deduce

$$\|\boldsymbol{\xi}_m(t)\|_{0,\Omega}^2 \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2\Big). \tag{2.33}$$

The estimates (2.33) and (2.31) yield: for all $t \in [0, T]$,

$$\begin{aligned}&\int_0^t \|\boldsymbol{\xi}_m(s)\|^2_{0,\Omega_1}\,ds + \int_0^t a_1[\boldsymbol{\xi}_m(s), \boldsymbol{\xi}_m(s)]\,ds \\ &\qquad \le Ce^{CT}\Big(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{v}_0\|^2_{1,\Omega_1} + \|\mathbf{u}_0\|^2_{1,\Omega_2} + \|\mathbf{u}_1\|^2_{1,\Omega_2}\Big)\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}&\|\textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds\|^2_{0,\Omega_2} + a_2[\int_0^t \boldsymbol{\xi}_m(s)\,ds, \int_0^t \boldsymbol{\xi}_m(s)\,ds] \\ &\qquad \le Ce^{CT}\Big(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{v}_0\|^2_{1,\Omega_1} + \|\mathbf{u}_0\|^2_{1,\Omega_2} + \|\mathbf{u}_1\|^2_{1,\Omega_2}\Big).\end{aligned} \tag{2.35}$$

Combining (2.33)–(2.35) and utilizing the coercivity conditions (2.2) or (2.3) we arrive at the desired energy estimate (2.27).

We next prove (2.28). For each given $\boldsymbol{\eta} \in \boldsymbol{\Psi}$ with $\|\boldsymbol{\eta}\|_{1,\Omega} \le 1$, we write $\boldsymbol{\eta}$ as $\boldsymbol{\eta} = P_m\boldsymbol{\eta} + (\boldsymbol{\eta} - P_m\boldsymbol{\eta})$ where P_m is defined by (2.20). Since $\boldsymbol{\xi}'_m(t) \in \boldsymbol{\Psi}_m$, we have

$$[[\boldsymbol{\xi}'_m(t), \boldsymbol{\eta}]] = [[\boldsymbol{\xi}'_m(t), P_m\boldsymbol{\eta}]] + [[\boldsymbol{\xi}'_m(t), \boldsymbol{\eta} - P_m\boldsymbol{\eta}]] = [[\boldsymbol{\xi}'_m(t), P_m\boldsymbol{\eta}]].$$

From the last equation and (2.15) we obtain

$$\begin{aligned}[[\boldsymbol{\xi}'_m(t), \boldsymbol{\eta}]] &= [[\boldsymbol{\xi}'_m(t), P_m\boldsymbol{\eta}]] \\ &= [[\mathbf{f}(t), P_m\boldsymbol{\eta}]] - a_1[\boldsymbol{\xi}_m(t), P_m\boldsymbol{\eta}] - a_2[\mathbf{u}_0, P_m\boldsymbol{\eta}] \\ &\qquad -a_2[\textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds, P_m\boldsymbol{\eta}] + a_1[\boldsymbol{\xi}_m(0) - \mathbf{v}_0, P_m\boldsymbol{\eta}] \\ &\le C\Big(\|\mathbf{f}(t)\|_{0,\Omega} + \|\boldsymbol{\xi}_m(t)\|_{1,\Omega_1} + \|\textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds\|_{1,\Omega_2} \\ &\qquad\qquad +\|\mathbf{u}_0\|_{1,\Omega_2} + \|\mathbf{u}_1\|_{1,\Omega_2} + \|\mathbf{v}_0\|_{1,\Omega_1}\Big)\|P_m\boldsymbol{\eta}\|_{1,\Omega}.\end{aligned} \tag{2.36}$$

Thus, (2.28) follows from (2.36), (2.27) and the fact that $\|P_m\boldsymbol{\eta}\|_{1,\Omega} \le \|\boldsymbol{\eta}\|_{1,\Omega} \le 1$ (see Lemma 2.2.)

To prove (2.29), we arbitrarily choose a $\mathbf{v} \in V_1$ with $\|\mathbf{v}\|_{1,\Omega_1} \le 1$. We then define an $\boldsymbol{\eta} \in \boldsymbol{\Psi}$ by $\boldsymbol{\eta}|_{\Omega_1} = \mathbf{v}$ and $\boldsymbol{\eta}|_{\Omega_2} = \mathbf{0}$. Evidently we have that $\|\boldsymbol{\eta}\|_{1,\Omega} = \|\mathbf{v}\|_{1,\Omega_1} \le 1$ and

$$[\boldsymbol{\xi}'_m(t), \mathbf{v}]_{\Omega_1} = [\boldsymbol{\xi}'_m(t), \boldsymbol{\eta}]_{\Omega} \le C\|\boldsymbol{\xi}'_m(t)\|_{\boldsymbol{\Psi}^*}.$$

The last estimate together with (2.28) implies (2.29).

We may prove (2.30) analogously. □

2.4. **Existence and uniqueness of a weak solution.** By passing to the limit in the Galerkin approximations, we may prove the existence of a solution $\boldsymbol{\xi}$ for the auxiliary problem (2.10)–(2.11) which in turn directly yields the existence and uniqueness of a solution to the fluid-structure interaction problem. Note that as in the Galerkin approximations, we consider in this subsection the existence of an auxiliary weak solution on the space $\boldsymbol{\Psi}$ so that the weak formulation does not contain the pressure term.

