DECAY OF VISCOUS SURFACE WAVES WITHOUT SURFACE TENSION IN HORIZONTALLY INFINITE DOMAINS

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Abstract. We consider a viscous fluid of finite depth below the air, occupying a three-dimensional domain bounded below by a fixed solid boundary and above by a free moving boundary. The fluid dynamics are governed by the gravity-driven incompressible Navier-Stokes equations, and the effect of surface tension is neglected on the free surface. The long time behavior of solutions near equilibrium has been an intriguing question since the work of Beale [4]. This paper is the second in a series of three [13, 14] that answers this question. Here we consider the case in which the free interface is horizontally infinite; we prove that the problem is globally well-posed and that solutions decay to equilibrium at an algebraic rate. In particular, the free interface decays to a flat surface.

Our framework contains several novel techniques, which include: (1) optimal a priori estimates that utilize a "geometric" reformulation of the equations; (2) a two-tier energy method that couples the boundedness of high-order energy to the decay of low-order energy, the latter of which is necessary to balance out the growth of the highest derivatives of the free interface; (3) control of both negative and positive Sobolev norms, which enhances interpolation estimates and allows for the decay of infinite surface waves. Our decay estimates lead to the construction of global-in-time solutions to the surface wave problem.

1. Introduction

1.1. **Formulation of the equations in Eulerian coordinates.**

We consider a viscous, incompressible fluid evolving in a moving domain

$$\Omega(t) = \{y \in \Sigma \times \mathbb{R} \mid -b < y_3 < \eta(y_1, y_2, t)\}. \tag{1.1}$$

Here we assume that $\Sigma = \mathbb{R}^2$. The lower boundary b is assumed to be fixed and given, but the upper boundary is a free surface that is the graph of the unknown function $\eta : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$. We assume that $b > 0$ is a fixed constant so that the lower boundary is flat. For each t, the fluid is described by its velocity and pressure functions $(u, p) : \Omega(t) \to \mathbb{R}^3 \times \mathbb{R}$. We require that (u, p, η) satisfy the gravity-driven incompressible Navier-Stokes equations in $\Omega(t)$ for $t > 0$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u & \text{in } \Omega(t) \\ \operatorname{div} u = 0 & \text{in } \Omega(t) \\ \partial_t \eta = u_3 - u_1 \partial_{y_1} \eta - u_2 \partial_{y_2} \eta & \text{on } \{y_3 = \eta(y_1, y_2, t)\} \\ (pI - \mu \mathbb{D}(u))\nu = g\eta\nu & \text{on } \{y_3 = \eta(y_1, y_2, t)\} \\ u = 0 & \text{on } \{y_3 = -b\} \end{cases} \tag{1.2}$$

for ν the outward-pointing unit normal on $\{y_3 = \eta\}$, I the 3×3 identity matrix, $(\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i$ the symmetric gradient of u, $g > 0$ the strength of gravity, and $\mu > 0$ the viscosity. The tensor $(pI - \mu\mathbb{D}(u))$ is known as the viscous stress tensor. The third equation in (1.2) implies that the free surface is advected with the fluid. Note that in (1.2) we have shifted the gravitational forcing to the boundary and eliminated the constant atmospheric pressure, p_{atm}, in the usual way by adjusting the actual pressure $\bar{p}$ according to $p = \bar{p} + gy_3 - p_{atm}$.

The problem is augmented with initial data (u_0, η_0) satisfying certain compatibility conditions, which for brevity we will not write now. We will assume that $\eta_0 > -b$ on Σ.

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Without loss of generality, we may assume that $\mu = g = 1$. Indeed, a standard scaling argument allows us to scale so that $\mu = g = 1$, at the price of multiplying b by a positive constant. This means that, up to renaming b, we arrive at the above problem with $\mu = g = 1$.

The problem (1.2) possesses a natural physical energy. For sufficiently regular solutions, we have an energy evolution equation that expresses how the change in physical energy is related to the dissipation:

$$\frac{1}{2}\int_{\Omega(t)} |u(t)|^2 + \frac{1}{2}\int_{\Sigma} |\eta(t)|^2 + \frac{1}{2}\int_0^t \int_{\Omega(s)} |\mathbb{D}u(s)|^2 \, ds = \frac{1}{2}\int_{\Omega(0)} |u_0|^2 + \frac{1}{2}\int_{\Sigma} |\eta_0|^2 . \tag{1.3}$$

The first two integrals constitute the kinetic and potential energies, while the third constitutes the dissipation. The structure of this energy evolution equation is the basis of the energy method we will use to analyze (1.2).

1.2. **Geometric form of the equations.** In order to work in a fixed domain, we want to flatten the free surface via a coordinate transformation. We will not use a Lagrangian coordinate transformation, but rather a flattening transformation introduced by Beale in [5]. To this end, we consider the fixed domain

$$\Omega := \{x \in \Sigma \times \mathbb{R} \mid -b < x_3 < 0\} \tag{1.4}$$

for which we will write the coordinates as $x \in \Omega$. We will think of Σ as the upper boundary of Ω, and we will write $\Sigma_b := \{x_3 = -b\}$ for the lower boundary. We continue to view η as a function on $\Sigma \times \mathbb{R}^+$. We then define

$$\bar{\eta} := \mathcal{P}\eta = \text{harmonic extension of } \eta \text{ into the lower half space,} \tag{1.5}$$

where $\mathcal{P}\eta$ is defined by (A.17). The harmonic extension $\bar{\eta}$ allows us to flatten the coordinate domain via the mapping

$$\Omega \ni x \mapsto (x_1, x_2, x_3 + \bar{\eta}(x,t)(1 + x_3/b)) = \Phi(x,t) = (y_1, y_2, y_3) \in \Omega(t). \tag{1.6}$$

Note that $\Phi(\Sigma, t) = \{y_3 = \eta(y_1, y_2, t)\}$ and $\Phi(\cdot, t)|_{\Sigma_b} = Id_{\Sigma_b}$, i.e. Φ maps Σ to the free surface and keeps the lower surface fixed. We have

$$\nabla\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & J \end{pmatrix} \text{ and } \mathcal{A} := (\nabla\Phi^{-1})^T = \begin{pmatrix} 1 & 0 & -AK \\ 0 & 1 & -BK \\ 0 & 0 & K \end{pmatrix} \tag{1.7}$$

for

$$\begin{aligned} &A = \partial_1\bar{\eta}\tilde{b}, \quad B = \partial_2\bar{\eta}\tilde{b}, \\ &J = 1 + \bar{\eta}/b + \partial_3\bar{\eta}\tilde{b}, \quad K = J^{-1}, \\ &\tilde{b} = (1 + x_3/b). \end{aligned} \tag{1.8}$$

Here $J = \det\nabla\Phi$ is the Jacobian of the coordinate transformation.

If η is sufficiently small (in an appropriate Sobolev space), then the mapping Φ is a diffeomorphism. This allows us to transform the problem to one on the fixed spatial domain Ω for $t \ge 0$. In the new coordinates, the PDE (1.2) becomes

$$\begin{cases} \partial_t u - \partial_t\bar{\eta}\tilde{b}K\partial_3 u + u\cdot\nabla_{\mathcal{A}}u - \Delta_{\mathcal{A}}u + \nabla_{\mathcal{A}}p = 0 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega \\ S_{\mathcal{A}}(p,u)\mathcal{N} = \eta\mathcal{N} & \text{on } \Sigma \\ \partial_t\eta = u\cdot\mathcal{N} & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b \\ u(x,0) = u_0(x), \eta(x',0) = \eta_0(x'). \end{cases} \tag{1.9}$$

Here we have written the differential operators $\nabla_{\mathcal{A}}$, $\operatorname{div}_{\mathcal{A}}$, and $\Delta_{\mathcal{A}}$ with their actions given by $(\nabla_{\mathcal{A}}f)_i := \mathcal{A}_{ij}\partial_j f$, $\operatorname{div}_{\mathcal{A}} X := \mathcal{A}_{ij}\partial_j X_i$, and $\Delta_{\mathcal{A}}f = \operatorname{div}_{\mathcal{A}}\nabla_{\mathcal{A}}f$ for appropriate f and X; for $u\cdot\nabla_{\mathcal{A}}u$ we mean $(u\cdot\nabla_{\mathcal{A}}u)_i := u_j\mathcal{A}_{jk}\partial_k u_i$. We have also written $\mathcal{N} := -\partial_1\eta e_1 - \partial_2\eta_2 e_2 + e_3$ for the non-unit normal to Σ, and we write $S_{\mathcal{A}}(p,u) = (pI - \mathbb{D}_{\mathcal{A}}u)$ for the stress tensor, where I the 3×3

identity matrix and $(\mathbb{D}_{\mathcal{A}} u)_{ij} = \mathcal{A}_{ik}\partial_k u_j + \mathcal{A}_{jk}\partial_k u_i$ is the symmetric $\mathcal{A}-$gradient. Note that if we extend $\operatorname{div}_{\mathcal{A}}$ to act on symmetric tensors in the natural way, then $\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) = \nabla_{\mathcal{A}} p - \Delta_{\mathcal{A}} u$ for vector fields satisfying $\operatorname{div}_{\mathcal{A}} u = 0$.

Recall that $\mathcal{A}$ is determined by η through the relation (1.7). This means that all of the differential operators in (1.9) are connected to η, and hence to the geometry of the free surface. This geometric structure is essential to our analysis, as it allows us to control high-order derivatives that would otherwise be out of reach.

1.3. **Previous results and Beale's non-decay theorem.** Many authors have considered problems similar to (1.2), both with and without viscosity and surface tension: [1, 3, 4, 5, 6, 7, 8, 9, 11, 15, 16, 17, 18, 21, 20, 22, 23, 25, 26, 27, 28, 29, 30, 31, 32]. We refer the reader to the introduction of our paper [13] for a discussion of how these results relate to ours. We will only mention the details of those papers most relevant to the present problem.

In [4], Beale developed a local existence theory for the problem (1.2) in Lagrangian coordinates, where the unknowns are replaced with $v = u \circ \zeta$, $q = p \circ \zeta$ for ζ the Lagrangian flow map, which satisfies $\partial_t \zeta = v$. The result showed that (roughly speaking), given $v_0 \in H^{r-1}$ for $r \in (3, 7/2)$, there exists a unique solution on a time interval $(0,T)$, with T depending on v_0, so that $v \in L^2 H^r \cap H^{r/2} L^2$. A second local existence theorem was then proved for small data near equilibrium. It showed that for any fixed $0 < T < \infty$, there exists a collection of sufficiently small data so that a unique solution exists on $(0,T)$.

The second result suggests that solutions should exist globally in time for small data. If global solutions do exist, it is natural to expect the free surface to decay to 0 as $t \to \infty$. However, Beale's third result in [4] was a non-decay theorem that showed that a "reasonable" extension to small-data global well-posedness with decay of the free surface fails. Among other things, the theorem's hypotheses require that

$$\begin{aligned} v &\in L^1([0,\infty); H^r(\Omega)) \text{ for } r \in (3,7/2), \\ \zeta_3|_\Sigma &\in L^2([0,\infty); L^2(\Sigma)), \\ v(x,0) &= 0,\ \zeta(x,0) = x + \varepsilon\Theta(x), \\ \lim_{t\to\infty} \zeta_3|_\Sigma &= 0, \end{aligned} \tag{1.10}$$

where Ω is given by (1.4), $\zeta(x,0)$ is the flow map that gives the geometry of the initial fluid domain, Θ is a specially chosen function satisfying certain conditions, and $\varepsilon > 0$ is a small parameter. Note that the third line in (1.10) implies that the system is initially close to equilibrium, and the fourth line implies that the free surface decays to 0 as $t \to \infty$.

The proof of the non-decay theorem, which is a reductio ad absurdum, hinges on the special conditions imposed on the map Θ and the fact that $v \in L^1 H^r$. In the discussion of this result, Beale pointed out that it does not imply the non-existence of global-in-time solutions but rather that establishing global-in-time results requires stronger or different hypotheses than those imposed in the non-decay theorem.

The non-decay theorem raises two intriguing questions. First, is viscosity alone capable of producing global well-posedness? Second, if global solutions exist, do they decay as $t \to \infty$? Our main result answers both questions in the affirmative. In order to avoid the applicability of the non-decay theorem, we must show why its hypotheses are not satisfied. We would like to highlight three crucial ways in which we do this. The first and most obvious is that we work in a different coordinate system and within a different functional framework. In particular this requires higher regularity of the initial data and imposes more compatibility conditions than are satisfied by the data in the non-decay theorem.

Second, we will find (see (1.20)) that u decays according to $\|u(t)\|_2^2 \le C/(1+t)^{1+\lambda}$ for $\lambda \in (0,1)$. This is not sufficiently rapid to guarantee that u belongs to the space $L^1([0,\infty); H^2(\Omega))$, which is in violation of a key assumption (1.10) in the non-decay result. Technically, our u is in Eulerian coordinates, but if we formally identify u with v, then we see the difficulty clearly: we cannot integrate the equation $\partial_t \zeta = v$ to obtain ζ as $t \to \infty$, which means that we cannot make sense of the fourth equation in (1.10). One of the advantages of the Eulerian and geometric formulations is that the free surface function η may be analyzed without regard to what is happening to the entire flow map ζ in Ω.

Third, we find that η decays in time according to $\|\eta(t)\|_0^2 \le C/(1+t)^\lambda$ for $\lambda \in (0,1)$. This is not fast enough to guarantee that η is in $L^2([0,\infty); L^2(\Sigma))$. If we identify η with $\zeta_3|_\Sigma$, then we see that we cannot guarantee that the second condition in (1.10) holds.

The above decay rates should be compared to those in the problem with surface tension, which in general allows for faster decay (see the discussion in Section 1.7) to equilibrium. In this context, Beale-Nishida [6] showed that the decay estimates $\|u(t)\|_2^2 \le C/(1+t)^2$ and $\|\eta(t))\|_0^2 \le C/(1+t)$ are sharp. As such, we should not expect $u \in L^1 H^2$ or $\eta \in L^2 L^2$ in our problem.

1.4. **Local well-posedness.** The a priori estimates we develop in this paper are done in different coordinates and in a different functional framework from those used by Beale in [4]. As such, we need a local well-posedness theory for (1.9) in our framework. We proved this in Theorem 1.1 of our companion paper [13]. Since we will need the result here, we record it now.

In order to state our result, we must explain our notation for Sobolev spaces and norms. We take $H^k(\Omega)$ and $H^k(\Sigma)$ for $k \ge 0$ to be the usual Sobolev spaces. When we write norms we will suppress the H and Ω or Σ. When we write $\left\|\partial_t^j u\right\|_k$ and $\left\|\partial_t^j p\right\|_k$ we always mean that the space is $H^k(\Omega)$, and when we write $\left\|\partial_t^j \eta\right\|_k$ we always mean that the space is $H^k(\Sigma)$.

In the following we write ${}_0H^1(\Omega) := \{u \in H^1(\Omega) \mid u|_{\Sigma_b} = 0\}$. The compatibility conditions for the initial data are the natural ones that would be satisfied for solutions in our functional framework. They are cumbersome to write, so we shall not record them here. We refer the reader to [13] for their precise definition.

Theorem 1.1. *Let $N \ge 3$ be an integer. Assume that u_0 and η_0 satisfy the bounds $\|u_0\|_{4N}^2 + \|\eta_0\|_{4N+1/2}^2 < \infty$ as well as the appropriate compatibility conditions. There exist $0 < \delta_0, T_0 < 1$ so that if*

$$0 < T \le T_0 \min\left\{1, \frac{1}{\|\eta_0\|_{4N+1/2}^2}\right\}, \tag{1.11}$$

and $\|u_0\|_{4N}^2 + \|\eta_0\|_{4N}^2 \le \delta_0$, then there exists a unique solution (u,p,η) to (1.9) on the interval $[0,T]$ that achieves the initial data. The solution obeys the estimates

$$\begin{aligned} &\sum_{j=0}^{2N} \sup_{0\le t\le T} \left\|\partial_t^j u\right\|_{4N-2j}^2 + \sum_{j=0}^{2N} \sup_{0\le t\le T} \left\|\partial_t^j \eta\right\|_{4N-2j}^2 + \sum_{j=0}^{2N-1} \sup_{0\le t\le T} \left\|\partial_t^j p\right\|_{4N-2j-1}^2 \\ &\quad + \int_0^T \left(\sum_{j=0}^{2N} \left\|\partial_t^j u\right\|_{4N-2j+1}^2 + \left\|\partial_t^{2N+1} u\right\|_{({}_0H^1(\Omega))^*}^2 + \sum_{j=0}^{2N} \left\|\partial_t^j p\right\|_{4N-2j}^2 \right) \\ &\quad + \int_0^T \left(\|\eta\|_{4N+1/2}^2 + \|\partial_t \eta\|_{4N-1/2}^2 + \sum_{j=2}^{2N+1} \left\|\partial_t^j \eta\right\|_{4N-2j+5/2}^2 \right) \\ &\quad \le C\left(\|u_0\|_{4N}^2 + \|\eta_0\|_{4N}^2 + T\|\eta_0\|_{4N+1/2}^2 \right) \end{aligned} \tag{1.12}$$

and

$$\sup_{0\le t\le T} \|\eta\|_{4N+1/2}^2 \le C\left(\|u_0\|_{4N}^2 + (1+T)\|\eta_0\|_{4N+1/2}^2 \right) \tag{1.13}$$

for a universal constant $C > 0$. The solution is unique among functions that achieve the initial data and for which the sum of the first three sums in (1.12) is finite. Moroever, η is such that the mapping $\Phi(\cdot,t)$, defined by (1.6), is a C^{4N-2} diffeomorphism for each $t \in [0,T]$.

