Category Theory: Notes & Errata

- Lecture 17, Example (a): It is faster and more natural to observe directly that reflexive coequalisers are preserved. To see this, note that the coequaliser in **Set** of $f, g : A \to B$ is B quotiented by the transitive closure of $f(x) \sim g(x)$ for $x \in A$, whilst in **Gp** it is B quotiented by the transitive closure of $h_1(x_1)...h_n(x_n) \sim h'_1(x_1)...h'_n(x_n)$ where $x_1...x_n \in A$ and each h_i and h'_i can be either f or g. Thus in general U will not preserve coequalisers. However we have a reflexive coequaliser, so for example given $f(x)g(y) \sim g(x)f(y)$:

$$
\begin{aligned}
g(rf(x).x^{-1}.rg(x).y) &= grf(x).g(x)^{-1}.grg(x).g(y) \\
&= f(x).g(x)^{-1}.g(x).g(y) \\
&= f(x)g(y) \\
f(rf(x).x^{-1}.rg(x).y) &= frf(x).f(x)^{-1}.frg(x).f(y) \\
&= f(x).f(x)^{-1}.g(x).f(y) \\
&= g(x).f(y)
\end{aligned}
$$

  Thus everything regarded as equal in the Gp coequaliser is also seen as equal in the Set coequalisers.

- 5.9 is the *Precise Monadicity Theorem*.

- Lecture 19 page 10: Given $A \in$Set with this structure, we want to define $\alpha : TA \to A$ to make a member of $\mathbf{Set}^{\mathbb{T}}$. Now T is finitary, so as in 6.8 take inclusion $I : \mathbf{Set}_f \to \mathbf{Set}$ and forgetful $U : (I \downarrow A) \to \mathbf{Set}_f$; then $TA = \mathrm{colim} TIU$, so we can define α by specifying maps:

$$
\begin{aligned}
\alpha_{\bar{x}} : TIU(\bar{x} : I\bar{n} \to A) &\to A \\
\omega &\mapsto \omega_A(\bar{x})
\end{aligned}
$$

  Then the derived algebra has operations $\omega(\bar{x}) = \alpha(T\bar{x}(\omega)) = \alpha_{\bar{x}}(\omega) = \omega_A(x)$ as desired. It remains to check that $(A, \alpha) \in \mathrm{Set}^{\mathbb{T}}$.

- Lemma 7.9. It is not clear to us how to modify the functor to give the desired left adjoint. However it is possible to construct directly a left adjoint for $f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(B)$ by combining the proofs of the previous two results.

- Lecture 24 construction: Note we quotient out the morphisms but not the objects, so the 'same' morphism will in fact occur repeatedly between different isomorphic objects.