Theorem 2.4. *Assume that $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2$, $\mathbf{u}_0$ and $\mathbf{u}_1$ satisfy* (2.4). *Then, there exists a unique $\boldsymbol{\xi} \in L^2(0,T;\mathbf{L}^2(\Omega)) \cap H^1(0,T;\boldsymbol{\Psi}^*)$ which satisfies*

$$\boldsymbol{\xi}|_{\Omega_1} \in L^2(0,T;X_1), \qquad \operatorname{div}\boldsymbol{\xi}|_{\Omega_1} = 0, \qquad \int_0^t \boldsymbol{\xi}(s)|_{\Omega_2}\,ds \in L^\infty(0,T;X_2) \tag{2.37}$$

and (2.10)–(2.12). *Moreover,*

$$\begin{aligned}&\|\boldsymbol{\xi}(t)\|_{0,\Omega}^2 + \|\boldsymbol{\xi}\|_{L^2(0,T;\mathbf{H}^1(\Omega_1))}^2 + \|\textstyle\int_0^t \boldsymbol{\xi}(s)\,ds\|_{\mathbf{H}^1(\Omega_2)}^2\\ &\qquad \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + \|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2\Big)\end{aligned} \tag{2.38}$$

for all $t \in [0,T]$,

$$\begin{aligned}&\|\boldsymbol{\xi}'\|_{L^2(0,T;\boldsymbol{\Psi}^*)}^2\\ &\qquad \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + \|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2\Big),\end{aligned} \tag{2.39}$$

$$\begin{aligned}&\|\partial_t[\boldsymbol{\xi}|_{\Omega_1}]\|_{L^2(0,T;V_1^*)}^2\\ &\qquad \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + \|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2\Big),\end{aligned} \tag{2.40}$$

and

$$\begin{aligned}&\|\partial_t[\boldsymbol{\xi}|_{\Omega_2}]\|_{L^2(0,T;\mathbf{H}^{-1}(\Omega_2))}^2\\ &\qquad \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + \|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2\Big).\end{aligned} \tag{2.41}$$

Proof: Let $\boldsymbol{\xi}_m$ of the form (2.14) be a solution of (2.15)–(2.16). Using the initial condition and the energy estimates (2.27)–(2.28) we may extract subsequence of $\{\boldsymbol{\xi}_m\}$, still denoted by $\{\boldsymbol{\xi}_m\}$, such that

$$\boldsymbol{\xi}_m \overset{*}{\rightharpoonup} \boldsymbol{\xi} \quad \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega)), \qquad \boldsymbol{\xi}_m \rightharpoonup \boldsymbol{\xi} \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega)),$$
$$\boldsymbol{\xi}_m|_{\Omega_1} \rightharpoonup \boldsymbol{\xi}|_{\Omega_1} \quad \text{in } L^2(0,T;\mathbf{H}^1(\Omega_1)),$$
$$\int_0^t [\boldsymbol{\xi}_m(s)]|_{\Omega_2}\,ds \overset{*}{\rightharpoonup} \int_0^t [\boldsymbol{\xi}(s)]|_{\Omega_2}\,ds \quad \text{in } L^\infty(0,T;\mathbf{H}^1(\Omega_2))$$

and

$$\partial_t\boldsymbol{\xi}_m \rightharpoonup \partial_t\boldsymbol{\xi} \quad \text{in } L^2(0,T;\boldsymbol{\Psi}^*)$$

for some $\boldsymbol{\xi} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap H^1(0,T;\boldsymbol{\Psi}^*)$ satisfying (2.37). These weak limits also imply

$$\int_0^t [\boldsymbol{\xi}_m(s)|_{\Omega_1}]|_{\Gamma_0}\,ds \rightharpoonup \int_0^t [\boldsymbol{\xi}(s)|_{\Omega_1}]|_{\Gamma_0}\,ds \quad \text{in } L^2(0,T;\mathbf{H}^{1/2}(\Gamma_0))$$

and

$$\int_0^t [\boldsymbol{\xi}_m(s)|_{\Omega_2}]|_{\Gamma_0}\,ds \rightharpoonup \int_0^t [\boldsymbol{\xi}(s)|_{\Omega_2}]|_{\Gamma_0}\,ds \quad \text{in } L^2(0,T;\mathbf{H}^{1/2}(\Gamma_0)).$$

Passing to the limit in (2.17), we obtain (2.12). Also, passing to the limit in (2.27)–(2.30) yields (2.38)–(2.41).

To prove $\boldsymbol{\xi}$ satisfies (2.10), we proceed as follows. We fix an integer N and choose a function $\boldsymbol{\eta} \in C^1([0,T];\boldsymbol{\Psi})$ having the form

$$\boldsymbol{\eta} = \sum_{j=1}^N d_j(t)\boldsymbol{\psi}_j. \tag{2.42}$$

For each $m > N$, we integrate (2.15) with respect to t to obtain

$$\begin{aligned}&\int_0^T \Big(\rho_1[\partial_t\boldsymbol{\xi}_m,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\partial_t\boldsymbol{\xi}_m,\boldsymbol{\eta}]_{\Omega_2} + a_1[\boldsymbol{\xi}_m,\boldsymbol{\eta}] + a_2[\textstyle\int_0^t \boldsymbol{\xi}_m(s)\,ds,\boldsymbol{\eta}]\Big)\,dt\\ &\qquad = \int_0^T \Big(\rho_1[\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2} - a_2[\mathbf{u}_0,\boldsymbol{\eta}] + a_1[\boldsymbol{\xi}_m(0)-\mathbf{v}_0,\boldsymbol{\eta}]\Big)\,dt.\end{aligned} \tag{2.43}$$