Remark 1.2. *All of the computations involved in the a priori estimates that we develop in this paper are justified by Theorem 1.1 and a specialization of it that we prove later, Theorem 10.7. In this sense, Theorem 1.1 is a necessary ingredient in the global analysis of* (1.9). *We do not believe that our a priori estimates could be justified within a high-regularity modification of the functional framework of* [4].

1.5. **Main result.** Sylvester [25] and Tani-Tanaka [26] studied the existence of small-data global-in-time solutions via the parabolic regularity method pioneered by Beale [4] and Solonnikov [23]. The results say nothing about the decay of the free surface, nor do they contradict Beale's non-decay theorem since they require higher regularity, more compatibility conditions, and do not allow for $\eta \in L^2([0,\infty); L^2(\Sigma))$.

To state our global well-posedness result, we must first define various energies and dissipations. The exact form of some of the energies is too complicated to write out here, so we will neglect to do so, referring to the proper definitions later in the paper, in Section 2.4. We assume that $\lambda \in (0,1)$ is a fixed constant and we define $\mathcal{I}_\lambda u$ according to (A.7) and $\mathcal{I}_\lambda \eta$ according to (A.8). The high-order energy is

$$\mathcal{E}_{10} := \|\mathcal{I}_\lambda u\|_0^2 + \sum_{j=0}^{10} \left\|\partial_t^j u\right\|_{20-2j}^2 + \sum_{j=0}^{9} \left\|\partial_t^j p\right\|_{19-2j}^2 + \|\mathcal{I}_\lambda \eta\|_0^2 + \sum_{j=0}^{10} \left\|\partial_t^j \eta\right\|_{20-2j}^2, \tag{1.14}$$

and the high-order dissipation rate is

$$\begin{aligned}\mathcal{D}_{10} := \|\mathcal{I}_\lambda u\|_1^2 + \sum_{j=0}^{10} \left\|\partial_t^j u\right\|_{21-2j}^2 + \|\nabla p\|_{19}^2 + \sum_{j=1}^{9} \left\|\partial_t^j p\right\|_{20-2j}^2 \\ + \|D\eta\|_{20-3/2}^2 + \|\partial_t \eta\|_{20-1/2}^2 + \sum_{j=2}^{11} \left\|\partial_t^j \eta\right\|_{20-2j+5/2}^2.\end{aligned} \tag{1.15}$$

We write the high-order spatial derivatives of η as

$$\mathcal{F}_{10} := \|\eta\|_{20+1/2}^2. \tag{1.16}$$

We define the low-order energies $\mathcal{E}_{7,1}$ and $\mathcal{E}_{7,2}$ according to (2.52) and (2.53) with $n = 7$. Here the index m in $\mathcal{E}_{7,m}$ is a "minimal derivative" count that is included in order to improve decay rates in our estimates. Finally, we define the total energy

$$\mathcal{G}_{10}(t) = \sup_{0\le r\le t} \mathcal{E}_{10}(r) + \int_0^t \mathcal{D}_{10}(r)dr + \sum_{m=1}^{2} \sup_{0\le r\le t} (1+r)^{m+\lambda} \mathcal{E}_{7,m}(r) + \sup_{0\le r\le t} \frac{\mathcal{F}_{10}(r)}{(1+r)}. \tag{1.17}$$

Notice that the low-order terms $\mathcal{E}_{7,m}$ are weighted, so bounds on $\mathcal{G}_{10}$ yield decay estimates for $\mathcal{E}_{7,m}$.

Theorem 1.3. *Suppose the initial data* (u_0, η_0) *satisfy the compatibility conditions of Theorem 1.1. There exists a* $\kappa > 0$ *so that if* $\mathcal{E}_{10}(0) + \mathcal{F}_{10}(0) < \kappa$, *then there exists a unique solution* (u, p, η) *on the interval* $[0,\infty)$ *that achieves the initial data. The solution obeys the estimate*

$$\mathcal{G}_{10}(\infty) \le C_1 \left(\mathcal{E}_{10}(0) + \mathcal{F}_{10}(0)\right) < C_1\kappa, \tag{1.18}$$

where $C_1 > 0$ *is a universal constant. For any* $0 \le \rho < \lambda$, *we have that*

$$\sup_{t\ge 0} \left[(1+t)^{2+\rho} \|u(t)\|_{C^2(\Omega)}^2\right] \le C(\rho)\kappa, \tag{1.19}$$

for $C(\rho) > 0$ *a constant depending on* ρ. *Also,*

$$\sup_{t\ge 0} \left[(1+t)^{1+\lambda} \|u(t)\|_2^2 + (1+t)^{1+\lambda} \|\eta(t)\|_{L^\infty}^2 + \sum_{j=0}^{1} (1+t)^{j+\lambda} \left\|D^j \eta(t)\right\|_0^2\right] \le C\kappa \tag{1.20}$$

for a universal constant $C > 0$.

Remark 1.4. *In our companion paper* [14], *where we analyze* (1.9) *in horizontally periodic domains, we will require* η_0 *to satisfy the "zero average condition"*

$$\int_\Sigma \eta_0 = 0. \tag{1.21}$$

For the horizontally periodic problem, this condition propagates in time (see Lemma 2.5, a variant of which holds in the periodic case), from which one sees that (1.21) *is a necessary condition decay in* L^2 *or* L^∞*. It also serves as an obstacle to applying Beale's non-decay theorem. For a complete discussion, we refer to our paper* [14].

In the present case, the bound $\mathcal{E}_{10}(0) < \kappa$ *requires, in particular, that the initial data satisfy* $\|\mathcal{I}_\lambda \eta_0\|_0^2 < \infty$*. This condition can be viewed as a sort of weak version of the zero average condition in the infinite case. To see this, note that if* η_0 *is sufficiently nice, say* $L^1(\Sigma)$*, then*

$$0 = \int_\Sigma \eta_0 \Leftrightarrow \hat{\eta}_0(0) = 0, \tag{1.22}$$

for $\hat{\cdot}$ *the Fourier transform. This means that the zero average condition is equivalent to requiring that* $\hat{\eta}_0$ *vanishes at the origin. We enforce a weak version of this by requiring that* $\mathcal{I}_\lambda \eta_0 \in L^2(\Sigma) = H^0(\Sigma)$*, which requires that* $|\xi|^{-2\lambda} |\hat{\eta}_0(\xi)|^2$ *is integrable near* $\xi = 0$*. Since* $\lambda < 1$*, this does not require* $\hat{\eta}_0(0) = 0$*, but it does prevent* $|\hat{\eta}_0|$ *from being "too big" at the origin. Note that the condition* $\mathcal{I}_\lambda \eta_0 \in L^2$ *is more general than* (1.21).

Remark 1.5. *The decay estimates* (1.19) *and* (1.20) *do not follow directly from the decay of* $\mathcal{E}_{7,2}(t)$ *implied by* (1.18)*. Rather, they are deduced via auxiliary arguments, employing* (1.18).

Remark 1.6. *The decay of* $\|u(t)\|_2^2$ *given in* (1.20) *is not fast enough to guarantee that* $u \in L^1([0,\infty); H^2(\Omega))$*. Even if we could take* $\lambda = 1$*, we would still get logarithmic blow-up of the* $L^1 H^2$ *norm.*

Remark 1.7. *The surface* η *is sufficiently small to guarantee that the mapping* $\Phi(\cdot, t)$*, defined in* (1.6)*, is a diffeomorphism for each* $t \geq 0$*. As such, we may change coordinates to* $y \in \Omega(t)$ *to produce a global-in-time, decaying solution to* (1.2).

Remark 1.8. *Later in the paper, we let* $N \geq 3$ *be an integer and perform our analysis in terms of estimates at the* $2N$ *and* $N+2$ *levels; we take* $N = 5$ *in the present case to get the* 10 *and* 7 *appearing above. This is not optimal. With somewhat more work, we can improve our results to* $N = 4$ *with the restriction that* $\lambda \in (3/5, 1)$*. It is likely that this can be further improved by adjusting the scheme from* $2N$ *and* $N+2$ *to something slightly different. We have sacrificed optimality in order to simplify the presentation and make our "two-tier energy method" clearer. The first tier is at the level* $2N$ *and the second at the level* $N+2$*, which is meant to be roughly half of the first tier. The extra* $+2$ *is added to aid in applying some Sobolev embeddings.*

The proof of Theorem 1.3 is completed in Section 11. We now present a summary of the principal difficulties we encounter in our analysis as well as a sketch of the key ideas used in our proof.

Principal difficulties

In the study of the unforced incompressible Navier-Stokes equations in a fixed bounded domain with Dirichlet boundary conditions, it is natural to use the energy method to prove that solutions decay in time. Indeed, one may prove an analogue of (1.3) for sufficiently smooth solutions, which relates the natural energy and dissipation:

$$\partial_t \mathcal{E} + \mathcal{D} := \partial_t \int_\Omega \frac{|u(t)|^2}{2} + \frac{1}{2}\int_\Omega |\mathbb{D}u(t)|^2 = 0. \tag{1.23}$$

Korn's inequality allows us to control $C\mathcal{E}(t) \leq \mathcal{D}(t)$ for a constant $C > 0$ independent of time, which shows that the dissipation is stronger than the energy. From this and Gronwall's lemma we may immediately deduce that the energy $\mathcal{E}$ decays exponentially in time and that we have the estimate $\mathcal{E}(t) \leq \mathcal{E}(0)\exp(-Ct)$.

If one seeks to similarly use the energy method to obtain decay estimates for solutions to (1.2), then one encounters a fundamental obstacle that may already be observed in the differential form of (1.3),

$$\partial_t \left(\int_{\Omega(t)} \frac{|u(t)|^2}{2} + \int_\Sigma \frac{|\eta(t)|^2}{2} \right) + \frac{1}{2} \int_{\Omega(t)} |\mathbb{D}u(t)|^2 = 0. \tag{1.24}$$

The difficulty is that the dissipation provides no direct control of the $\eta-$term in the energy. As such, we must resort to using the equations (1.2) to try to control $\|\eta(t)\|_0$ in terms of $\|\mathbb{D}u(t)\|_0$. From (1.2) we see that there are only two available routes: solving for η in the fourth equation; or using the third equation, which is the kinetic transport equation. If we pursue the first route, then we must be able to control

$$\|p(t)\|^2_{H^0(\Sigma)} + \|\mathbb{D}u(t)\nu \cdot \nu\|^2_{H^0(\Sigma)} \lesssim \|\mathbb{D}u(t)\|^2_{H^0(\Omega(t))} , \tag{1.25}$$

which is not possible. If instead we pursue the second route, then we must estimate η as a solution to the kinematic transport equation. Such an estimate (see Lemma A.9) only allows us to estimate $\|\eta(t)\|_0$ in terms of $\int_0^t \|\mathbb{D}u(s)\|_0 \, ds$. That is, transport estimates do not provide control of the $\eta-$part of the energy in terms of the "instantaneous" dissipation, but rather in terms of the "cumulative" integrated dissipation. From this we see that in our problem the dissipation is actually weaker than the energy, so we cannot argue as above to deduce exponential decay.

We might hope that we could avoid this problem by working with a high-regularity energy method, but we will always encounter the same type of problem as above. Regardless of the level of regularity in the energy, the instantaneous dissipation is always weaker than the instantaneous energy, which prevents us from deducing exponential decay of the energy. Instead we pursue a strategy similar to one employed in [24] for another problem where the dissipation is weaker than the energy. We first show that high-order energies are bounded by using an integrated version or (1.24) for derivatives of the solution. Then we consider a low-order energy and show that an equation of the form (1.24) holds, i.e. $\partial_t \mathcal{E}_{\mathrm{low}} + C\mathcal{D}_{\mathrm{low}} \le 0$. Now, instead of trying to estimate (1.25) for low-order derivatives, we instead interpolate between low-order derivatives and high-order derivatives, which are bounded. Instead of an estimate $C\mathcal{E}_{\mathrm{low}} \le \mathcal{D}_{\mathrm{low}}$, we must prove one of the form $C\mathcal{E}_{\mathrm{low}}^{1+\theta} \le \mathcal{D}_{\mathrm{low}}$ for some $\theta > 0$. We can then use this to derive the differential inequality $\partial_t \mathcal{E}_{\mathrm{low}} + C\mathcal{E}_{\mathrm{low}}^{1+\theta} \le 0$, which can be integrated to see that $\mathcal{E}_{\mathrm{low}}(t) \lesssim \mathcal{E}_{\mathrm{low}}(0)/(1+t)^{1/\theta}$. We would then find that the low-order energy decays algebraically in time rather than exponentially.

To complete this program, we must overcome a pair of intertwined difficulties. First, to close the high-order energy estimates with, say $\|u\|^2_{4N+1}$ for an integer $N \ge 0$ in the dissipation, we have to control η in $H^{4N+1/2}$. The only option for this is to again appeal to estimates for solutions to the transport equation, which say (roughly speaking) that

$$\sup_{0\le t\le T} \|\eta\|^2_{4N+1/2} \le C \exp\left(C \int_0^T \|Du(t)\|_{H^2(\Sigma)} \, dt \right) \left[\|\eta_0\|^2_{4N+1/2} + T \int_0^T \|u(t)\|^2_{4N+1} \, dt \right]. \tag{1.26}$$

Without knowing a priori that u decays, the right side of this estimate has the potential to grow at the rate of $(1+T)e^{\sqrt{T}}$. Even if u decays rapidly, the right side can still grow like $(1+T)$. This growth is potentially disastrous in closing the high-order, global-in-time estimates. To manage the growth, we must identify a special decaying term that always appears in products with the highest derivatives of η. If the special term decays quickly enough, then we can hope to balance the growth and close the high-order estimates. Due to the growth in (1.26), we believe that it is not possible to construct global-in-time solutions without also deriving a decay result.

This leads us to the second difficulty in this program. The decay rate of the special term is dictated by the decay rate of the low-order energy, so we must make the low-order energy decay sufficiently quickly. This amounts to making the constant $\theta > 0$ appearing in the interpolation estimates above sufficiently small. We must then carefully choose the terms that will appear in the low-order and high-order energies in order to keep θ small enough. It turns out that this requires us enforce a minimal derivative count in the low-order energy, i.e. only terms with m

derivatives or more are allowed. It also requires us to extend the high-order energy to include estimates of negative derivatives up to order $\lambda \in (0,1)$. Then $\theta = \theta(m,\lambda)$, and only by taking $m=2$, $\lambda > 0$ can we make θ small enough to achieve the desired decay rate.

The resolution of these intertwined difficulties requires a delicate and involved analysis. We now sketch some of the techniques we will employ.

Horizontal energy evolution estimates

In order to use the natural energy structure of the problem (given in Eulerian coordinates by (1.3)) to study high-order derivatives, we can only apply derivatives that do not break the structure of the boundary condition $u=0$ on Σ_b. Since Σ_b is flat, any differential operator $\partial^\alpha = \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ is allowed. We apply these operators for various choices of α and sum the resulting energy evolution equations. After estimating the nonlinear terms that appear from differentiating (1.9), we are eventually led to evolution equations for these "horizontal" energies and dissipations, $\bar{\mathcal{E}}_{10}$, $\bar{\mathcal{D}}_{10}$, $\bar{\mathcal{E}}_{7,m}$, and $\bar{\mathcal{D}}_{7,m}$ for $m=1,2$ (see Section 2.4 for precise definitions). Here we write bars to indicate "horizontal" derivatives. Roughly speaking, at high-order we have the estimate

$$\bar{\mathcal{E}}_{10}(t) + \int_0^t \bar{\mathcal{D}}_{10}(r)dr \lesssim \mathcal{E}_{10}(0) + \int_0^t (\mathcal{E}_{10}(r))^\theta \mathcal{D}_{10}(r)dr + \int_0^t \sqrt{\mathcal{D}_{10}(r)\mathcal{K}(r)\mathcal{F}_{10}(r)}dr, \tag{1.27}$$

where $\mathcal{K}$ is of the form

$$\mathcal{K} = \|\nabla u\|_{C^1}^2 + \|Du\|_{H^2(\Sigma)}^2, \tag{1.28}$$

and $\theta > 0$; and at the low-order we have

$$\partial_t \bar{\mathcal{E}}_{7,m} + \bar{\mathcal{D}}_{7,m} \lesssim \mathcal{E}_{10}^\theta \mathcal{D}_{7,m}, \tag{1.29}$$

where $\mathcal{D}_{7,m}$ is the low-order dissipation. Notice that the product $\mathcal{K}\mathcal{F}_{10}$ in (1.27) multiplies low-order norms of u against the highest-order norm of η. Technically, the estimate (1.27) also involves $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$ in addition to horizontal derivatives. For the moment let us ignore these terms and continue with the discussion of our energy method. We will discuss $\mathcal{I}_\lambda$ in detail below.