By passing to the limit as $m \to \infty$, we find

$$\begin{aligned}\int_0^T &\Big(\langle\langle \partial_t \boldsymbol{\xi}, \boldsymbol{\eta}\rangle\rangle + a_1[\boldsymbol{\xi}, \boldsymbol{\eta}) + a_2[\textstyle\int_0^t \boldsymbol{\xi}(s)\, ds, \boldsymbol{\eta}]\Big)\, dt \\ &= \int_0^T \Big(\rho_1[\mathbf{f}_1, \boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2, \boldsymbol{\eta}]_{\Omega_2} - a_2[\mathbf{u}_0, \boldsymbol{\eta}]\Big)\, dt.\end{aligned} \tag{2.44}$$

Here we have used the fact that (see Lemma 2.2)

$$\|\boldsymbol{\xi}_m(0) - \mathbf{v}_0\|_{1,\Omega_1} = \|P_m \boldsymbol{\xi}_0 - \boldsymbol{\xi}_0\|_{1,\Omega_1} \le \|P_m \boldsymbol{\xi}_0 - \boldsymbol{\xi}_0\|_{1,\Omega} \to 0 \quad \text{as } m \to \infty$$

Equality (2.44) then holds for all $\boldsymbol{\eta} \in L^2(0,T;\boldsymbol{\Psi})$, since functions of the form (2.42) are dense in $L^2(0,T;\boldsymbol{\Psi})$. In particular, (2.44) implies (2.10).

To verify the initial condition (2.11), we first note that the regularity $\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{\Psi})$ and $\partial_t \boldsymbol{\xi} \in L^2(0,T;\boldsymbol{\Psi}^*)$ implies that $\boldsymbol{\xi} \in C([0,T];\boldsymbol{\Psi}^*)$. On the one hand, by choosing an $\boldsymbol{\eta} \in C^1([0,T];\boldsymbol{\Psi})$ in (2.44) and integrating by parts we have

$$\begin{aligned}\int_0^T &\Big(-\langle\langle \boldsymbol{\xi}, \partial_t \boldsymbol{\eta}\rangle\rangle + a_1[\boldsymbol{\xi}, \boldsymbol{\eta}] + a_2[\textstyle\int_0^t \boldsymbol{\xi}_m(s)\, ds, \boldsymbol{\eta}]\Big)\, dt \\ &= \int_0^T \Big([[\mathbf{f}, \boldsymbol{\eta}]] - a_2[\mathbf{u}_0, \boldsymbol{\eta}]\Big)\, dt + \langle\langle \boldsymbol{\xi}(0), \boldsymbol{\eta}(0)\rangle\rangle.\end{aligned} \tag{2.45}$$

On the other hand, from (2.43) we deduce

$$\begin{aligned}\int_0^T &\Big(-\rho_1[\boldsymbol{\xi}_m, \partial_t \boldsymbol{\eta}]_{\Omega_1} - \rho_2[\boldsymbol{\xi}_m, \partial_t \boldsymbol{\eta}]_{\Omega_2} + a_1[\boldsymbol{\xi}_m, \boldsymbol{\eta}] + a_2[\textstyle\int_0^t \boldsymbol{\xi}_m(s)\, ds, \boldsymbol{\eta}]\Big) dt \\ &= \int_0^T \Big([[\mathbf{f}, \boldsymbol{\eta}]] - a_2[\mathbf{u}_0, \boldsymbol{\eta}] + a_1[\boldsymbol{\xi}_m(0) - \mathbf{v}_0, \boldsymbol{\eta}]\Big)\, dt + [[\boldsymbol{\xi}_m(0), \boldsymbol{\eta}(0)]].\end{aligned} \tag{2.46}$$

Passing to the limit in (2.46) yields

$$\begin{aligned}\int_0^T &\Big(-\langle\langle \boldsymbol{\xi}, \partial_t \boldsymbol{\eta}\rangle\rangle + a_1[\boldsymbol{\xi}, \boldsymbol{\eta}] + a_2[\textstyle\int_0^t \boldsymbol{\xi}(s)\, ds, \boldsymbol{\eta}]\Big)\, dt \\ &= \int_0^T \Big(\rho_1[\mathbf{f}_1, \boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2, \boldsymbol{\eta}]_{\Omega_2} - a_2[\mathbf{u}_0, \boldsymbol{\eta}]\Big)\, dt + [[\boldsymbol{\xi}_0, \boldsymbol{\eta}(0)]].\end{aligned} \tag{2.47}$$

Comparing (2.45) and (2.47) we obtain

$$\langle\langle \boldsymbol{\xi}(0) - \boldsymbol{\xi}_0, \boldsymbol{\eta}(0)\rangle\rangle = 0 \qquad \forall\, \boldsymbol{\eta}(0) \in \boldsymbol{\Psi}, \tag{2.48}$$

which is precisely (2.11) as $\boldsymbol{\eta}(0) \in \boldsymbol{\Psi}$ is arbitrary.