The actual derivation of bounds like (1.27)–(1.29) is rather delicate and depends crucially on the geometric structure of the equations given in (1.9). Indeed, if we attempted rewrite (1.9) as a perturbation of the usual constant-coefficient Navier-Stokes equations, then we would fail to achieve the estimate (1.27) because we would be unable to control the interaction between $\partial_t^{10} p$ and $\operatorname{div} \partial_t^{10} u$, the latter of which does not vanish in the geometric form of the equations.

Comparison estimates

The next step in the analysis is to replace the horizontal energies and dissipations with the full energies and dissipations. We prove that there is a universal $0 < \delta < 1$ so that if $\mathcal{E}_{10} \le \delta$, then

$$\begin{gathered} \mathcal{E}_{10} \lesssim \bar{\mathcal{E}}_{10}, \quad \mathcal{D}_{10} \lesssim \bar{\mathcal{D}}_{10} + \mathcal{K}\mathcal{F}_{10}, \\ \mathcal{E}_{7,m} \lesssim \bar{\mathcal{E}}_{7,m}, \quad \mathcal{D}_{7,m} \lesssim \bar{\mathcal{D}}_{7,m} \end{gathered} \tag{1.30}$$

This estimate is extremely delicate and can only be obtained by carefully using the structure of the equations. We make use of every bit of information from the boundary conditions and the vorticity equations to establish it. There are two structural components of the estimates that are of such importance that we mention them now. First, the equation $\operatorname{div}_{\mathcal{A}} u = 0$ allows us to write $\partial_3 u_3 = -(\partial_1 u_1 + \partial_2 u_2) + G^2$ for some quadratic nonlinearity G^2. This allows us to "trade" a vertical derivative of u_3 for horizontal derivatives of u_1 and u_2, an indispensable trick in our analysis. Second, the interaction between the parabolic scaling of u ($\partial_t u \sim \Delta u$) and the transport scaling of η ($\partial_t \eta \sim u_3|_\Sigma$) allows us to gain regularity for the temporal derivatives of η in the dissipation, and it also gives us control of $\partial_t^{11}\eta$, which is one more time derivative than appears in the energy.

Two-tier energy method

Suppose we know that

$$\mathcal{K}(r) \le \frac{\delta}{(1+r)^{2+\gamma}} \tag{1.31}$$

for some $0 < \delta < 1$ and $\gamma > 0$. Since η satisfies a transport equation, we may use Lemma A.9 to derive an estimate of the form

$$\sup_{0 \le r \le t} \mathcal{F}_{10}(r) \lesssim \exp\left(C \int_0^t \sqrt{\mathcal{K}(r)} dr \right) \left[\mathcal{F}_{10}(0) + t \int_0^t \mathcal{D}_{10}(r) dr \right]. \tag{1.32}$$

Although the right side of this equation could potentially blow up exponentially in time, the decay of $\mathcal{K}$ in (1.31) implies that

$$\sup_{0 \le r \le t} \mathcal{F}_{10}(r) \lesssim \mathcal{F}_{10}(0) + t \int_0^t \mathcal{D}_{10}(r) dr. \tag{1.33}$$

Note that $\gamma > 0$ in (1.31) is essential; we would not be able to tame the exponential term in (1.32) without it, and then (1.33) would not hold. This estimate allows for $\mathcal{F}_{10}(t)$ to grow linearly in time, but in the product $\mathcal{K}(r)\mathcal{F}_{10}(r)$ that appears in (1.27), we can use the decay of $\mathcal{K}$ to balance this growth. Then if $\sup_{0 \le r \le t} \mathcal{E}_{10}(r) \le \delta$ with δ small enough, we can combine (1.27), (1.30), (1.31), and (1.33) to get an estimate

$$\mathcal{E}_{10}(t) + \int_0^t \mathcal{D}_{10}(r) dr \lesssim \mathcal{E}_{10}(0) + \mathcal{F}_{10}(0). \tag{1.34}$$

This highlights the first step of our two-tier energy method: the decay of low-order terms (i.e. $\mathcal{K}$) can balance the growth of $\mathcal{F}_{10}$, yielding boundedness of the high-order terms. In order to close this argument, we must use a second step: the boundedness of the high-order terms implies the decay of low-order terms, and in particular the decay of $\mathcal{K}$.

To attain this decay, we combine (1.29) and (1.30) to see that

$$\partial_t \bar{\mathcal{E}}_{7,m} + \frac{1}{2} \mathcal{D}_{7,m} \le 0 \tag{1.35}$$

if $\mathcal{E}_{10} \le \delta$ for δ small enough. If we could show that $\bar{\mathcal{E}}_{7,m} \lesssim \mathcal{D}_{7,m}$, then this estimate would yield exponential decay of $\bar{\mathcal{E}}_{7,m}$ and $\mathcal{E}_{7,m}$. An inspection of $\bar{\mathcal{E}}_{7,m}$ and $\mathcal{D}_{7,m}$ (see Section 2.4) shows that $\mathcal{D}_{7,m}$ can control every term in $\bar{\mathcal{E}}_{7,m}$ except $\|\eta\|_0^2$ (and $\|\partial_t \eta\|_0^2$ when $m = 2$). In a sense, this means that exponential decay fails precisely because the dissipation fails to control η at the lowest order. In lieu of $\bar{\mathcal{E}}_{7,m} \lesssim \mathcal{D}_{7,m}$, we instead interpolate between $\mathcal{E}_{10}$ (which can control all the lowest-order terms of η) and $\mathcal{D}_{7,m}$:

$$\bar{\mathcal{E}}_{7,m} \lesssim \mathcal{E}_{10}^{1/(m+\lambda+1)} \mathcal{D}_{7,m}^{(m+\lambda)/(m+\lambda+1)}. \tag{1.36}$$

Combining (1.35) with (1.36) and the boundedness of $\mathcal{E}_{10}$ in terms of the data, (1.34), then allows us to deduce that

$$\partial_t \bar{\mathcal{E}}_{7,m} + \frac{C}{(\mathcal{E}_{10}(0) + \mathcal{F}_{10}(0))^{1/(m+\lambda)}} (\bar{\mathcal{E}}_{7,m})^{1+1/(m+\lambda)} \le 0. \tag{1.37}$$

Gronwall's inequality (along with some auxiliary estimates) then leads us to the bound

$$\mathcal{E}_{7,m}(t) \lesssim \bar{\mathcal{E}}_{7,m}(t) \lesssim \frac{\mathcal{E}_{10}(0) + \mathcal{F}_{10}(0)}{(1+t)^{m+\lambda}}. \tag{1.38}$$

We thus use the boundedness of high-order terms to deduce the decay of low-order terms, completing the second step of the two-tier energy estimates.

Negative Sobolev estimates via $\mathcal{I}_\lambda$

Notice that the decay rate in (1.38) is enhanced by $\lambda \in (0,1)$. As we will see below, the parameter $\gamma > 0$ in the decay of $\mathcal{K}$, given in (1.31), is determined by the rate $m + \lambda$. If we took $\lambda = 0$, then we would not get $\gamma > 0$, and we would be unable to balance the growth of $\mathcal{F}_{10}$. Then estimates (1.33) and (1.34) would fail, and we would be unable to close our estimates. We thus see the necessity of introducing the "negative Sobolev" estimates via the horizontal Riesz potential $\mathcal{I}_\lambda$.

The difficulty, then, is that we must apply the non-local operator $\mathcal{I}_\lambda$ to a nonlinear PDE and then study the evolution of $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$. The flatness of the lower boundary Σ_b is essential here since it allows us to have $\mathcal{I}_\lambda u = 0$ on Σ_b. This means that the operator $\mathcal{I}_\lambda$ does not break the boundary conditions, and we can use the natural energy structure to include $\|\mathcal{I}_\lambda u\|_0^2$ and

$\|\mathcal{I}_\lambda \eta\|_0^2$ in the energy and $\|\mathcal{I}_\lambda u\|_1^2$ in the dissipation. To close the estimates for these terms, we must be able to estimate $\mathcal{I}_\lambda$ acting on various nonlinearities in terms of $\mathcal{E}_{10}^\theta \mathcal{D}_{10}$ for some $\theta > 0$. These estimates turn out to be rather delicate, and we must again employ almost all of the structure of the equations and boundary conditions in order to derive them. They are also responsible for the constraint $\lambda < 1$. For $\lambda \geq 1$, the nonlinear estimates would not work as we need them to.

We should point out that a priori, we do not know that $\mathcal{I}_\lambda u(t)$ or $\mathcal{I}_\lambda \eta(t)$ even make sense for $t > 0$, since this is not provided by Theorem 1.1. To show that these terms are well-defined, which then justifies applying $\mathcal{I}_\lambda$ to the equations, we must actually prove a specialization of the local well-posedness theorem that includes the boundedness of $\mathcal{I}_\lambda u$, $\mathcal{I}_\lambda p$, and $\mathcal{I}_\lambda \eta$. We do this in Theorem 10.7.

Interpolation estimates and minimal derivative counts

The negative Sobolev estimates alone do not close the overall estimates in our two-tier energy method. To do that, we must verify that $\mathcal{K}$ decays as in (1.31) for some $\gamma > 0$. An inspection of $\mathcal{E}_{7,m}$ shows that we cannot directly control $\mathcal{K} \lesssim \mathcal{E}_{7,m}$ for either $m = 1, 2$, so we must resort to an interpolation argument. We show that through interpolation it is actually possible to control $\mathcal{K} \lesssim \mathcal{E}_{7,1}$, but the $\mathcal{E}_{7,1}$ only decays like $(1+t)^{-1-\lambda}$, which is not fast enough for (1.31). The energy $\mathcal{E}_{7,2}$ decays at a faster rate, but we cannot show that $\mathcal{K} \lesssim \mathcal{E}_{7,2}$. Instead, we show that if $\mathcal{E}_{7,2}(t) \leq \varepsilon(1+t)^{-2-\lambda}$, then

$$\mathcal{K} \lesssim \mathcal{E}_{7,2}^{(8+2\lambda)/(8+4\lambda)} \lesssim \varepsilon^{(8+2\lambda)/(8+4\lambda)} \frac{1}{(1+t)^{2+\lambda/2}}, \tag{1.39}$$

so that after renaming $\delta = C\varepsilon^{(8+2\lambda)/(8+4\lambda)}$ and $\gamma = \lambda/2 > 0$ we find that (1.31) does hold.

The parameters m and λ interact in an important way. The decay rate increases with m and with λ. As mentioned above, we are technically constrained to $\lambda < 1$, so we must increase m to 2 in order to hit the target decay rate in (1.31). It is tempting, then, to consider abandoning the $\mathcal{I}_\lambda$ operators and simply use a third energy with $m \geq 3$, which should decay like $(1+t)^{-m}$. However, if one were to do this for any $m \geq 3$, one would find that there is a corresponding decrease in the interpolation power: $\mathcal{K} \lesssim \mathcal{E}_{7,m}^{\theta(m)}$, where $\theta(m)$ decreases with m in such a way that $m\theta(m) \leq 2$ so that (1.31) would fail. We thus see that the negative estimates are not just a convenience, but rather a necessity.

The derivation of (1.39) is delicate, requiring a two-step bootstrap process to iteratively improve the interpolation powers. We again crucially make use of the structure of the equations and boundary conditions. We extensively interpolate between our negative Sobolev estimates and our positive Sobolev estimates. The utility of the negative estimates is quite clear here: the interpolation powers improve when we interpolate with negative derivatives (as opposed to say, no derivatives).

To complete the proof of (1.39), we crucially use an estimate for $\mathcal{I}_1 \partial_t \eta$. This corresponds to $\lambda = 1$, so we are not able to apply $\mathcal{I}_1 \partial_t$ to the equations to attain the estimate. Rather, the estimate comes for free from the transport equation for η, which allows us to write $\partial_t \eta = -\partial_1 U_1 - \partial_2 U_2$ for $U_i \in H^1$. In our analysis of the horizontally periodic problem in [14], where we can take $\Sigma = \mathbb{T}^2$, this identity and (1.21) give rise to a Poincaré inequality $\|\eta(t)\|_0^2 \lesssim \|D\eta(t)\|_0^2$ for $t \geq 0$, which is crucial in our analysis there. From this we see that the estimate for $\mathcal{I}_1 \partial_t \eta$ is of analytic importance for the problem (1.2).

The interpolation of negative and positive Sobolev estimates provides a completely new tool in the study of time decay in dissipative PDE problems in the whole (or semi-infinite) space. For the viscous surface wave problem, a particular advantage of the negative-positive method is that, unlike the usual $L^p - L^q$ machinery, our norms are preserved along the time evolution. We anticipate that this method will prove useful in the analysis of other dissipative equations.

1.6. **Comparison to the periodic problem.** In our companion paper [14], we prove the analogue of Theorem 1.3 for horizontally periodic domains. In this context we take $N \geq 3$ to be an integer and consider energies and dissipations $\mathcal{E}_{2N}$, $\mathcal{D}_{2N}$, $\mathcal{F}_{2N}$, and $\mathcal{G}_{2N}$; these are modifications of what we use here (with $N = 5$) that include temporal derivatives up to order

$2N$. See the paper [14] for the precise definitions. By increasing N, we can achieve arbitrarily fast algebraic rates for the solutions, which we identify as "almost exponential decay."

In order to compare with Theorem 1.3, we record a version of the periodic result now.

Theorem 1.9. *Suppose the initial data (u_0, η_0) satisfy the compatibility conditions of Theorem 1.1 and that η_0 satisfies the zero average condition (1.21). Let $N \geq 3$ be an integer. There exists a $0 < \kappa = \kappa(N)$ so that if $\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa$, then there exists a unique solution (u, p, η) on the interval $[0, \infty)$ that achieves the initial data. The solution obeys the estimates*

$$\mathcal{G}_{2N}(\infty) \leq C_1 \left(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)\right) < C_1 \kappa, \tag{1.40}$$

and

$$\sup_{t \geq 0} (1+t)^{4N-8} \left[\|u(t)\|_{2N+4}^2 + \|\eta(t)\|_{2N+4}^2 \right] \leq C_1 \kappa, \tag{1.41}$$

where $C_1 > 0$ is a universal constant.

Remark 1.10. *A key difference between the periodic result, Theorem 1.9, and the non-periodic result, Theorem 1.3, is that in the periodic case, increasing N also increases the decay rate. No such gain is possible in the non-periodic case, which is why we specialize to the case $N = 5$ there. In the periodic case, we do not use the same type of interpolation arguments that we use in the infinite case. This allows us to relax to $N \geq 3$.*

Remark 1.11. *In [17], Hataya studied the periodic problem with a flat bottom. Using the Beale-Solonnikov parabolic theory, it was shown that*

$$\int_0^\infty (1+t)^2 \|u(t)\|_{r-1}^2 \, dt + \sup_{t \geq 0} (1+t)^2 \|\eta(t)\|_{r-2}^2 < \infty \tag{1.42}$$

for $r \in (5, 11/2)$. Our result on the periodic problem is an improvement of this in two important ways. First, we establish faster decay rates by working in a higher regularity context. Second, we allow for a more general non-flat bottom geometry (see [14] for details).

Remark 1.12. *The reader interested in a unified presentation of Theorems 1.1, 1.3, and 1.9 may consult [12].*

1.7. **Comparison to the case with surface tension.** If the effect of surface tension is included at the air-fluid free interface, then the formulation of the PDE must be changed. Surface tension is modeled by modifying the fourth equation in (1.2) to be

$$(pI - \mu \mathbb{D}(u))\nu = g\eta\nu - \sigma H \nu, \tag{1.43}$$

where $H = \partial_i(\partial_i \eta / \sqrt{1 + |D\eta|^2})$ is the mean curvature of the surface $\{y_3 = \eta(t)\}$ and $\sigma > 0$ is the surface tension.

In [5], Beale proved small-data global well-posedness for the problem with surface tension in horizontally infinite domains. The flattened coordinate system we employ was introduced in [5] and used in place of Lagrangian coordinates. However, Beale employed a change of unknown velocities that is more complicated than just a coordinate change. Well-posedness was demonstrated with $u \in L^2 H^r$ and $\eta \in L^2 H^{r+1/2}$, given that $u_0 \in H^{r-1/2}$, $\eta_0 \in H^r$ are sufficiently small for $r \in (3, 7/2)$. In this context it is understood that surface tension leads to the decay of certain modes, thereby aiding global existence.