It remains to prove the uniqueness. The following proof is adapted from [9, pp.385-387]. Assume $\boldsymbol{\xi}$ and $\widetilde{\boldsymbol{\xi}}$ are two solutions of (2.10)–(2.11). Setting $\boldsymbol{\theta} = \boldsymbol{\xi} - \widetilde{\boldsymbol{\xi}}$ we have that

$$\boldsymbol{\theta} \in L^2(0,T;\boldsymbol{\Phi}), \quad \boldsymbol{\theta}' \in L^2(0,T;\boldsymbol{\Psi}^*),$$

$$\langle\langle \boldsymbol{\theta}', \boldsymbol{\eta}\rangle\rangle + a_1[\boldsymbol{\theta}, \boldsymbol{\eta}] + a_2[\textstyle\int_0^t \boldsymbol{\theta}(s)\, ds, \boldsymbol{\eta}] = 0 \qquad \forall\, \boldsymbol{\eta} \in \boldsymbol{\Psi}, \text{ a.e. } t, \tag{2.49}$$

and

$$\boldsymbol{\theta}(0) = \mathbf{0}. \tag{2.50}$$

We define $\boldsymbol{\zeta} = \int_0^t \boldsymbol{\theta}(s)\, ds$. Then

$$\boldsymbol{\zeta} \in L^2(0,T;\boldsymbol{\Psi}), \quad \boldsymbol{\zeta}' \in L^2(0,T;\boldsymbol{\Phi}), \quad \boldsymbol{\zeta}'' \in L^2(0,T;\boldsymbol{\Psi}^*),$$

$$\langle\langle \boldsymbol{\zeta}'', \boldsymbol{\eta}\rangle\rangle + a_1[\boldsymbol{\zeta}', \boldsymbol{\eta}] + a_2[\boldsymbol{\zeta}, \boldsymbol{\eta}] = 0 \qquad \forall\, \boldsymbol{\eta} \in \boldsymbol{\Psi}, \text{ a.e. } t, \tag{2.51}$$

$$\boldsymbol{\zeta}(0) = \mathbf{0} \quad \text{and} \quad \boldsymbol{\zeta}'(0) = \mathbf{0}, \tag{2.52}$$

where $(\cdot)'$ and $(\cdot)''$ denotes the first and second time derivatives, respectively. We note that from the regularity $\boldsymbol{\zeta} \in L^2(0,T;\boldsymbol{\Psi}) \cap H^1(0,T;\boldsymbol{\Psi}^*)$ and [30, p.176, Lemma

1.2] we deduce that $\boldsymbol{\zeta} \in C([0,T];\boldsymbol{\Phi})$; indeed, the spaces $\boldsymbol{\Psi}$, $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}^*$ satisfy that $\boldsymbol{\Psi} \hookrightarrow\hookrightarrow \boldsymbol{\Phi} \hookrightarrow \boldsymbol{\Psi}^*$ and that the duality pairing between $\boldsymbol{\Psi}^*$ and $\boldsymbol{\Psi}$ is generated from the weighted $\mathbf{L}^2$ inner product on $\boldsymbol{\Phi}$.

We arbitrarily fix a $\bar{t} \in (0,T]$ and define

$$\boldsymbol{\eta}(t) = \begin{cases} \int_t^{\bar{t}} \boldsymbol{\zeta}(\tau)\,d\tau, & t \in [0,\bar{t}] \\ \mathbf{0}, & t \in [\bar{t},T]. \end{cases}$$

Then $\boldsymbol{\eta}(t) \in \boldsymbol{\Psi}$ for every $t \in [0,T]$ and equation (2.51) yields

$$\int_0^{\bar{t}} \Big(\langle\langle \boldsymbol{\zeta}'', \boldsymbol{\eta} \rangle\rangle + a_1[\boldsymbol{\zeta}', \boldsymbol{\eta}] + a_2[\boldsymbol{\zeta}, \boldsymbol{\eta}] \Big) dt = 0.$$

Since $\boldsymbol{\zeta}'(0) = \mathbf{0}$ and $\boldsymbol{\eta}(\bar{t}) = \mathbf{0}$, we obtain after integrating by parts in the first two terms that

$$\int_0^{\bar{t}} \Big(-[[\boldsymbol{\zeta}', \boldsymbol{\eta}']] - a_1[\boldsymbol{\zeta}, \boldsymbol{\eta}'] - a_2[\boldsymbol{\eta}', \boldsymbol{\eta}] \Big) dt = 0.$$

Noting that $\boldsymbol{\eta}'(t) = -\boldsymbol{\zeta}(t)$ for $t \in [0,\bar{t}]$ we are led to

$$\int_0^{\bar{t}} \frac{1}{2}\frac{d}{dt}\Big([[\boldsymbol{\zeta}, \boldsymbol{\zeta}]] - a_2[\boldsymbol{\eta}, \boldsymbol{\eta}] \Big) dt = -\int_0^{\bar{t}} a_1[\boldsymbol{\zeta}, \boldsymbol{\zeta}]\,dt$$

so that

$$\frac{1}{2}[[\boldsymbol{\zeta}(\bar{t}), \boldsymbol{\zeta}(\bar{t})]] + \frac{1}{2}a_2[\boldsymbol{\eta}(0), \boldsymbol{\eta}(0)] = -\int_0^{\bar{t}} a_1[\boldsymbol{\zeta}, \boldsymbol{\zeta}]\,dt \le 0.$$

This implies that

$$\boldsymbol{\zeta}(\bar{t}) = \mathbf{0}.$$

As $\bar{t}$ is arbitrary, we deduce that

$$\boldsymbol{\zeta} = \mathbf{0} \quad \text{in } [0,T] \times \Omega.$$

Hence,

$$\boldsymbol{\xi} - \widetilde{\boldsymbol{\xi}} = \boldsymbol{\zeta}' = \mathbf{0} \quad \text{in } [0,T] \times \Omega. \quad \Box$$

Remark: In the proof for uniqueness, it is not known whether $\boldsymbol{\xi}(t)$ and $\widetilde{\boldsymbol{\xi}}(t)$ belong to $\boldsymbol{\Psi}$ for almost every t. Thus we cannot simply set $\boldsymbol{\eta} = \boldsymbol{\theta}(t)$ in (2.49).

If $\boldsymbol{\xi}$ is the solution of (2.10)–(2.12) that is guaranteed to exist by Theorem 2.4, then by setting $\mathbf{v} = \boldsymbol{\xi}|_{\Omega_1}$ and $\mathbf{u} = \mathbf{u}_0 + \int_0^t \boldsymbol{\xi}(s)|_{\Omega_2}\,ds$ we immediately obtain the existence of a weak solution $(\mathbf{v},\mathbf{u})$ for the divergence-free formulation of the fluid-structure interaction problem.

Theorem 2.5. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$ *and* $\mathbf{u}_1$ *satisfy* (2.4). *Then, there exists a unique pair* $(\mathbf{v},\mathbf{u}) \in L^2(0,T;X_1) \times L^2(0,T;X_2)$ *which satisfies* (2.6)–(2.8), *where* (2.6) *holds in the sense of distributions on* $(0,T)$. *Moreover,*

$$\begin{aligned} &\|\mathbf{v}(t)\|_{0,\Omega}^2 + \|\mathbf{u}_t(t)\|_{0,\Omega}^2 + \|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^1(\Omega_1))}^2 + \|\mathbf{u}(t)\|_{\mathbf{H}^1(\Omega_2)}^2 \\ &\quad \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + \|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2 \Big) \end{aligned} \tag{2.53}$$

for all $t \in [0,T]$ *and*

$$\begin{aligned} &\|\mathbf{v}_t\|_{L^2(0,T;V_1^*)}^2 + \|\mathbf{u}_{tt}\|_{L^2(0,T;\mathbf{H}^{-1}(\Omega_2))}^2 \\ &\quad \le Ce^{CT}\Big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + \|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2 \Big). \end{aligned} \tag{2.54}$$

3. The existence of a strong solution and an L^2 pressure.

Under additional regularity and compatibility assumptions on the data, we may prove stronger energy estimates for the Galerkin solutions $\{\boldsymbol{\xi}_m\}$. Such a priori estimates will allow us to derive regularity results for the solution of (2.10)–(2.11) and to show the existence of a corresponding pressure field.

3.1. Strong energy estimates.

Theorem 3.1. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$ *and* $\mathbf{u}_1$ *satisfy* (2.4) *and*

$$\begin{aligned} &\partial_t \mathbf{f}_i \in L^2(0,T;\mathbf{L}^2(\Omega_i)),\ i=1,2, \\ &\mathbf{v}_0 \in \mathbf{H}^2(\Omega_1), \quad \mathbf{u}_1 \in \mathbf{H}^2(\Omega_2), \quad \mathbf{u}_0 \in \mathbf{H}^2(\Omega_2). \end{aligned} \tag{3.1}$$

Assume further that there exists a $p_0 \in H^1(\Omega_1)$ *such that*

$$\begin{aligned} &\Big(p_0\mathbf{n}_1 - \mu_1(\nabla\mathbf{v}_0 + \nabla\mathbf{v}_0^T)\cdot\mathbf{n}_1\Big)\Big|_{\Gamma_0} \\ &\quad = \Big(\mu_2(\nabla\mathbf{u}_0 + \nabla\mathbf{u}_0^T)\cdot\mathbf{n}_2 + (\lambda_2+\mu_2)(\operatorname{div}\mathbf{u}_0)\mathbf{n}_2\Big)\Big|_{\Gamma_0} \end{aligned} \tag{3.2}$$

where $\mathbf{n}_i$ *denotes the outward-pointing normal along* $\partial\Omega_i$*. Then, for each integer* $m>0$*, the solution* $\boldsymbol{\xi}_m$ *of* (2.15)*–*(2.16) *satisfies the estimate*

$$\begin{aligned} &\|\boldsymbol{\xi}'_m(t)\|^2_{0,\Omega} + \|\boldsymbol{\xi}'_m\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_1))} + \|\textstyle\int_0^t \boldsymbol{\xi}'_m(s)\,ds\|^2_{\mathbf{H}^1(\Omega_2)} \\ &\le Ce^{CT}\Big(\|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{1,\Omega_2}\Big) \end{aligned} \tag{3.3}$$

for all $t\in[0,T]$.

PROOF: Defining $\boldsymbol{\zeta}_m = \partial_t\boldsymbol{\xi}_m$ and differentiating (2.15) we obtain that for each $t\in[0,T]$,

$$\begin{aligned} &\rho_1[\partial_t\boldsymbol{\zeta}_m,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\partial_t\boldsymbol{\zeta}_m,\boldsymbol{\eta}]_{\Omega_2} + a_1[\boldsymbol{\zeta}_m,\boldsymbol{\eta}] + a_2[\textstyle\int_0^t\boldsymbol{\zeta}_m(s)\,ds,\boldsymbol{\eta}] \\ &\quad = \rho_1[\partial_t\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\partial_t\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2} - a_2[\boldsymbol{\xi}_m(0),\boldsymbol{\eta}] \qquad \forall\,\boldsymbol{\eta}\in\boldsymbol{\Psi}_m. \end{aligned} \tag{3.4}$$