In [6], Beale-Nishida studied the asymptotic properties of the solutions constructed in [5]. They showed that if $\eta_0 \in L^1(\Sigma)$, then

$$\sup_{t \geq 0} (1+t)^2 \|u(t)\|_2^2 + \sup_{t \geq 0} \sum_{j=1}^2 (1+t)^{1+j} \left\| D^j \eta(t) \right\|_0^2 < \infty, \tag{1.44}$$

and that this decay rate is optimal. Taking $\lambda \approx 1$ in our Theorem 1.3, the estimates (1.20) yield almost the same decay rates.

In [20], Nishida-Teramoto-Yoshihara showed that in horizontally periodic domains with surface tension and a flat bottom, if η_0 has zero average, then there exists a $\gamma > 0$ so that

$$\sup_{t \geq 0} e^{\gamma t} \left[\|u(t)\|_2^2 + \|\eta(t)\|_3^2 \right] < \infty. \tag{1.45}$$

In this case, the equation (1.43) gives a third way of estimating η in terms of the dissipation; using this, it is possible to show that the dissipation is stronger than the energy. Thus, if surface tension is added in the periodic case, fully exponential decay is possible, whereas without surface tension we only recover algebraic decay of arbitrary order in Theorem 1.9.

The comparison of these two results with ours establishes a nice contrast between the surface tension and non-surface tension cases. Without surface tension we can recover "almost" the same decay rate as in the case with surface tension. This shows that viscosity is the basic decay mechanism and that the effect of surface tension serves to enhance the decay rate.

1.8. **Definitions and terminology.** We now mention some of the definitions, bits of notation, and conventions that we will use throughout the paper.

Einstein summation and constants

We will employ the Einstein convention of summing over repeated indices for vector and tensor operations. Throughout the paper $C > 0$ will denote a generic constant that can depend on the parameters of the problem, N, and Ω, but does not depend on the data, etc. We refer to such constants as "universal." They are allowed to change from one inequality to the next. When a constant depends on a quantity z we will write $C = C(z)$ to indicate this. We will employ the notation $a \lesssim b$ to mean that $a \leq Cb$ for a universal constant $C > 0$.

Norms

We write $H^k(\Omega)$ with $k \geq 0$ and and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for the usual Sobolev spaces. We will typically write $H^0 = L^2$; the exception to this is when we use $L^2([0,T]; H^k)$ notation to indicate the space of square-integrable functions with values in H^k.

To avoid notational clutter, we will avoid writing $H^k(\Omega)$ or $H^k(\Sigma)$ in our norms and typically write only $\|\cdot\|_k$. Since we will do this for functions defined on both Ω and Σ, this presents some ambiguity. We avoid this by adopting two conventions. First, we assume that functions have natural spaces on which they "live." For example, the functions u, p, and $\bar{\eta}$ live on Ω, while η itself lives on Σ. As we proceed in our analysis, we will introduce various auxiliary functions; the spaces they live on will always be clear from the context. Second, whenever the norm of a function is computed on a space different from the one in which it lives, we will explicitly write the space. This typically arises when computing norms of traces onto Σ of functions that live on Ω.

Derivatives

We write $\mathbb{N} = \{0, 1, 2, \dots\}$ for the collection of non-negative integers. When using space-time differential multi-indices, we will write $\mathbb{N}^{1+m} = \{\alpha = (\alpha_0, \alpha_1, \dots, \alpha_m)\}$ to emphasize that the $0-$index term is related to temporal derivatives. For just spatial derivatives we write $\mathbb{N}^m$. For $\alpha \in \mathbb{N}^{1+m}$ we write $\partial^\alpha = \partial_t^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$. We define the parabolic counting of such multi-indices by writing $|\alpha| = 2\alpha_0 + \alpha_1 + \dots + \alpha_m$. We will write Df for the horizontal gradient of f, i.e. $Df = \partial_1 f e_1 + \partial_2 f e_2$, while ∇f will denote the usual full gradient.

For a given norm $\|\cdot\|$ and integers $k, m \geq 0$, we introduce the following notation for sums of spatial derivatives:

$$\left\| D_m^k f \right\|^2 := \sum_{\substack{\alpha \in \mathbb{N}^2 \\ m \leq |\alpha| \leq k}} \|\partial^\alpha f\|^2 \text{ and } \left\| \nabla_m^k f \right\|^2 := \sum_{\substack{\alpha \in \mathbb{N}^3 \\ m \leq |\alpha| \leq k}} \|\partial^\alpha f\|^2 . \tag{1.46}$$

The convention we adopt in this notation is that D refers to only "horizontal" spatial derivatives, while ∇ refers to full spatial derivatives. For space-time derivatives we add bars to our notation:

$$\left\| \bar{D}_m^k f \right\|^2 := \sum_{\substack{\alpha \in \mathbb{N}^{1+2} \\ m \leq |\alpha| \leq k}} \|\partial^\alpha f\|^2 \text{ and } \left\| \bar{\nabla}_m^k f \right\|^2 := \sum_{\substack{\alpha \in \mathbb{N}^{1+3} \\ m \leq |\alpha| \leq k}} \|\partial^\alpha f\|^2 . \tag{1.47}$$

When $k = m \geq 0$ we will write

$$\left\| D^k f \right\|^2 = \left\| D_k^k f \right\|^2, \left\| \nabla^k f \right\|^2 = \left\| \nabla_k^k f \right\|^2, \left\| \bar{D}^k f \right\|^2 = \left\| \bar{D}_k^k f \right\|^2, \left\| \bar{\nabla}^k f \right\|^2 = \left\| \bar{\nabla}_k^k f \right\|^2. \tag{1.48}$$

We allow for composition of derivatives in this counting scheme in a natural way; for example, we write

$$\left\| D D_m^k f \right\|^2 = \left\| D_m^k D f \right\|^2 = \sum_{\substack{\alpha \in \mathbb{N}^2 \\ m \leq |\alpha| \leq k}} \left\| \partial^\alpha D f \right\|^2 = \sum_{\substack{\alpha \in \mathbb{N}^2 \\ m+1 \leq |\alpha| \leq k+1}} \left\| \partial^\alpha f \right\|^2. \tag{1.49}$$

1.9. **Plan of paper.** Throughout the paper we assume that $N \geq 5$ and $\lambda \in (0,1)$ are both fixed. Notice that Theorem 1.3 is phrased with the choice $N = 5$.

In Section 2 we prove some preliminary lemmas and we define the energies and dissipations. In Section 3 we perform our bootstrap interpolation argument to control various quantities in terms of $\mathcal{E}_{N+2,m}$ and $\mathcal{D}_{N+2,m}$. In Section 4 we present estimates of the nonlinear forcing terms G^i (as defined in (2.24)–(2.31)) and some other nonlinearities. In Section 5 we use the geometric form of the equations to estimate the evolution of the highest-order temporal derivatives. We also analyze the natural (no derivatives) energy in this context. Section 6 concerns similar energy evolution estimates for the other horizontal derivatives. For these we employ the linear perturbed framework with the G^i forcing terms. In Section 7 we assemble the estimates of Sections 5 and 6 into unified estimates. Section 8 concerns the comparison estimates, where we show how to estimate the full energies and dissipations in terms of their horizontal counterparts. Section 9 combines all of the analysis of Sections 3–8 into our a priori estimates for solutions to (1.9). Section 10 concerns a specialized version of the local well-posedness theorem that includes the boundedness of $\mathcal{I}_\lambda$ terms. Finally, in Section 11 we record our global well-posedness and decay result, proving Theorem 1.3.

Below, in (2.58), we will define the total energy $\mathcal{G}_{2N}$ that we use in the global well-posedness analysis. For the purposes of deriving our a priori estimates, we will assume throughout Sections 3–9 that solutions are given on the interval $[0,T]$ and that $\mathcal{G}_{2N}(T) \leq \delta$ for $0 < \delta < 1$ as small as in Lemma 2.4 so that its conclusions hold. This also means that $\mathcal{E}_{2N}(t) \leq 1$ for $t \in [0,T]$. We should remark that Theorem 1.1 does not produce solutions that necessarily satisfy $\mathcal{G}_{2N}(T) < \infty$. All of the terms in $\mathcal{G}_{2N}(T)$ are controlled by Theorem 1.1 except those involving the Riesz operator: $\|\mathcal{I}_\lambda u\|_0^2$, $\|\mathcal{I}_\lambda \eta\|_0^2$, and $\int_0^T \|\mathcal{I}_\lambda u(t)\|_1^2 \, dt$. To guarantee that these terms are well-defined, we must prove a specialized version of the local well-posedness result, Theorem 10.7. In principle, we should record this before the a priori estimates, but the technique we use to control the $\mathcal{I}_\lambda$ terms is based on one we develop for the a priori estimates, so we present the theorem in Section 10 after the a priori estimates. Note that the bounds of Theorem 10.7 control more than just $\mathcal{G}_{2N}(T)$ (in particular, $\partial_t^{2N+1} u$, $\partial_t^{2N} p$, and $\mathcal{I}_\lambda p$), and the extra control it provides guarantees that all of the calculations used in the a priori estimates are justified.

2. Preliminaries for the a priori estimates

In this section we present some preliminary results that we will use in our a priori estimates. We first present two forms of equations similar to (1.9) and describe the corresponding energy evolution structure. Then we record some useful lemmas.

2.1. **Geometric form.** We now give a linear formulation of the PDE (1.9) in its geometric form. Suppose that η, u are known and that $\mathcal{A}, \mathcal{N}, J$, etc are given in terms of η as usual ((1.7), etc). We then consider the linear equation for (v, q, ζ) given by

$$\begin{cases} \partial_t v - \partial_t \bar{\eta} \tilde{b} K \partial_3 v + u \cdot \nabla_{\mathcal{A}} v + \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(q,v) = F^1 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} v = F^2 & \text{in } \Omega \\ S_{\mathcal{A}}(q,v)\mathcal{N} = \zeta \mathcal{N} + F^3 & \text{on } \Sigma \\ \partial_t \zeta - \mathcal{N} \cdot v = F^4 & \text{on } \Sigma \\ v = 0 & \text{on } \Sigma_b. \end{cases} \tag{2.1}$$

Now we record the natural energy evolution associated to solutions v, q, ζ of the geometric form equations (2.1).

Lemma 2.1. *Suppose that u and η are given solutions to* (1.9). *Suppose (v, q, ζ) solve* (2.1). *Then*

$$\partial_t \left(\frac{1}{2} \int_\Omega J \left|v\right|^2 + \frac{1}{2} \int_\Sigma \left|\zeta\right|^2 \right) + \frac{1}{2} \int_\Omega J \left|\mathbb{D}_{\mathcal{A}} v\right|^2 = \int_\Omega J(v \cdot F^1 + qF^2) + \int_\Sigma -v \cdot F^3 + \zeta F^4. \tag{2.2}$$

Proof. We multiply the i^{th} component of the first equation of (2.1) by Jv_i, sum over i and integrate over Ω to find that

$$I + II = III \tag{2.3}$$

for

$$I = \int_\Omega \partial_t v_i J v_i - \partial_t \bar{\eta} \tilde{b} \partial_3 v_i v_i + u_j \mathcal{A}_{jk} \partial_k v_i J v_i, \tag{2.4}$$

$$II = \int_\Omega \mathcal{A}_{jk} \partial_k S_{ij}(v, q) J v_i, \text{ and } III = \int_\Omega F^1 \cdot v J. \tag{2.5}$$

In order to integrate by parts in I, II we will utilize the geometric identity $\partial_k (J \mathcal{A}_{ik}) = 0$ for each i.

Then

$$I = \partial_t \int_\Omega \frac{\left|v\right|^2 J}{2} + \int_\Omega -\frac{\left|v\right|^2 \partial_t J}{2} - \partial_t \bar{\eta} \tilde{b} \partial_3 \frac{\left|v\right|^2}{2} + u_j \partial_k \left(J \mathcal{A}_{jk} \frac{\left|v\right|^2}{2} \right) := I_1 + I_2. \tag{2.6}$$

Since $\tilde{b} = 1 + x_3/b$, an integration by parts and an application of the boundary condition $v = 0$ on Σ_b reveals that

(2.7)
$$I_2 = \int_\Omega -\frac{\left|v\right|^2 \partial_t J}{2} - \partial_t \bar{\eta} \tilde{b} \partial_3 \frac{\left|v\right|^2}{2} + u_j \partial_k \left(J \mathcal{A}_{jk} \frac{\left|v\right|^2}{2} \right) = \int_\Omega -\frac{\left|v\right|^2 \partial_t J}{2} + \frac{\left|v\right|^2}{2} \left(\frac{\partial_t \bar{\eta}}{b} + \tilde{b} \partial_t \partial_3 \bar{\eta} \right)$$
$$- \int_\Omega \partial_k u_j J \mathcal{A}_{jk} \frac{\left|v\right|^2}{2} + \frac{1}{2} \int_\Sigma -\partial_t \eta \left|v\right|^2 + u_j J \mathcal{A}_{jk} e_3 \cdot e_k \left|v\right|^2.$$

It is straightforward to verify that $\partial_t J = \partial_t \bar{\eta}/b + \tilde{b} \partial_t \partial_3 \bar{\eta}$ in Ω and that $J \mathcal{A}_{jk} e_3 \cdot e_k = \mathcal{N}_j$ on Σ. Then since u, η satisfy $\partial_k u_j \mathcal{A}_{jk} = 0$ and $\partial_t \eta = u \cdot \mathcal{N}$, we have $I_2 = 0$. Hence

$$I = \partial_t \int_\Omega \frac{\left|v\right|^2 J}{2}. \tag{2.8}$$

A similar integration by parts shows that

$$\begin{aligned} II &= \int_\Omega -\mathcal{A}_{jk} S_{ij}(v, q) J \partial_k v_i + \int_\Sigma J \mathcal{A}_{j3} S_{ij}(v, q) v_i \\ &= \int_\Omega -q \mathcal{A}_{ik} \partial_k v_i J + J \frac{\left|\mathbb{D}_{\mathcal{A}} v\right|^2}{2} + \int_\Sigma S_{ij}(v, q) \mathcal{N}_j v_i \end{aligned} \tag{2.9}$$

so that

$$II = \int_\Omega -qJF^2 + J \frac{\left|\mathbb{D}_{\mathcal{A}} v\right|^2}{2} + \int_\Sigma \zeta \mathcal{N} \cdot v + v \cdot F^3. \tag{2.10}$$

But

$$\int_\Sigma \zeta \mathcal{N} \cdot v = \int_\Sigma \zeta (\partial_t \zeta - F^4) = \partial_t \int_\Sigma \frac{\left|\zeta\right|^2}{2} + \int_\Sigma -\zeta F^4, \tag{2.11}$$

which means

$$II = \int_\Omega -qJF^2 + J \frac{\left|\mathbb{D}_{\mathcal{A}} v\right|^2}{2} + \partial_t \int_\Sigma \frac{\left|\zeta\right|^2}{2} + \int_\Sigma -\zeta F^4. \tag{2.12}$$

Now (2.2) follows from (2.3), (2.8), and (2.12). □

In order to utilize (2.1) we apply the differential operator $\partial^\alpha = \partial_t^{\alpha_0}$ to (1.9). The resulting equations are (2.1) for $v = \partial^\alpha u$, $q = \partial^\alpha p$, and $\zeta = \partial^\alpha \eta$, where

$$F^1 = F^{1,1} + F^{1,2} + F^{1,3} + F^{1,4} + F^{1,5} + F^{1,6} \tag{2.13}$$

for

$$F_i^{1,1} = \sum_{0<\beta<\alpha} C_{\alpha,\beta} \partial^\beta (\partial_t \bar{\eta} \tilde{b} K) \partial^{\alpha-\beta} \partial_3 u_i + \sum_{0<\beta\le\alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} \partial_t \bar{\eta} \partial^\beta (\tilde{b} K) \partial_3 u_i \tag{2.14}$$

$$F_i^{1,2} = -\sum_{0<\beta\le\alpha} C_{\alpha,\beta} \left(\partial^\beta (u_j \mathcal{A}_{jk}) \partial^{\alpha-\beta} \partial_k u_i + \partial^\beta \mathcal{A}_{ik} \partial^{\alpha-\beta} \partial_k p \right) \tag{2.15}$$

$$F_i^{1,3} = \sum_{0<\beta\le\alpha} C_{\alpha,\beta} \partial^\beta \mathcal{A}_{j\ell} \partial^{\alpha-\beta} \partial_\ell (\mathcal{A}_{im} \partial_m u_j + \mathcal{A}_{jm} \partial_m u_i) \tag{2.16}$$

$$F_i^{1,4} = \sum_{0<\beta<\alpha} C_{\alpha,\beta} \mathcal{A}_{jk} \partial_k (\partial^\beta \mathcal{A}_{i\ell} \partial^{\alpha-\beta} \partial_\ell u_j + \partial^\beta \mathcal{A}_{j\ell} \partial^{\alpha-\beta} \partial_\ell u_i) \tag{2.17}$$

$$F_i^{1,5} = \partial^\alpha \partial_t \bar{\eta} \tilde{b} K \partial_3 u_i \text{ and } F_i^{1,6} = \mathcal{A}_{jk} \partial_k (\partial^\alpha \mathcal{A}_{i\ell} \partial_\ell u_j + \partial^\alpha \mathcal{A}_{j\ell} \partial_\ell u_i). \tag{2.18}$$

In these equations, the terms $C_{\alpha,\beta}$ are constants that depend on α and β. The term $F^2 = F^{2,1} + F^{2,2}$ for

$$F^{2,1} = -\sum_{0<\beta<\alpha} C_{\alpha,\beta} \partial^\beta \mathcal{A}_{ij} \partial^{\alpha-\beta} \partial_j u_i \text{ and } F^{2,2} = -\partial^\alpha \mathcal{A}_{ij} \partial_j u_i. \tag{2.19}$$

We write $F^3 = F^{3,1} + F^{3,2}$ for

$$F^{3,1} = \sum_{0<\beta\le\alpha} C_{\alpha,\beta} \partial^\beta D\eta (\partial^{\alpha-\beta} \eta - \partial^{\alpha-\beta} p) \tag{2.20}$$

$$F_i^{3,2} = \sum_{0<\beta\le\alpha} C_{\alpha,\beta} (\partial^\beta (\mathcal{N}_j \mathcal{A}_{im}) \partial^{\alpha-\beta} \partial_m u_j + \partial^\beta (\mathcal{N}_j \mathcal{A}_{jm}) \partial^{\alpha-\beta} \partial_m u_i). \tag{2.21}$$

Finally,

$$F^4 = \sum_{0<\beta\le\alpha} C_{\alpha,\beta} \partial^\beta D\eta \cdot \partial^{\alpha-\beta} u. \tag{2.22}$$

2.2. **Perturbed linear form.** Writing the equations in the form (1.9) is more faithful to the geometry of the free boundary problem, but it is inconvenient for many of our a priori estimates. This stems from the fact that if we want to think of the coefficients of the equations for u, p as being frozen for a fixed free boundary given by η, then the underlying linear operator has non-constant coefficients. This makes it unsuitable for applying differential operators.