Setting $\boldsymbol{\eta} = \boldsymbol{\zeta}_m(t)$ in (2.15) and repeating the steps for the derivation of (2.31) we have

$$\begin{aligned} &\|\boldsymbol{\zeta}_m(t)\|^2_{0,\Omega} + \int_0^t a_1[\boldsymbol{\zeta}_m(t),\boldsymbol{\zeta}_m(t)]\,dt + a_2[\textstyle\int_0^t\boldsymbol{\zeta}_m(s)\,ds, \int_0^t\boldsymbol{\zeta}_m(s)\,ds] \\ &\quad \le C\Big(\|\boldsymbol{\zeta}_m(0)\|^2_{0,\Omega} + \|\partial_t\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\xi}_m(0)\|^2_{1,\Omega_2}\Big) \end{aligned} \tag{3.5}$$

so that using Gronwall's inequality (2.32) and (2.16) we deduce

$$\|\boldsymbol{\zeta}_m(t)\|^2_{0,\Omega} \le Ce^{CT}\Big(\|\boldsymbol{\xi}'_m(0)\|^2_{0,\Omega_1} + \|\partial_t\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{v}_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{1,\Omega_2}\Big). \tag{3.6}$$

The term $\|\boldsymbol{\xi}'_m(0)\|^2_{0,\Omega_1}$ can be estimated as follows. Evaluating (2.15) at $t=0$ then setting $\boldsymbol{\eta} = \boldsymbol{\xi}'_m(0)$ and using (2.16) and the divergence-free property of $\boldsymbol{\Psi}_m$, we deduce

$$\begin{aligned} [[\boldsymbol{\xi}'_m(0),\boldsymbol{\xi}'_m(0)]] &= [[\mathbf{f}(0),\boldsymbol{\xi}'_m(0)]] - a_2[\mathbf{u}_0,\boldsymbol{\xi}'_m(0)] - a_1[\mathbf{v}_0,\boldsymbol{\xi}'_m(0)] - b[\boldsymbol{\xi}'_m(0),p_0] \\ &= [[\mathbf{f}(0),\boldsymbol{\xi}'_m(0)]] + [\Delta\mathbf{u}_0 + \nabla(\operatorname{div}\mathbf{u}_0),\boldsymbol{\xi}'_m(0)]_{\Omega_2} + [\Delta\mathbf{v}_0 - \nabla p_0,\boldsymbol{\xi}'_m(0)]_{\Omega_1} \\ &\quad + \int_{\Gamma_0}\Big(-\mu_2(\nabla\mathbf{u}_0+\nabla\mathbf{u}_0^T)\cdot\mathbf{n}_2 - (\lambda_2+\mu_2)(\operatorname{div}\mathbf{u}_0)\mathbf{n}_2 \\ &\qquad + p_0\mathbf{n}_1 - (\nabla\mathbf{v}_0+\nabla\mathbf{v}_0^T)\cdot\mathbf{n}_1\Big)\cdot\boldsymbol{\xi}'_m(0)\,d\Gamma \end{aligned}$$

so that upon substituting (3.2) into the last estimate we have

$$[[\boldsymbol{\xi}'_m(0),\boldsymbol{\xi}'_m(0)]] \le C\Big(\|\mathbf{f}(0)\|^2_{\mathbf{L}^2(\Omega)}+\|\mathbf{v}_0\|^2_{2,\Omega}+\|p_0\|^2_{1,\Omega_1}+\|\mathbf{u}_0\|^2_{2,\Omega}\Big)+\frac{1}{2}[[\boldsymbol{\xi}'_m(0),\boldsymbol{\xi}'_m(0)]].$$

This last relation and the estimate

$$\|\mathbf{f}(0)\|^2_{\mathbf{L}^2(\Omega)} \le C\Big(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\partial_t \mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}\Big)$$

lead us to

$$[[\boldsymbol{\xi}'_m(0),\boldsymbol{\xi}'_m(0)]] \le C\Big(\|\mathbf{v}_0\|^2_{2,\Omega} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_0\|^2_{2,\Omega} + \|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))}\Big).$$

From (3.5), (3.6) and the last inequality we obtain (3.3). □

Remark: The term $a_1[\boldsymbol{\xi}_m(0)-\mathbf{v}_0,\boldsymbol{\eta}]$ is added to the Galerkin approximation (2.15) to provide the cancellation of $a_1[\boldsymbol{\xi}_m(0),\boldsymbol{\eta}]$ in the estimation of $[[\boldsymbol{\xi}'_m(0),\boldsymbol{\xi}'_m(0)]]$. This extra term vanishes in the limit as $m\to\infty$ and thus does not affect the continuous weak form.

As an immediate consequence of Theorems 2.5 and 3.1 we obtain the following strong a priori estimates for the solution to (2.6)–(2.8).

Theorem 3.2. *Assume the hypotheses of Theorem* 3.1 *hold. Then, the solution* $(\mathbf{v},\mathbf{u})$ *to* (2.6)–(2.8) *satisfies the estimate*

$$\begin{aligned}&\|\partial_t\mathbf{v}(t)\|^2_{0,\Omega_1} + \|\partial_{tt}\mathbf{u}(t)\|^2_{0,\Omega_2} + \|\partial_t\mathbf{v}\|^2_{L^2(0,T;X_1)} + \|\partial_t\mathbf{u}(t)\|^2_{1,\Omega_2)}\\ &\le Ce^{CT}\Big(\|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{1,\Omega_2}\Big).\end{aligned} \tag{3.7}$$

3.2. **The existence of a pressure.** With the existence of a strong solution guaranteed by Theorem 3.2, it is now possible for us to establish the existence of an L^2 integrable pressure.

To find a pressure that satisfies (2.5), we need to show the existence of a $p\in L^2(0,T;L^2(\Omega_1))$ such that

$$\begin{aligned}b[\boldsymbol{\eta},p] = &-\rho_1[\mathbf{v}_t,\boldsymbol{\eta}]_{\Omega_1} - a_1[\mathbf{v},\boldsymbol{\eta}] - \rho_2[\mathbf{u}_{tt},\boldsymbol{\eta}]_{\Omega_2} - a_2[\mathbf{u},\boldsymbol{\eta}]\\ &+\rho_1[\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2} \qquad \forall\,\boldsymbol{\eta}\in\mathbf{H}^1_0(\Omega),\ \text{a.e. } t.\end{aligned} \tag{3.8}$$

Using the auxiliary formulation (2.10), we see that this is equivalent to showing there exists a $p\in L^2(0,T;L^2(\Omega_1))$ which satisfies

$$\begin{aligned}&\rho_1[\boldsymbol{\xi}_t,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\boldsymbol{\xi}_t,\boldsymbol{\eta}]_{\Omega_2} + b[\boldsymbol{\eta},p] + a_1[\boldsymbol{\xi},\boldsymbol{\eta}] + a_2[\textstyle\int_0^t \boldsymbol{\xi}(s)\,ds,\boldsymbol{\eta}]\\ &\qquad = \rho_1[\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2} - a_2[\mathbf{u}_0,\boldsymbol{\eta}] \qquad \forall\,\boldsymbol{\eta}\in\mathbf{H}^1_0(\Omega),\ \text{a.e. } t.\end{aligned} \tag{3.9}$$

By virtue of [17, p.58, Lemma 4.1], the proof of the existence of a pressure p for (3.9) is reduced to showing the following inf-sup condition for the pair of function spaces $\{\mathbf{H}^1_0(\Omega),Q_1\}$, where $Q_1=L^2(\Omega_1)$:

$$\inf_{q\in Q_1}\sup_{\boldsymbol{\eta}\in\mathbf{H}^1_0(\Omega)} \frac{b[\boldsymbol{\eta},q]}{\|\boldsymbol{\eta}\|_{1,\Omega}\|q\|_{0,\Omega_1}} \ge C. \tag{3.10}$$

The proof of this inf-sup condition follows exactly that of [1, Lemma 3.1], though the inf-sup condition established in that Lemma is for a slightly different pair of function spaces.

Theorem 3.3. *The inf-sup condition* (3.10) *holds.*

PROOF: Obviously, it suffices to show that

$$\forall\, q \in Q_1,\ \exists \text{ an } \boldsymbol{\eta} \in \boldsymbol{\Psi} \text{ such that } \operatorname{div} \boldsymbol{\eta}|_{\Omega_1} = q \text{ and } \|\boldsymbol{\eta}\|_{1,\Omega} \le C\|q\|_{0,\Omega_1}. \tag{3.11}$$

Let $q \in Q_1$ be given. We define $\widetilde{q} \in L^2_0(\Omega) = \{s \in L^2(\Omega) \mid \int_\Omega s\, d\Omega = 0\}$ by $\widetilde{q}|_{\Omega_1} = q$ and $\widetilde{q}|_{\Omega_2} = -(1/|\Omega_2|)\int_{\Omega_1} q\, d\mathbf{x}$. By virtue of [17, p.24, Corollary 2.4], there exists a unique $\boldsymbol{\eta} \in \mathbf{H}^1_0(\Omega)$ such that

$$\operatorname{div} \boldsymbol{\eta} = \widetilde{q}, \qquad \|\boldsymbol{\eta}\|_{1,\Omega} \le C\|\widetilde{q}\|_{0,\Omega} \le C\|q\|_{0,\Omega_1}.$$

This proves (3.11), which in turn implies (3.10). □

From [17, p.58, Lemma 4.1] and Theorems 2.5, 3.2 and 3.3 we readily deduce the following result.

Theorem 3.4. *Assume the hypotheses of Theorem* 3.1 *hold. Then, there exists a unique triplet* $(\mathbf{v}, p, \mathbf{u})$ *which possesses the regularity*

$$\mathbf{v} \in L^\infty(0,T;\mathbf{L}^2(\Omega_1)) \cap L^2(0,T;X_1), \quad \mathbf{u} \in L^\infty(0,T;X_2), \quad p \in L^2(0,T;L^2(\Omega_1))$$
$$\mathbf{v}_t \in L^\infty(0,T;\mathbf{L}^2(\Omega_1)) \cap L^2(0,T;X_1),\ \mathbf{u}_t \in L^\infty(0,T;X_2),\ \mathbf{u}_{tt} \in L^\infty(0,T;\mathbf{L}^2(\Omega_2))$$

and satisfies equations (2.5)*,* (2.7)*,* (2.8) *and*

$$b[\mathbf{v}, q] = 0 \qquad \forall\, q \in L^2(\Omega_1),\ \textit{a.e. } t.$$

Moreover, the estimates (2.53)*,* (3.7) *and*

$$\|p\|_{L^2(0,T;L^2(\Omega_1))} \le Ce^{CT}\Big(\|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{1,\Omega_2}\Big)$$

hold.