To get around this problem, in many parts of the paper we will analyze the PDE in a different formulation, which looks like a perturbation of the linearized problem. The utility of this form of the equations lies in the fact that the linear operators have constant coefficients. The equations in this form are

$$\begin{cases} \partial_t u + \nabla p - \Delta u = G^1 & \text{in } \Omega \\ \operatorname{div} u = G^2 & \text{in } \Omega \\ (pI - \mathbb{D}u - \eta I) e_3 = G^3 & \text{on } \Sigma \\ \partial_t \eta - u_3 = G^4 & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b. \end{cases} \tag{2.23}$$

Here we have written $G^1 = G^{1,1} + G^{1,2} + G^{1,3} + G^{1,4} + G^{1,5}$ for

$$G_i^{1,1} = (\delta_{ij} - \mathcal{A}_{ij}) \partial_j p \tag{2.24}$$

$$G_i^{1,2} = u_j \mathcal{A}_{jk} \partial_k u_i \tag{2.25}$$

$$G_i^{1,3} = [K^2(1 + A^2 + B^2) - 1]\partial_{33} u_i - 2AK\partial_{13} u_i - 2BK\partial_{23} u_i \tag{2.26}$$

$$G_i^{1,4} = [-K^3(1+A^2+B^2)\partial_3 J + AK^2(\partial_1 J + \partial_3 A) + BK^2(\partial_2 J + \partial_3 B) - K(\partial_1 A + \partial_2 B)]\partial_3 u_i \tag{2.27}$$

$$G_i^{1,5} = \partial_t \bar{\eta}(1 + x_3/b) K \partial_3 u_i. \tag{2.28}$$

G^2 is the function

$$G^2 = AK\partial_3 u_1 + BK\partial_3 u_2 + (1-K)\partial_3 u_3, \tag{2.29}$$

and G^3 is the vector

$$G^3 := \partial_1 \eta \begin{pmatrix} p - \eta - 2(\partial_1 u_1 - AK\partial_3 u_1) \\ -\partial_2 u_1 - \partial_1 u_2 + BK\partial_3 u_1 + AK\partial_3 u_2 \\ -\partial_1 u_3 - K\partial_3 u_1 + AK\partial_3 u_3 \end{pmatrix} + \partial_2 \eta \begin{pmatrix} -\partial_2 u_1 - \partial_1 u_2 + BK\partial_3 u_1 + AK\partial_3 u_2 \\ p - \eta - 2(\partial_2 u_2 - BK\partial_3 u_2) \\ -\partial_2 u_3 - K\partial_3 u_2 + BK\partial_3 u_3 \end{pmatrix} + \begin{pmatrix} (K-1)\partial_3 u_1 + AK\partial_3 u_3 \\ (K-1)\partial_3 u_2 + BK\partial_3 u_3 \\ 2(K-1)\partial_3 u_3 \end{pmatrix}. \tag{2.30}$$

Finally,

$$G^4 = -D\eta \cdot u. \tag{2.31}$$

Remark 2.2. *The appearance of the term $(p - \eta)$ in the first two rows of the first two vectors in the definition of G^3 can cause some technical problems later when we attempt to estimate G^3. Notice though, that according to* (2.23)*, we may write*

$$(p - \eta) = 2\partial_3 u_3 + G^3 \cdot e_3 = \partial_1 \eta(-\partial_1 u_3 - K\partial_3 u_1 + AK\partial_3 u_3) + \partial_2 \eta(-\partial_2 u_3 - K\partial_3 u_2 + BK\partial_3 u_3) + 2K\partial_3 u_3 \tag{2.32}$$

on Σ. We may then replace the appearances of $(p - \eta)$ in (2.30) *with the right side of* (2.32)*.*

At several points in our analysis, we will need to localize (2.23) by multiplying by a cutoff function. This leads us to consider the energy evolution for a minor modification of (2.23).

Lemma 2.3. *Suppose (v, q, ζ) solve*

$$\begin{cases} \partial_t v + \nabla q - \Delta v = \Phi^1 & \text{in } \Omega \\ \operatorname{div} v = \Phi^2 & \text{in } \Omega \\ (qI - \mathbb{D}v)e_3 = a\zeta e_3 + \Phi^3 & \text{on } \Sigma \\ \partial_t \zeta - v_3 = \Phi^4 & \text{on } \Sigma \\ v = 0 & \text{on } \Sigma_b, \end{cases} \tag{2.33}$$

where either $a = 0$ or $a = 1$. Then

$$\partial_t \left(\frac{1}{2}\int_\Omega |v|^2 + \frac{1}{2}\int_\Sigma a\,|\zeta|^2 \right) + \frac{1}{2}\int_\Omega |\mathbb{D}v|^2 = \int_\Omega v \cdot \Phi^1 + q\Phi^2 + \int_\Sigma -v \cdot \Phi^3 + a\zeta \Phi^4. \tag{2.34}$$

Proof. We take the inner-product of the first equation in (2.33) with v and integrate over Ω to find

$$\partial_t \int_\Omega \frac{|v|^2}{2} - \int_\Omega (qI - \mathbb{D}v) : \nabla u + \int_\Sigma (qI - \mathbb{D}v)e_3 \cdot u = \int_\Omega v \cdot \Phi^1. \tag{2.35}$$

We then use the second equation in (2.33) to compute

$$\int_\Omega -(qI - \mathbb{D}v) : \nabla u = \int_\Omega -q \operatorname{div} v + \frac{|\mathbb{D}v|^2}{2} = \int_\Omega -q\Phi^2 + \frac{|\mathbb{D}v|^2}{2}. \tag{2.36}$$

The boundary conditions in (2.33) provide the equality

$$\int_\Sigma (qI - \mathbb{D}v)e_3 \cdot v = \int_\Sigma a\zeta v_3 + v \cdot \Phi^3 = \partial_t \int_\Sigma a\frac{|\zeta|^2}{2} + \int_\Sigma -a\zeta\Phi^4 + v \cdot \Phi^3. \tag{2.37}$$

Combining (2.35)–(2.37) then yields (2.34). □

2.3. **Some initial lemmas.** The following result is useful for removing the appearance of J factors.

Lemma 2.4. *There exists a universal* $0 < \delta < 1$ *so that if* $\|\eta\|_{5/2}^2 \le \delta$, *then*

$$\|J-1\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \le \frac{1}{2}, \text{ and } \|K\|_{L^\infty}^2 + \|\mathcal{A}\|_{L^\infty}^2 \lesssim 1. \tag{2.38}$$

Proof. According to the definitions of A, B, J given in (1.8) and Lemma A.5, we may bound

$$\|J-1\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \lesssim \|\bar{\eta}\|_3^2 \lesssim \|\eta\|_{5/2}^2 . \tag{2.39}$$

Then if δ is sufficiently small, we find that the first inequality in (2.38) holds. As a consequence $\|K\|_{L^\infty}^2 + \|\mathcal{A}\|_{L^\infty}^2 \lesssim 1$, which is the second inequality in (2.38). □

We now compute $\partial_t \eta$ in terms of a pair of auxiliary functions, U_1, U_2 defined on Σ. Note that in our analysis later, u and η will always be sufficiently smooth to justify the calculations in the next Lemma, and it will always hold that $U_i \in H^1(\Sigma)$.

Lemma 2.5. *For* $i = 1, 2$, *define* $U_i : \Sigma \to \mathbb{R}$ *by*

$$U_i(x') = \int_{-b}^0 J(x', x_3) u_i(x', x_3) dx_3. \tag{2.40}$$

Then $\partial_t \eta = -\partial_1 U_1 - \partial_2 U_2$ *on* Σ.

Proof. Let $\varphi \in \mathscr{S}(\Sigma)$. On Σ we have that $u \cdot \mathcal{N} = u \cdot (J\mathcal{A} e_3) = J\mathcal{A}^T u \cdot e_3 = J\mathcal{A}^T u \cdot \nu$, where $\nu = e_3$ is the unit normal to Σ. We may use the equation for $\partial_t \eta$ in (1.9) and the divergence theorem to compute

$$\begin{aligned} \int_\Sigma \partial_t \eta \varphi &= \int_\Sigma (-u_1 \partial_1 \eta - u_2 \partial_2 \eta + u_3)\varphi = \int_\Sigma \varphi J \mathcal{A}_{ij} u_i \nu_j = \int_\Omega \partial_j (\varphi J \mathcal{A}_{ij} u_i) \\ &= \int_\Omega \partial_j \varphi J \mathcal{A}_{ij} u_i + \varphi \partial_j (J\mathcal{A}_{ij}) u_i + \varphi J \mathcal{A}_{ij} \partial_j u_i = \int_\Omega \partial_j \varphi J \mathcal{A}_{ij} u_i, \end{aligned} \tag{2.41}$$

where the last equality follows from the geometric identity $\partial_j (J\mathcal{A}_{ij}) = 0$ and the equation $\mathcal{A}_{ij} \partial_j u_i = 0$, which is the second equation in (1.9). According to the definition of $\mathcal{A}$ given by (1.7), we may write $\mathcal{A}_{ij} = \delta_{ij} + \delta_{j3} Z_i$ for δ_{ij} the Kronecker delta and $Z = K(-Ae_1 - Be_2 + e_3)$. Then

$$\int_\Omega \partial_j \varphi J \mathcal{A}_{ij} u_i = \int_\Omega \partial_j \varphi J u_i (\delta_{ij} + \delta_{j3} Z_j) = \int_\Omega \partial_i \varphi J u_i + \int_\Omega \partial_3 \varphi J u_i Z_i = \int_\Omega \partial_i \varphi J u_i \tag{2.42}$$

since $\partial_3 \varphi = 0$, a consequence of the fact that $\varphi = \varphi(x_1, x_2)$ is independent of x_3. Again because φ depends only on $(x_1, x_2) = x' \in \Sigma$, we may write

$$\int_\Omega \partial_i \varphi J u_i = \int_\Sigma \partial_i \varphi(x') \int_{-b}^0 J(x', x_3) u_i(x', x_3) dx_3 dx' = \int_\Sigma \partial_i \varphi(x') U_i(x') dx'. \tag{2.43}$$

Now we chain together (2.41), (2.42), and (2.43) and integrate by parts to deduce that

$$\int_\Sigma \partial_t \eta \varphi = \int_\Sigma -\varphi \partial_i U_i. \tag{2.44}$$

Since this holds for any $\varphi \in \mathscr{S}(\Sigma)$ (resp. $C^\infty(\Sigma)$), we then have that $\partial_t \eta = -\partial_i U_i$. □

2.4. **Energies and dissipations.** Below we define the energies and dissipations we will use in our analysis. We state them in general in terms of two integers $n, m \in \mathbb{N}$ with $n \geq m$. In our actual analysis we will take $n = 2N$ and $n = N + 2$ for $N \geq 5$ and $m = 1, 2$. Recall that we employ the derivative conventions described in Section 1.8. We define the horizontal instantaneous energy with minimal derivative count m (or just horizontal energy, for short) by

$$\bar{\mathcal{E}}_{n,m} := \left\|\bar{D}_m^{2n-1} u\right\|_0^2 + \left\|D\bar{D}^{2n-1} u\right\|_0^2 + \left\|\sqrt{J}\partial_t^n u\right\|_0^2 + \left\|\bar{D}_m^{2n}\eta\right\|_0^2. \tag{2.45}$$

Here the first three terms are split in this manner for the technical convenience of adding the $\sqrt{J}$ term to only the highest temporal derivative.

Remark 2.6. *In light of Lemma 2.4, we see that* $\bar{\mathcal{E}}_{n,m}$ *satisfies*

$$\frac{1}{2}\left(\left\|\bar{D}_m^{2n} u\right\|_0^2 + \left\|\bar{D}_m^{2n}\eta\right\|_0^2\right) \leq \bar{\mathcal{E}}_{n,m} \leq \frac{3}{2}\left(\left\|\bar{D}_m^{2n} u\right\|_0^2 + \left\|\bar{D}_m^{2n}\eta\right\|_0^2\right) \tag{2.46}$$

We define the horizontal dissipation rate with minimal derivative count m (horizontal dissipation) by

$$\bar{\mathcal{D}}_{n,m} := \left\|\bar{D}_m^{2n}\mathbb{D}u\right\|_0^2. \tag{2.47}$$

Let $\mathcal{I}_\lambda$ be defined by (A.7)–(A.8). The horizontal energy without a minimal derivative restriction is

$$\bar{\mathcal{E}}_n := \|\mathcal{I}_\lambda u\|_0^2 + \left\|\bar{D}_0^{2n} u\right\|_0^2 + \|\mathcal{I}_\lambda \eta\|_0^2 + \left\|\bar{D}_0^{2n}\eta\right\|_0^2, \tag{2.48}$$

and the horizontal dissipation without a minimal derivative restriction is

$$\bar{\mathcal{D}}_n := \|\mathbb{D}\mathcal{I}_\lambda u\|_0^2 + \left\|\bar{D}_0^{2n}\mathbb{D}u\right\|_0^2. \tag{2.49}$$

In addition to the horizontal energy and dissipation, we must also define full energies and dissipations, which involve full derivatives. We write the full energy as

$$\mathcal{E}_n := \|\mathcal{I}_\lambda u\|_0^2 + \sum_{j=0}^{n}\left\|\partial_t^j u\right\|_{2n-2j}^2 + \sum_{j=0}^{n-1}\left\|\partial_t^j p\right\|_{2n-2j-1}^2 + \|\mathcal{I}_\lambda \eta\|_0^2 + \sum_{j=0}^{n}\left\|\partial_t^j \eta\right\|_{2n-2j}^2, \tag{2.50}$$

and we define the full dissipation rate by

$$\begin{aligned}\mathcal{D}_n := \|\mathcal{I}_\lambda u\|_1^2 + \sum_{j=0}^{n}\left\|\partial_t^j u\right\|_{2n-2j+1}^2 + \|\nabla p\|_{2n-1}^2 + \sum_{j=1}^{n-1}\left\|\partial_t^j p\right\|_{2n-2j} \\ + \|D\eta\|_{2n-3/2}^2 + \|\partial_t \eta\|_{2n-1/2}^2 + \sum_{j=2}^{n+1}\left\|\partial_t^j \eta\right\|_{2n-2j+5/2}^2.\end{aligned} \tag{2.51}$$

We define a similar energy with a minimal derivative count of one by

$$\begin{aligned}\mathcal{E}_{n,1} := \bar{\mathcal{E}}_{n,1} + \left\|\nabla^2 u\right\|_{2n-2}^2 + \sum_{j=1}^{n}\left\|\partial_t^j u\right\|_{2n-2j}^2 \\ + \|\nabla p\|_{2n-2}^2 + \sum_{j=1}^{n-1}\left\|\partial_t^j p\right\|_{2n-2j-1}^2 + \|D\eta\|_{2n-1}^2 + \sum_{j=1}^{n}\left\|\partial_t^j \eta\right\|_{2n-2j}^2,\end{aligned} \tag{2.52}$$

and with a minimal derivative count of two by

$$\begin{aligned}\mathcal{E}_{n,2} := \bar{\mathcal{E}}_{n,2} + \left\|\nabla^3 u\right\|_{2n-3}^2 + \sum_{j=1}^{n}\left\|\partial_t^j u\right\|_{2n-2j}^2 \\ + \left\|\nabla^2 p\right\|_{2n-3}^2 + \sum_{j=1}^{n-1}\left\|\partial_t^j p\right\|_{2n-2j-1}^2 + \left\|D^2\eta\right\|_{2n-2}^2 + \sum_{j=1}^{n}\left\|\partial_t^j \eta\right\|_{2n-2j}^2.\end{aligned} \tag{2.53}$$

Similarly, the dissipation with a minimal derivative count of one is

$$\mathcal{D}_{n,1} := \bar{\mathcal{D}}_{n,1} + \left\|\nabla^3 u\right\|_{2n-2}^2 + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|_{2n-2j+1}^2 + \left\|\nabla^2 p\right\|_{2n-2}^2 + \sum_{j=1}^{n-1} \left\|\partial_t^j p\right\|_{2n-2j}^2 + \left\|D^2\eta\right\|_{2n-5/2}^2 + \left\|\partial_t \eta\right\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \left\|\partial_t^j \eta\right\|_{2n-2j+5/2}^2, \tag{2.54}$$

while the dissipation with a minimal derivative count of two is

$$\mathcal{D}_{n,2} := \bar{\mathcal{D}}_{n,2} + \left\|\nabla^4 u\right\|_{2n-3}^2 + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|_{2n-2j+1}^2 + \left\|\nabla^3 p\right\|_{2n-3}^2 + \left\|\partial_t \nabla p\right\|_{2n-3}^2 + \sum_{j=2}^{n-1} \left\|\partial_t^j p\right\|_{2n-2j}^2 + \left\|D^3\eta\right\|_{2n-7/2}^2 + \left\|D\partial_t \eta\right\|_{2n-3/2}^2 + \sum_{j=2}^{n+1} \left\|\partial_t^j \eta\right\|_{2n-2j+5/2}^2. \tag{2.55}$$

Note that by definition $\mathcal{E}_{n,m} \ge \bar{\mathcal{E}}_{n,m}$ and $\mathcal{D}_{n,m} \ge \bar{\mathcal{D}}_{n,m}$. In all of these definitions, the index n counts the highest number of time derivatives used.

Certain norms of η and u will play a special role in our analysis; we write

$$\mathcal{F}_{2N} := \|\eta\|_{4N+1/2}^2 \tag{2.56}$$

and

$$\mathcal{K} := \|\nabla u\|_{L^\infty}^2 + \left\|\nabla^2 u\right\|_{L^\infty}^2 + \sum_{i=1}^{2} \|D u_i\|_{H^2(\Sigma)}^2. \tag{2.57}$$

Note that the regularity of u will always be sufficiently high for the L^∞ norms in $\mathcal{K}$ to be considered as $L^\infty(\bar{\Omega})$ norms, where $\bar{\Omega}$ is the closure of Ω. Finally, we define the total energy we will use in our analysis:

$$\mathcal{G}_{2N}(t) = \sup_{0\le r\le t} \mathcal{E}_{2N}(r) + \int_0^t \mathcal{D}_{2N}(r)dr + \sum_{m=1}^{2} \sup_{0\le r\le t} (1+r)^{m+\lambda} \mathcal{E}_{N+2,m}(r) + \sup_{0\le r\le t} \frac{\mathcal{F}_{2N}(r)}{(1+r)}. \tag{2.58}$$

2.5. **Some initial estimates.** We have the following Lemma that constrains N.

Lemma 2.7. *If $N \ge 4$, then for $m = 1, 2$ we have that $\mathcal{E}_{N+2,m} \lesssim \mathcal{E}_{2N}$ and $\mathcal{D}_{N+2,m} \lesssim \mathcal{E}_{2N}$.*

Proof. The proof follows by simply comparing the definitions of these terms. □

Now we present an estimate of $\mathcal{I}_1 \partial_t \eta$.

Lemma 2.8. *We have the estimate $\|\mathcal{I}_1 \partial_t \eta\|_0^2 \lesssim \|u\|_0^2 \le \mathcal{E}_{2N}$.*

Proof. According to Lemma 2.5, we have that $\partial_t \eta = -\partial_i U_i$, where U_i, $i = 1, 2$, is defined in the lemma. It is easy to see that $U_i \in H^1(\Sigma)$. Taking the Fourier transform, we find that

$$\|\mathcal{I}_1 \partial_t \eta\|_0^2 = \int_\Sigma |\xi|^{-2} \left|\widehat{\partial_t \eta}(\xi)\right|^2 d\xi \lesssim \int_\Sigma |\xi|^{-2} \left|\xi \cdot \widehat{U}(\xi)\right|^2 d\xi \lesssim \int_\Sigma \left|\widehat{U}(\xi)\right|^2 d\xi = \|U\|_{H^0(\Sigma)}^2. \tag{2.59}$$

However, Hölder's inequality and Lemma 2.4 imply that $\|U\|_{H^0(\Sigma)} \lesssim \|J\|_{L^\infty} \|u\|_0 \lesssim \|u\|_0$, so the desired estimate follows. □

3. Interpolation estimates at the $N+2$ level

3.1. **Initial interpolation estimates for $\eta, \bar{\eta}, u$ and ∇p.** The fact that $\mathcal{E}_{N+2,m}$ and $\mathcal{D}_{N+2,m}$, $m = 1, 2$, have a minimal count of derivatives creates numerous problems when we try to estimate terms with fewer derivatives in terms of $\mathcal{E}_{N+2,m}$ and $\mathcal{D}_{N+2,m}$. Our way around this is to interpolate between $\mathcal{E}_{N+2,m}$ (or $\mathcal{D}_{N+2,m}$) and $\mathcal{E}_{2N}$. In Sections 3.1–3.5 we will prove various interpolation inequalities of the form

$$\|X\|^2 \lesssim (\mathcal{E}_{N+2,m})^\theta (\mathcal{E}_{2N})^{1-\theta} \text{ and } \|X\|^2 \lesssim (\mathcal{D}_{N+2,m})^\theta (\mathcal{E}_{2N})^{1-\theta}, \tag{3.1}$$

where $\theta \in (0,1]$, X is some quantity, and $\|\cdot\|$ is some norm (usually either H^0 or L^∞).

In the interest of brevity, we will record these estimates in tables that only list the value of θ in the estimate. Before each table we will tell which norms are being considered and give a rough summary of the terms X that appear in the table. For example, we might write "the following table encodes the power in the $H^0(\Sigma)$ and $H^0(\Omega)$ interpolation estimates for η and $\bar{\eta}$ and their derivatives," before the following table.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\eta, \bar{\eta}$	θ_1	θ_2	θ_3
$D\eta, \nabla\bar{\eta}$	θ_4	θ_5	θ_6

We understand this to mean that

$$\|\eta\|_0^2 \lesssim (\mathcal{E}_{N+2,1})^{\theta_1} (\mathcal{E}_{2N})^{1-\theta_1}, \ \|\eta\|_0^2 \lesssim (\mathcal{D}_{N+2,1})^{\theta_2} (\mathcal{E}_{2N})^{1-\theta_2}, \ \|\eta\|_0^2 \lesssim (\mathcal{E}_{N+2,2})^{\theta_2} (\mathcal{E}_{2N})^{1-\theta_2}, \tag{3.2}$$

$$\|\eta\|_0^2 \lesssim (\mathcal{D}_{N+2,2})^{\theta_3} (\mathcal{E}_{2N})^{1-\theta_3}, \ \|\nabla\bar{\eta}\|_{H^0(\Omega)}^2 \lesssim (\mathcal{E}_{N+2,1})^{\theta_4} (\mathcal{E}_{2N})^{1-\theta_4}, \\ \|\nabla\bar{\eta}\|_{H^0(\Omega)}^2 \lesssim (\mathcal{D}_{N+2,1})^{\theta_5} (\mathcal{E}_{2N})^{1-\theta_5}, \tag{3.3}$$

etc. When we write $\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$ in a table, it means that θ is the same when interpolating between $\mathcal{D}_{N+2,1}$ and $\mathcal{E}_{2N}$ and between $\mathcal{E}_{N+2,2}$ and $\mathcal{E}_{2N}$. When we write multiple entries for X, we mean that the same interpolation estimates hold for each item listed. Often, we will have a θ appearing in a table of the form $\theta = 1/(1+r)$. When we write this, we mean that the desired interpolation inequality holds with this θ for any fixed $r \in (0,1)$, and the constant in the inequality then depends on r.

We must record estimates for too many choices of X to allow us to write the full details of each estimate. However, most of the estimates are straightforward, so in our proofs we will frequently present only a sketch of how to obtain them, providing details only for the most delicate estimates. The terms we estimate are often linear combinations of several terms, each of which would get a different interpolation power. When this occurs, we will record the lowest power achieved by a term in the sum. According to Lemma 2.7, this is justified by the estimate

$$\mathcal{E}_{2N}^{1-\theta} \mathcal{E}_{N+2,m}^{\theta} + \mathcal{E}_{2N}^{1-\kappa} \mathcal{E}_{N+2,m}^{\kappa} = \mathcal{E}_{2N}^{1-\theta} \mathcal{E}_{N+2,m}^{\theta} + \mathcal{E}_{2N}^{1-\kappa} \mathcal{E}_{N+2,m}^{\kappa-\theta} \mathcal{E}_{N+2,m}^{\theta} \\ \lesssim \mathcal{E}_{2N}^{1-\theta} \mathcal{E}_{N+2,m}^{\theta} + \mathcal{E}_{2N}^{1-\kappa} \mathcal{E}_{2N}^{\kappa-\theta} \mathcal{E}_{N+2,m}^{\theta} \lesssim \mathcal{E}_{2N}^{1-\theta} \mathcal{E}_{N+2,m}^{\theta} \tag{3.4}$$

for $0 \le \theta \le \kappa \le 1$. A similar estimate holds with $\mathcal{E}_{N+2,m}$ replaced by $\mathcal{D}_{N+2,m}$. It may happen that in estimating a product of two or more terms, we end up with estimates of the form

$$\|X\|^2 \lesssim (\mathcal{E}_{N+2,m})^{\theta_1} (\mathcal{E}_{2N})^{1-\theta_1} (\mathcal{E}_{N+2,m})^{\theta_2} (\mathcal{E}_{2N})^{1-\theta_2} \tag{3.5}$$

with $\theta_1 + \theta_2 > 1$. In this case, Lemma 2.7 again allows us to bound

$$\|X\|^2 \lesssim (\mathcal{E}_{N+2,m})^1 (\mathcal{E}_{N+2,m})^{\theta_1+\theta_2-1} (\mathcal{E}_{2N})^{2-\theta_1-\theta_2} \lesssim \mathcal{E}_{N+2,m} \mathcal{E}_{2N} \le \mathcal{E}_{N+2,m}, \tag{3.6}$$

where we have used the bound $\mathcal{E}_{2N} \le 1$. It might also happen that (3.5) occurs with $\theta_1 < 1$ and $\theta_2 = 1/(1+r)$, in which case we always understand that r is chosen so that $\theta_1 + \theta_2 = 1$.

Now that our notation is explained, we turn to the estimates themselves We begin with estimates of η.

Lemma 3.1. *The following table encodes the power in the $L^\infty(\Sigma)$ and $L^\infty(\Omega)$ interpolation estimates for η and $\bar{\eta}$ and their derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\eta, \bar{\eta}$	$(\lambda+1)/(\lambda+1+r)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$D\eta, \nabla\bar{\eta}$	1	$(\lambda+2)/(\lambda+2+r)$	$(\lambda+2)/(\lambda+3)$
$D^2\eta, \nabla^2\bar{\eta}$	1	1	$(\lambda+3)/(\lambda+3+r)$
$D^3\eta, \nabla^3\bar{\eta}$	1	1	1
$\partial_t\eta, \partial_t\bar{\eta}$	1	1	$2/(2+r)$
$D\partial_t\eta, \nabla\partial_t\bar{\eta}$	1	1	1

The following table encodes the power in the $H^0(\Sigma)$ and $H^0(\Omega)$ interpolation estimates for η and $\bar{\eta}$ and their derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\eta, \bar{\eta}$	$\lambda/(\lambda+1)$	$\lambda/(\lambda+2)$	$\lambda/(\lambda+3)$
$D\eta, \nabla\bar{\eta}$	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$D^2\eta, \nabla^2\bar{\eta}$	1	1	$(\lambda+2)/(\lambda+3)$
$D^3\eta, \nabla^3\bar{\eta}$	1	1	1
$\partial_t\eta, \partial_t\bar{\eta}$	1	1	1/2
$D\partial_t\eta, \nabla\partial_t\bar{\eta}$	1	1	1

Proof. The estimates follow directly from the Sobolev embeddings and Lemmas A.6 and A.7, using the bounds $\|\mathcal{I}_\lambda \eta\|_0^2 \le \mathcal{E}_{2N}$ and $\|\mathcal{I}_1 \partial_t \eta\|_0^2 \lesssim \mathcal{E}_{2N}$, the latter of which is a consequence of Lemma 2.8. □

Now we record some estimates involving u.

Lemma 3.2. *The following table encodes the power in the $L^\infty(\Omega)$ and $L^\infty(\Sigma)$ interpolation estimates for u and its derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
u	$1/(1+r)$	$1/2$	$1/3$
Du	1	$2/(2+r)$	$2/3$
∇u	$1/(1+r)$	$1/2$	$1/3$
D^2u	1	1	$1/(1+r)$
$D\nabla u$	1	$2/(2+r)$	$2/3$
$\nabla^2 u$	1	$1/(1+r)$	$1/2$
$\nabla^3 u$	1	1	$1/(1+r)$
$\nabla^4 u$	1	1	1
$\partial_t u$	1	1	1

The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for u and its derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
u	$\lambda/(\lambda+1)$	$\lambda/(\lambda+1)$	$\lambda/(\lambda+2)$	$\lambda/(\lambda+2)$
Du	1	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$
D^2u	1	1	1	1
$\nabla D^2 u$	1	1	1	1
$\partial_t u$	1	1	1	1

The following table encodes the power in some improved $L^\infty(\Sigma)$ interpolation estimates for u and its tangential derivatives on Σ. Here we restrict to $r \in (0, 1/2)$.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
u	$1/(1+r)$	$1/(1+r)$	$1/2$	$1/2$
Du	1	$2/(2+r)$	$2/(2+r)$	$2/(2+r)$

Proof. The estimates of the first two tables follow directly from Sobolev embeddings and Lemmas A.8 and A.13. For the $L^\infty(\Sigma)$ estimates of the last table, we use $r \in [0, 1/2)$ in (A.34) of Lemma A.7 along with trace estimates and Lemma A.13 to bound

$$
\begin{aligned}
(3.7) \quad \|u\|_{L^\infty(\Sigma)}^2 &\lesssim (\|u\|_{H^0(\Sigma)}^2)^{(s+r-1)/(s+r)} (\|D^s u\|_{H^r(\Sigma)})^{1/(s+r)} \\
&\lesssim (\|u\|_{1/2}^2)^{(s+r-1)/(s+r)} (\|D^s u\|_1^2)^{1/(s+r)} \\
&\lesssim (\|u\|_{1/2}^2)^{(s+r-1)/(s+r)} (\|D^s \nabla u\|_0^2)^{1/(s+r)}.
\end{aligned}
$$

For $\mathcal{E}_{N+2,1}$ and $\mathcal{D}_{N+2,1}$ we choose $s = 1$ and $r \in (0, 1/2)$, while for $\mathcal{E}_{N+2,2}$ and $\mathcal{D}_{N+2,m}$ we choose $s = 2$ and $r = 0$. In both cases, $\|u\|_{1/2}^2 \le \mathcal{E}_{2N}$ and $\|D^s \nabla u\|_0^2 \le \mathcal{E}_{N+2,m}$. A similar argument works for the Du estimates in $L^\infty(\Sigma)$. □

Now we estimate ∇p in L^∞.

Lemma 3.3. *The following table encodes the power in the $L^\infty(\Omega)$ interpolation estimates for ∇p and its derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
∇p	1	$1/(1+r)$	$1/2$
$\nabla^2 p$	1	1	$1/(1+r)$
$\nabla^3 p$	1	1	1
$\partial_t \nabla p$	1	1	1

Proof. The estimates follow directly from the Sobolev embeddings and Lemma A.8. □

3.2. **Interpolation estimates for G^i, $i = 1, 2, 3, 4$.** Now that we have some preliminary estimates for $u, \eta, \bar{\eta}$, and ∇p (plus some of their derivatives), we can estimate the G^i forcing terms defined in (2.24)–(2.31).