For a strong solution in the sense of Theorem 3.4, the velocity interface condition holds in the strong sense

$$\mathbf{v}|_{\Gamma_0} = \mathbf{u}_t|_{\Gamma_0} \quad \text{in } L^2(0,T;\mathbf{H}^{1/2}(\Gamma_0))$$

and

$$\mathbf{v}|_{\Gamma_0} = \mathbf{u}_t|_{\Gamma_0} \quad \text{in } \mathbf{H}^{1/2}(\Gamma_0),\ \text{a.e. } t.$$

Also, it can be inferred that

$$\mathbf{v} \in C([0,T];X_1), \quad \mathbf{u} \in C([0,T];X_2) \quad \text{and} \quad \partial_t \mathbf{u} \in C([0,T];\mathbf{L}^2(\Omega_2))$$

so that the initial conditions (2.7) hold in the respective strong senses.

Remark. The inf-sup condition (3.10) is equivalent to the following inf-sup condition for the space pair $\{X_1, Q_1\}$:

$$\inf_{q\in Q_1} \sup_{\mathbf{v}\in X_1} \frac{b[\mathbf{v}, q]}{\|\mathbf{v}\|_{1,\Omega_1}\|q\|_{0,\Omega_1}} \ge C. \tag{3.12}$$

To see this, let the extension operator $E : X_1 \to \mathbf{H}^1_0(\Omega)$ be defined as follows: for every $\mathbf{z} \in X_1$, $(E\mathbf{z})|_{\Omega_1} = \mathbf{z}$ and $(E\mathbf{z})|_{\Omega_2} = \widetilde{\mathbf{z}}$ where $\widetilde{\mathbf{z}} \in X_2$ is the solution of

$$[\nabla\widetilde{\mathbf{z}}, \nabla\mathbf{w}]_\Omega = 0 \quad \forall\, \mathbf{w} \in \mathbf{H}^1_0(\Omega_2), \qquad \widetilde{\mathbf{z}}|_{\Gamma_2} = \mathbf{0}, \qquad \widetilde{\mathbf{z}}|_{\Gamma_0} = \mathbf{z}|_{\Gamma_0}. \tag{3.13}$$

(For each given $\mathbf{z} \in X_1$, the definition of X_1 ensures that there exists an $\boldsymbol{\eta} \in \mathbf{H}^1_0(\Omega)$ such that $\boldsymbol{\eta}|_{\Gamma_0} = \mathbf{z}|_{\Gamma_0}$ so that (3.13) possesses a unique solution $\widetilde{\mathbf{z}} \in X_2$.) Standard elliptic estimates (e.g., [17, p.12]) yields

$$\|E\mathbf{z}\|_{1,\Omega} \le C\|(E\mathbf{z})|_{\Omega_1}\|_{1,\Omega_1} + \|(E\mathbf{z})|_{\Omega_2}\|_{1,\Omega_2} \le C(\|\mathbf{z}\|_{1,\Omega_1} + \|\mathbf{z}\|_{1/2,\Gamma_0}) \le C\|\mathbf{z}\|_{1,\Omega_1}$$

for all $\mathbf{z} \in X_1$. Then, for every $q \in Q_1$ we have

$$\sup_{\boldsymbol{\eta} \in \mathbf{H}_0^1(\Omega)} \frac{b[\boldsymbol{\eta}, q]}{\|q\|_{0,\Omega_1} \|\boldsymbol{\eta}\|_{1,\Omega}} \geq \sup_{\mathbf{z} \in X_1} \frac{b[E\mathbf{z}, q]}{\|q\|_{0,\Omega_1} \|E\mathbf{z}\|_{1,\Omega}}$$
$$\geq C \sup_{\mathbf{z} \in X_1} \frac{b[E\mathbf{z}, q]}{\|q\|_{0,\Omega_1} \|\mathbf{z}\|_{1,\Omega_1}} = C \sup_{\mathbf{z} \in X_1} \frac{b[\mathbf{z}, q]}{\|q\|_{0,\Omega_1} \|\mathbf{z}\|_{1,\Omega_1}}$$

and

$$\sup_{\boldsymbol{\eta} \in \mathbf{H}_0^1(\Omega)} \frac{b[\boldsymbol{\eta}, q]}{\|q\|_{0,\Omega_1} \|\boldsymbol{\eta}\|_{1,\Omega}} \leq \sup_{\boldsymbol{\eta} \in \mathbf{H}_0^1(\Omega)} \frac{b[\boldsymbol{\eta}, q]}{\|q\|_{0,\Omega_1} \|\boldsymbol{\eta}\|_{1,\Omega_1}} \leq \sup_{\mathbf{z} \in X_1} \frac{b[\mathbf{z}, q]}{\|q\|_{0,\Omega_1} \|\mathbf{z}\|_{1,\Omega_1}};$$

in other words, (3.10) and (3.12) are indeed equivalent. The inf-sup condition (3.12) is useful for recovering the pressure field from the fluid equations only. It is also useful for analyzing the Stokes problems with mixed velocity/stress boundary conditions.

4. **Concluding remarks.** We have demostrated the existence and uniqueness of weak (fluid velocity and solid displacement) solutions of a fluid-structure interaction problem involving a linear, viscous, incompressible fluid and an elastic solid. We have considered the particular case of a solid that undergoes only infinitesimal elastic displacements but whose velocity is large enough so that the fluid and structure remain fully coupled. Under additional smoothness assumptions on the data, we have also demonstrated the existence of an L^2 integrable fluid pressure. In forthcoming work, we will consider finite element approximations of our model fluid-structure interaction problem and, in future work, we will consider the extension of our results to nonlinear fluid models such as the Navier-Stokes equations and to optimization and control problems involving the interactions between fluids and structures.

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