Lemma 3.4. *The following table encodes the power in the $L^\infty(\Omega)$ interpolation estimates for $G^{1,i}$, $i = 1, \dots, 5$ and G^1 and their spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$G^{1,1}$	1	1	$(3\lambda+5)/(2\lambda+6)$
$\nabla G^{1,1}$	1	1	1
$G^{1,2}$	1	1	$2/3$
$DG^{1,2}$	1	1	1
$\nabla G^{1,2}$	1	1	$2/3$
$G^{1,3}$	1	1	$(3\lambda+5)/(2\lambda+6)$
$\nabla G^{1,3}$	1	1	1
$G^{1,4}$	1	1	1
$\nabla G^{1,4}$	1	1	1
$G^{1,5}$	1	1	1
$\nabla G^{1,5}$	1	1	1
G^1	1	1	$2/3$
DG^1	1	1	1
∇G^1	1	1	$2/3$

The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for $G^{1,i}$, $i = 1, \dots, 5$ and G^1 and their spatial derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$G^{1,1}$	1	1	1	$(3\lambda+3)/(2\lambda+6)$
$\nabla G^{1,1}$	1	1	1	$(3\lambda+5)/(2\lambda+6)$
$G^{1,2}$	1	$(3\lambda+1)/(2\lambda+2)$	$(3\lambda+2)/(2\lambda+4)$	$(5\lambda+2)/(4\lambda+8)$
$DG^{1,2}$	1	1	1	$(5\lambda+4)/(3\lambda+6)$
$G^{1,3}$	1	1	1	$(3\lambda+3)/(2\lambda+6)$
$\nabla G^{1,3}$	1	1	1	$(3\lambda+5)/(2\lambda+6)$
$G^{1,4}$	1	1	1	$(4\lambda+6)/(3\lambda+9)$
$DG^{1,4}$	1	1	1	1
$G^{1,5}$	1	1	1	$5/6$
$\nabla G^{1,5}$	1	1	1	1
G^1	1	$(3\lambda+1)/(2\lambda+2)$	$(3\lambda+2)/(2\lambda+4)$	$(5\lambda+2)/(4\lambda+8)$
DG^1	1	1	1	$(5\lambda+4)/(3\lambda+6)$

Proof. The definitions of $G^{1,i}$ show that these terms are linear combinations of products of one or more terms that can be estimated in either L^∞ or H^0 by using Sobolev embeddings and Lemmas 3.1, 3.2, and 3.3. For the L^∞ table we estimate products using the usual algebra of L^∞: $\|XY\|_{L^\infty} \le \|X\|_{L^\infty} \|Y\|_{L^\infty}$. For the H^0 table, we estimate products with both

$$\|XY\|_0^2 \le \|X\|_0^2 \|Y\|_{L^\infty} \text{ and } \|XY\|_0^2 \le \|Y\|_0^2 \|X\|_{L^\infty}, \tag{3.8}$$

and then take the larger value of θ produced by these two bounds. □

Now we estimate G^2. The proof works as in Lemma 3.4, so we omit it.

Lemma 3.5. *The following table encodes the power in the $L^\infty(\Omega)$ and $L^\infty(\Sigma)$ interpolation estimates for G^2 and its spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^2	1	1	$(4\lambda+6)/(3\lambda+9)$
DG^2	1	1	1
∇G^2	1	1	$(3\lambda+5)/(2\lambda+6)$
$\nabla^2 G^2$	1	1	1

The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for G^2 and its spatial derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^2	1	$(3\lambda+2)/(2\lambda+4)$	$(4\lambda+3)/(3\lambda+9)$
DG^2	1	1	$(4\lambda+6)/(3\lambda+9)$
∇G^2	1	1	$(3\lambda+3)/(2\lambda+6)$
$\nabla^2 G^2$	1	1	$(3\lambda+5)/(2\lambda+6)$

Now we record G^3 estimates. Recall that by Remark 2.2, we may remove the appearance of $(p-\eta)$ in G^3. This allows us to perform the estimates of G^3 terms as in Lemmas 3.4 and 3.5, so we again omit the proof.

Lemma 3.6. *The following table encodes the power in the $L^\infty(\Sigma)$ interpolation estimates for G^3 and its spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^3	1	1	$(4\lambda+6)/(3\lambda+9)$
DG^3	1	1	1
D^2G^3	1	1	1

The following table encodes the power in the $H^0(\Sigma)$ interpolation estimates for G^3 and its spatial derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^3	1	$(3\lambda+2)/(2\lambda+4)$	$(4\lambda+3)/(3\lambda+9)$
DG^3	1	1	$(4\lambda+6)/(3\lambda+9)$
D^2G^3	1	1	1

Now for G^4 estimates. We again omit the proof.

Lemma 3.7. *The following table encodes the power in the $L^\infty(\Sigma)$ interpolation estimates for G^4 and its spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^4	1	1	1
DG^4	1	1	1
D^2G^4	1	1	1

The following table encodes the power in the $H^0(\Sigma)$ interpolation estimates for G^4 and its spatial derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^4	1	1	$(3\lambda+5)/(2\lambda+6)$
DG^4	1	1	1
D^2G^4	1	1	1

3.3. **Improved estimates for $u, \nabla p$.** Now we will use the structure of the equations (2.23) to improve our estimates for $u, \nabla p$, etc. Our first estimate is for Dp. It constitutes an improvement of our existing L^∞ estimate, Lemma 3.3, as well as a first H^0 estimate.

Lemma 3.8. *The following table encodes the power in an $L^\infty(\Omega)$ interpolation estimate.*

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
Dp	1	$1/(1+r)$	$(\lambda+2)/(\lambda+3)$

The following table encodes the power in an $H^0(\Omega)$ interpolation estimate.

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
Dp	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$

Proof. In order to record the proof of both the H^0 and L^∞ estimates at the same time, we will generically write $\|\cdot\|$ to refer to either the $H^0(\Omega)$ or $L^\infty(\Omega)$ norm. Similarly, we will write $\|\cdot\|_\Sigma$ to refer to the $H^0(\Sigma)$ or $L^\infty(\Sigma)$ norm. The starting point is an application of Lemma A.10 to bound

$$\|Dp\|^2 \lesssim \|Dp\|_\Sigma^2 + \|\partial_3 Dp\|^2 . \tag{3.9}$$

We will estimate both of the terms on the right hand side in order to prove the lemma.

In order to estimate Dp on Σ we utilize the boundary conditions in (2.23) to write

$$\partial_i p = \partial_i \eta + 2\partial_i \partial_3 u_3 + \partial_i (G^3 \cdot e_3) \tag{3.10}$$

for $i = 1, 2$. From this we easily see that

$$\|Dp\|_\Sigma^2 \lesssim \|D\eta\|_\Sigma^2 + \left\|DG^3\right\|_\Sigma^2 + \|D\partial_3 u_3\|_\Sigma^2 . \tag{3.11}$$

The first two terms may be estimated with Lemmas 3.1 and 3.6, but we must further exploit the structure of the equations in order to control the last term. For the H^0 estimate we use trace theory and the relation

$$\partial_3 u_3 = G^2 - \partial_1 u_1 - \partial_2 u_2 \tag{3.12}$$

to find

$$\|D\partial_3 u_3\|_{H^0(\Sigma)}^2 \lesssim \|D\partial_3 u_3\|_1^2 \lesssim \left\|DG^2\right\|_1^2 + \left\|D^2 u\right\|_1^2 . \tag{3.13}$$

Since $D^2 u = 0$ on Σ_b we may use Poincaré, Lemma A.13, to bound $\left\|D^2 u\right\|_1^2 \lesssim \left\|\nabla D^2 u\right\|_0^2$, so that upon replacing in the previous inequality we find

$$\|D\partial_3 u_3\|_{H^0(\Sigma)}^2 \lesssim \left\|DG^2\right\|_0^2 + \left\|D\nabla G^2\right\|_0^2 + \left\|D^2 \nabla u\right\|_0^2 . \tag{3.14}$$

For the corresponding L^∞ estimate we again use (3.12) to bound

$$\|D\partial_3 u_3\|_{L^\infty(\Sigma)}^2 \lesssim \left\|DG^2\right\|_{L^\infty(\Sigma)}^2 + \left\|D^2 u\right\|_{L^\infty(\Sigma)}^2 . \tag{3.15}$$

By Lemma A.13 we know that $\left\|D^2 u\right\|_{L^\infty(\Sigma)}^2 \lesssim \left\|\nabla D^2 u\right\|_{L^\infty(\Omega)}^2$, and also Lemma 3.5 guarantees that $\left\|DG^2\right\|_{L^\infty(\Sigma)}^2 \lesssim \left\|DG^2\right\|_{L^\infty(\Omega)}^2$, so we may replace these to arrive at the bound

$$\|D\partial_3 u_3\|_{L^\infty(\Sigma)}^2 \lesssim \left\|DG^2\right\|_{L^\infty(\Omega)}^2 + \left\|\nabla D^2 u\right\|_{L^\infty(\Omega)}^2 . \tag{3.16}$$

Then from (3.14) and (3.16) we know that

$$\|D\partial_3 u_3\|_\Sigma^2 \lesssim \left\|DG^2\right\|^2 + \left\|D\nabla G^2\right\|^2 + \left\|D^2 \nabla u\right\|^2 . \tag{3.17}$$

Combining (3.11) with (3.17) yields

$$\|Dp\|_\Sigma^2 \lesssim \|D\eta\|_\Sigma^2 + \left\|DG^3\right\|_\Sigma^2 + \left\|DG^2\right\|^2 + \left\|D\nabla G^2\right\|^2 + \left\|D^2\nabla u\right\|^2. \tag{3.18}$$

We may then employ Lemmas 3.1, 3.2, 3.3, 3.5, 3.6 to derive the interpolation power for $\|Dp\|_\Sigma^2$; we record this power in the following table. Both the L^∞ and H^0 powers are determined by $D\eta$, but the L^∞ estimate only improves the result of Lemma 3.3 for $\mathcal{D}_{N+2,2}$.

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\|Dp\|_{L^\infty(\Sigma)}^2$	1	$1/(1+r)$	$(\lambda+2)/(\lambda+3)$
$\|Dp\|_{H^0(\Sigma)}^2$	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$

Now we will estimate the term $\|\partial_3 Dp\|^2$. For this we use (2.23) to write

$$\partial_i\partial_3 p = \partial_i[(\partial_1^2+\partial_2^2-\partial_t)u_3 + \partial_3^2 u_3 + G^1\cdot e_3]. \tag{3.19}$$

for $i = 1, 2$. Again using (3.12), we may write

$$\partial_i\partial_3^2 u_3 = \partial_i\partial_3(G^2 - \partial_1 u_1 - \partial_2 u_2). \tag{3.20}$$

Combining these two equations then shows that

$$\|D\partial_3 p\|^2 \lesssim \left\|D^3 u\right\|^2 + \left\|D^2\nabla u\right\|^2 + \|D\partial_t u\|^2 + \left\|DG^1\right\|^2 + \left\|D\nabla G^2\right\|^2. \tag{3.21}$$

We may then employ Lemmas 3.2, 3.3, 3.4, and 3.5 to derive the interpolation power for $\|D\partial_3 p\|^2$; we record this power in the following table. The H^0 powers are determined by DG^1, but note that the L^∞ estimate does not improve the result of Lemma 3.3.

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\|D\partial_3 p\|_{L^\infty}^2$	1	1	$1/(1+r)$
$\|D\partial_3 p\|_0^2$	1	1	$(5\lambda+4)/(3\lambda+6)$

Now we return to (3.9) and employ our estimates of $\|Dp\|_\Sigma^2$ and $\|D\partial_3 p\|^2$ to deduce the desired interpolation powers for $\|Dp\|^2$. □

With this lemma in hand, we can now derive improved estimates for u.

Proposition 3.9. *Let*

$$\theta_1(\lambda) = \min\left\{\frac{5\lambda+2}{4\lambda+8}, \frac{\lambda+1}{\lambda+3}\right\} \text{ and } \theta_2(\lambda) = \min\left\{\frac{9\lambda+10}{8\lambda+16}, \frac{\lambda+2}{\lambda+3}\right\}. \tag{3.22}$$

The following table encodes the improved power in the $L^\infty(\Omega)$ interpolation estimate for u and its derivatives.

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1} \sim \mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
u	1	$2/(2+r)$	$2/3$
$\partial_3 u_i, i = 1, 2$	1	1	$2/3$
$\partial_3 u_3$	1	$2/(2+r)$	$2/3$
∇u	1	$2/(2+r)$	$2/3$
$\nabla^2 u$	1	$2/(2+r)$	$2/3$

The following table encodes the power in the $H^0(\Omega)$ interpolation estimate for u and its derivatives.

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
u	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$\theta_1(\lambda)$
$\partial_3 u_i, i=1,2$	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$\theta_1(\lambda)$
$\partial_3 u_3$	1	$(3\lambda+2)/(2\lambda+4)$	$(3\lambda+2)/(2\lambda+4)$	$(4\lambda+3)/(3\lambda+9)$
Du	1	1	$(2\lambda+3)/(2\lambda+4)$	$\theta_2(\lambda)$
∇u	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$\theta_1(\lambda)$
$D\nabla u$	1	1	$(2\lambda+3)/(2\lambda+4)$	$\theta_2(\lambda)$
$D\partial_3 u_3$	1	1	1	$(4\lambda+6)/(3\lambda+9)$
$\nabla\partial_3 u_3$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(3\lambda+3)/(2\lambda+6)$
$\nabla^2 u$	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$\theta_1(\lambda)$

The following table encodes the improved power in the $L^\infty(\Omega)$ interpolation estimate for ∇p.

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}\sim\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
∇p	1	$2/(2+r)$	$2/3$

The following table encodes the power in the $H^0(\Omega)$ interpolation estimate for derivatives of p.

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\partial_3 p$	1	$(3\lambda+1)/(2\lambda+2)$	$(3\lambda+2)/(2\lambda+4)$	$(5\lambda+2)/(4\lambda+8)$
∇p	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$\theta_1(\lambda)$

Proof. As in Lemma 3.8 we will write $\|\cdot\|$ and $\|\cdot\|_\Sigma$ to refer to both the H^0 and L^∞ norms on Ω and Σ respectively. We divide the proof into several steps, beginning with estimates of ∇u. With these established, we can extend to estimates of u, $D\nabla u$, Du, $D\partial_3 u_3$, and $\nabla\partial_3 u_3$ by employing Poincaré's inequality and interpolation. This in turn leads to estimates for $\partial_3 p$ and $\nabla^2 u$.

Step 1 – Estimates of ∇u

To begin the ∇u estimates, we split the components of ∇u into those involving x_1, x_2 derivatives and those involving x_3 derivatives. Indeed, we have

$$\|\nabla u\|^2 \lesssim \|Du\|^2 + \|\partial_3 u_3\|^2 + \sum_{i=1}^2 \|\partial_3 u_i\|^2 . \tag{3.23}$$

Lemma 3.2 provides an estimate of Du but not of $\partial_3 u$, so we must use the structure of the equations (2.23) to estimate the latter two terms.

To estimate $\partial_3 u_3$ we use equation (2.23) to bound

$$\|\partial_3 u_3\|^2 \lesssim \left\|G^2\right\|^2 + \|Du\|^2 . \tag{3.24}$$

Then Lemmas 3.2 and 3.5 provide interpolation estimates of G^2 and Du and hence the estimates of $\partial_3 u_3$ listed in the tables. The Du term determines the power for L^∞, while the power is determined by G^2 for H^0.

To estimate $\partial_3 u_i$ for $i=1,2$ we first apply Lemma A.10 to get

$$\|\partial_3 u_i\|^2 \lesssim \|\partial_3 u_i\|_\Sigma^2 + \left\|\partial_3^2 u_i\right\|^2 . \tag{3.25}$$

For the first term on the right we use equation (2.23) to bound

$$\|\partial_3 u_i\|_\Sigma^2 \lesssim \|Du_3\|_\Sigma^2 + \left\|G^3\right\|_\Sigma^2 . \tag{3.26}$$

Since $Du=0$ on Σ_b we can use trace theory, Lemma A.13, and the equation $\operatorname{div} u = G^2$ for

$$\|Du_3\|_\Sigma^2 \lesssim \|\nabla Du_3\|^2 \lesssim \left\|D^2 u\right\|^2 + \left\|DG^2\right\|^2 \tag{3.27}$$

For the second term on the right side of (3.25) we use (2.23) to bound

$$\left\|\partial_3^2 u_i\right\|^2 \lesssim \|\partial_t u\|^2 + \left\|D^2 u\right\|^2 + \|Dp\|^2 + \left\|G^1\right\|^2 . \tag{3.28}$$

We may then combine estimates (3.25)–(3.28) to deduce that

$$\|\partial_3 u_i\|^2 \lesssim \|\partial_t u\|^2 + \left\|D^2 u\right\|^2 + \|Dp\|^2 + \left\|G^1\right\|^2 + \left\|DG^2\right\|^2 + \left\|G^3\right\|_\Sigma^2 . \tag{3.29}$$

Now we use Lemma 3.2, 3.4–3.6, and 3.8 to find the interpolation powers for $\partial_3 u_i, i = 1, 2$ listed in the tables. For L^∞ the power is determined by G^1, while for H^0 the power is determined by Dp for $\mathcal{E}_{N+2,1}, \mathcal{E}_{N+2,2}$, and $\mathcal{D}_{N+2,1}$ but by the smaller of the powers of Dp and G^1 for $\mathcal{D}_{N+2,2}$.

With estimates for Du, $\partial_3 u_3$, and $\partial_3 u_i$ for $i = 1, 2$ in hand, we return to (3.23) to derive the estimates for ∇u listed in the tables. For the L^∞ estimate the power is determined by Du, while for H^0 it is determined by $\partial_3 u_i$, $i = 1, 2$.

Step 2 – Extensions to estimates of u, $D\nabla u$, $D\partial_3 u_3$, and $\nabla \partial_3 u_3$

Now we apply Lemma A.13 to control u in terms of ∇u:

$$\|u\|^2 \lesssim \|\nabla u\|^2 . \tag{3.30}$$

Our estimates for ∇u then provide the estimates for u listed in the tables.

We now turn to $D\nabla u$. Clearly $\|D\nabla u\|_0^2$ is conrolled by both $\mathcal{E}_{N+2,1}$ and $\mathcal{D}_{N+2,1}$, which yields the powers of 1 in the tables. An application of (A.38) from Lemma A.8 with $\lambda = 0$, $q = 1$, and $s = 1$ shows that

$$\|D\nabla u\|_0^2 \lesssim \left(\|\nabla u\|_0^2\right)^{1/2} \left(\left\|D^2 \nabla u\right\|_0^2\right)^{1/2} . \tag{3.31}$$

We employ this in conjunction with our estimate for ∇u and the estimate of $D^2 \nabla u$ from Lemma 3.2 to get the interpolation powers for $D\nabla u$ listed in the tables for $\mathcal{E}_{N+2,2}$ and $\mathcal{D}_{N+2,2}$. The estimates for Du listed in the tables follow immediately from the estimates for $D\nabla u$ via Poincaré:

$$\|Du\|^2 \lesssim \|D\nabla u\|^2 . \tag{3.32}$$

In order to estimate $D\partial_3 u_3$ and $\nabla\partial_3 u_3$ in H^0 we use that $\operatorname{div} u = G^2$ for

$$\|\nabla \partial_3 u_3\|_0^2 \lesssim \left\|\nabla G^2\right\|_0^2 + \|D\nabla u\|_0^2 . \tag{3.33}$$

and

$$\|D\partial_3 u_3\|_0^2 \lesssim \left\|DG^2\right\|_0^2 + \left\|D^2 u\right\|_0^2 . \tag{3.34}$$

Then our estimate for $D\nabla u$ and Lemmas 3.2 and 3.5 yield the estimates listed in the tables. For $\nabla\partial_3 u_3$ the power is determined by $D\nabla u$ for $\mathcal{E}_{N+2,1}, \mathcal{D}_{N+2,1}, \mathcal{E}_{N+2,2}$ and by ∇G^2 for $\mathcal{D}_{N+2,2}$. For $D\partial_3 u_3$ the power is determined by DG^2.

Step 3 – Estimates of $\partial_3 p$ and ∇p

Lemma 3.8 provides estimates for Dp, so to complete an estimate for ∇p we only need to consider $\partial_3 p$. For this we again use (2.23) to bound

$$\|\partial_3 p\|^2 \lesssim \left\|\partial_3^2 u_3\right\|^2 + \left\|D^2 u\right\|^2 + \|\partial_t u\|^2 + \left\|G^1\right\|^2 . \tag{3.35}$$

This and (3.33) then imply that

$$\|\partial_3 p\|^2 \lesssim \|D\nabla u\|^2 + \left\|D^2 u\right\|^2 + \|\partial_t u\|^2 + \left\|G^1\right\|^2 + \left\|\nabla G^2\right\|^2 , \tag{3.36}$$

and we may use Lemmas 3.2, 3.4, and 3.5 along with our new $D\nabla u$ estimate to determine the powers in the tables for $\partial_3 p$. In the L^∞ estimate the power is determined by $D\nabla u$, and in the H^0 estimate the power is determined by G^1. Then the estimates for ∇p follow by comparing the Dp estimates of Lemma 3.8 to the $\partial_3 p$ estimates.

Step 4 – Estimates of $\nabla^2 u$

Finally we consider $\nabla^2 u$, which we decompose according to x_1, x_2 and x_3 derivatives:

$$\left\|\nabla^2 u\right\|^2 \lesssim \left\|D^2 u\right\|^2 + \|D\nabla u\|^2 + \left\|\partial_3^2 u_3\right\|^2 + \sum_{i=1}^{2} \left\|\partial_3^2 u_i\right\|^2 . \tag{3.37}$$

According to our bounds (3.28) and (3.33) we may replace this with

$$\left\|\nabla^2 u\right\|^2 \lesssim \|\partial_t u\|^2 + \left\|D^2 u\right\|^2 + \|D\nabla u\|^2 + \|Dp\|^2 + \left\|G^1\right\|^2 + \left\|\nabla G^2\right\|^2 . \tag{3.38}$$

Then Lemmas 3.2, 3.4, 3.5, and 3.8 with our new estimate of $D\nabla u$ provide the estimates in the table for $\nabla^2 u$. The power in the L^∞ estimate is determined by $D\nabla u$, while for H^0 it is determined by Dp for $\mathcal{E}_{N+2,1}, \mathcal{E}_{N+2,2}$, and $\mathcal{D}_{N+2,1}$ but by the smaller of the powers of Dp and G^1 for $\mathcal{D}_{N+2,2}$. □

3.4. **Bootstrapping: first iteration.** We now use the improved estimates of Proposition 3.9 to improve the estimates of G^i, $i = 1, \dots, 4$ recorded in Lemmas 3.4–3.7. We will only record the improvements for the $H^0(\Omega)$ estimates.

Lemma 3.10. *The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for $G^{1,i}$, $i = 1, \dots, 5$ and G^1 and their spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$G^{1,1}$	1	1	1	$(5\lambda+6)/(3\lambda+9)$
$\nabla G^{1,1}$	1	1	1	1
$G^{1,2}$	1	1	1	$(23\lambda+22)/(12\lambda+24)$
$\nabla G^{1,2}$	1	1	1	$(23\lambda+22)/(12\lambda+24)$
$G^{1,3}$	1	1	1	$(5\lambda+6)/(3\lambda+9)$
$\nabla G^{1,3}$	1	1	1	1
$G^{1,4}$	1	1	1	1
$\nabla G^{1,4}$	1	1	1	1
$G^{1,5}$	1	1	1	1
$\nabla G^{1,5}$	1	1	1	1
G^1	1	1	1	$(5\lambda+6)/(3\lambda+9)$
∇G^1	1	1	1	$(23\lambda+22)/(12\lambda+24)$

Proof. We perform the estimates as in Lemma 3.4, except that now we use the improved interpolation estimates of Lemma 3.8 and Proposition 3.9. □

Now for G^2 estimates. We omit the proof.

Lemma 3.11. *The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for G^2 and its spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^2	1	1	1	$(7\lambda+6)/(3\lambda+9)$
DG^2	1	1	1	1
∇G^2	1	1	1	$(5\lambda+5)/(2\lambda+6)$
$\nabla^2 G^2$	1	1	1	1

Now for G^3 estimates. Again we omit the proof.

Lemma 3.12. *The following table encodes the power in the $H^0(\Sigma)$ interpolation estimates for G^3 and its spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^3	1	1	1	$(5\lambda+6)/(3\lambda+9)$
DG^3	1	1	1	$(5\lambda+6)/(3\lambda+9)$
$D^2 G^3$	1	1	1	1

Now for G^4 estimates. The proof is omitted.

Lemma 3.13. *The following table encodes the power in the $H^0(\Sigma)$ interpolation estimates for G^4 and its spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
G^4	1	1	1	1
DG^4	1	1	1	1
$D^2 G^4$	1	1	1	1

The improved estimates for G^i, $i = 1, \dots, 4$ now allow us to improve the H^0 estimates for u and its derivatives in Proposition 3.9.

Theorem 3.14. *The following table encodes the power in the $H^0(\Omega)$ interpolation estimate for u and its derivatives.*

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
u	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$\partial_3 u_3$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
Du	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
∇u	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$
$D\nabla u$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
$\nabla\partial_3 u_3$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
$\nabla^2 u$	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$

The following table encodes the power in the $H^0(\Omega)$ interpolation estimate for derivatives of p.

	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\partial_3 p$	1	1	$(2\lambda+3)/(2\lambda+4)$	$(\lambda+2)/(\lambda+3)$
∇p	1	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+2)$	$(\lambda+1)/(\lambda+3)$

Proof. The argument is essentially identical to that employed in Proposition 3.9, except that now we use Lemmas 3.10–3.13 for estimates of G^i and Proposition 3.9 for estimates of Du, D^2u. As such, we will only mention which terms determine the power for each estimate.

For ∇u the power is determined by Dp, and then Poincaré and interpolation give the estimates for u, $D\nabla u$, and Du. In the $\partial_3 p$ estimate the power is determined by $D\nabla u$, and in the ∇p estimate the power is determined by Dp. The power in the $\nabla^2 u$ estimate is determined by Dp.

The only estimate not modeled on one in Proposition 3.9 is the one for $\partial_3 u_3$. We employ the equation $\operatorname{div} u = G^2$ to bound

$$\|\partial_3 u_3\|^2 \lesssim \left\|G^2\right\|^2 + \|Du\|^2 \text{ and } \|\nabla\partial_3 u_3\|^2 \lesssim \left\|\nabla G^2\right\|^2 + \|D\nabla u\|^2 . \tag{3.39}$$

The estimates of $\partial_3 u_3$ and $\nabla\partial_3 u_3$ in the table follow from these, with the power of the former determined by Du and the latter determined by $D\nabla u$. □

3.5. **Bootstrapping: second iteration.** We now use the improved estimates of Theorem 3.14 to improve the estimates of G^i, $i = 1, 2$ recorded in Lemmas 3.10–3.11. We once again omit the proof.

Theorem 3.15. *The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for $G^{1,i}$, $i = 1, \dots, 5$ and G^1 and their spatial derivatives.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$G^{1,1}$	1	1	1	$(2\lambda+2)/(\lambda+3)$
$\nabla G^{1,1}, \nabla^2 G^{1,1}$	1	1	1	1
$G^{1,2}, \nabla G^{1,2}, \nabla^2 G^{1,2}$	1	1	1	1
$G^{1,3}$	1	1	1	$(2\lambda+2)/(\lambda+3)$
$\nabla G^{1,3}, \nabla^2 G^3$	1	1	1	1
$G^{1,4}, \nabla G^{1,4}, \nabla^2 G^{1,4}$	1	1	1	1
$G^{1,5}, \nabla G^{1,5}, \nabla^2 G^{1,5}$	1	1	1	1
G^1	1	1	1	$(2\lambda+2)/(\lambda+3)$
$\nabla G^1, \nabla^2 G^1$	1	1	1	1

The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for G^2 and its spatial derivatives.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$G^2, \nabla G^2, \nabla^2 G^2$	1	1	1	1

Now we make final improvements to our estimates.

Proposition 3.16. *The following table encodes the power in the $H^0(\Omega)$ interpolation estimates for $D\partial_3 u_i$ for $i = 1, 2$.*

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$D\partial_3 u_i, i = 1, 2$	1	1	1	$(\lambda+2)/(\lambda+3)$

The following table encodes the power in an $H^2(\Sigma)$ estimates for Du_i for $i = 1, 2$.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$Du_i, i = 1, 2$	1	1	1	$(\lambda+2)/(\lambda+3)$

The following table encodes the power in the improved $H^0(\Sigma)$ interpolation estimates for $\partial_t \eta$.

X	$\mathcal{E}_{N+2,1}$	$\mathcal{D}_{N+2,1}$	$\mathcal{E}_{N+2,2}$	$\mathcal{D}_{N+2,2}$
$\partial_t \eta$	1	1	1	$(\lambda+2)/(\lambda+3)$

Proof. We may argue exactly as in Lemma 3.8 to bound

$$\left\|D^2 p\right\|^2 \lesssim \left\|D^2\eta\right\|^2 + \left\|D^2\partial_t u\right\|^2 + \left\|D^4 u\right\|^2 + \left\|D^3\nabla u\right\|^2 + \left\|D^2 G^1\right\|^2 + \left\|D^2 G^2\right\|^2 + \left\|D^2\nabla G^2\right\|^2 + \left\|D^2 G^3\right\|_\Sigma^2 . \tag{3.40}$$

We may also argue as in Proposition 3.9 to bound

$$\left\|D\partial_3 u_i\right\|^2 \lesssim \left\|D\partial_t u\right\|^2 + \left\|D^3 u\right\|^2 + \left\|D^2 p\right\|^2 + \left\|DG^1\right\|^2 + \left\|DG^2\right\|^2 + \left\|DG^3\right\|_\Sigma^2 \tag{3.41}$$

for $i = 1, 2$. Combining (3.40) and (3.41) and employing Theorems 3.14 and 3.15 and Lemmas 3.12 and 3.13, we then find the $H^0(\Omega)$ estimates for $D\partial_3 u_i$, $i = 1, 2$ listed in the table. The power is determined by $D^2\eta$.

We now turn to the $\|Du_i\|_{H^2(\Sigma)}^2$ estimate for $i = 1, 2$. We employ trace theory and the Poincaré inequality to bound

$$\|Du_i\|_{H^0(\Sigma)}^2 \lesssim \|D\partial_3 u_i\|_0^2 \text{ and } \left\|D^3 u_i\right\|_{H^0(\Sigma)}^2 \lesssim \left\|D^3\partial_3 u_i\right\|_0^2 , \tag{3.42}$$

and then we utilize our new estimate for $D\partial_3 u_i$ to deduce the $H^2(\Sigma)$ estimates listed in the table. The power is determined by $D\partial_3 u_i$ since $D^3\partial_3 u_i$ has four derivatives and hence has a power of 1.

Finally, for the $\partial_t\eta$ estimate we use (2.23), trace theory, and Lemma A.13 to bound

$$\|\partial_t\eta\|_{H^0(\Sigma)}^2 \lesssim \|u_3\|_{H^0(\Sigma)}^2 + \left\|G^4\right\|_{H^0(\Sigma)}^2 \lesssim \|\nabla u_3\|_0^2 + \left\|G^4\right\|_{H^0(\Sigma)}^2 . \tag{3.43}$$

Then Theorem 3.14 and Lemma 3.13 provide the $\partial_t\eta$ estimate for $\mathcal{D}_{N+2,2}$ listed in the table, with the power determined by ∇u_3; the estimates for $\mathcal{E}_{N+2,1}, \mathcal{E}_{N+2,2}, \mathcal{D}_{N+2,1}$ come from Lemma 3.1. □

Now we record an interpolation estimate for $\mathcal{K}$, as defined by (2.57).

Lemma 3.17. *We have that $\mathcal{K} \lesssim \mathcal{E}_{N+2,2}^{(8+2\lambda)/(8+4\lambda)}$.*

Proof. By definition, $\mathcal{K} = \|\nabla u\|_{L^\infty}^2 + \left\|\nabla^2 u\right\|_{L^\infty}^2 + \sum_{i=1}^2 \|Du_i\|_{H^2(\Sigma)}^2$. We may then use the $H^2(\Sigma)$ interpolation estimate of Proposition 3.16 and the L^∞ interpolation estimate of Proposition 3.9 with $r = 2\lambda/(4+\lambda)$ to bound $\mathcal{K} \lesssim \mathcal{E}_{N+2,2}^{2/(2+r)}$. The choice of r implies that $2/(2+r) = (8+2\lambda)/(8+4\lambda)$, and the result follows. □

3.6. **Estimates at the high end.** Our analysis in Sections 3.1–3.5 dealt with the problems associated with estimating terms involving fewer derivatives than appear in $\mathcal{E}_{N+2,m}, \mathcal{D}_{N+2,m}$. We now turn to the problem of estimating terms involving more derivatives than are controlled by $\mathcal{D}_{N+2,m}$. We accomplish such an estimate by interpolating between $\mathcal{D}_{N+2,m}$ and $\mathcal{E}_{2N}$, which controls more derivatives since $N \geq 5$. Fortunately, the only term we must concern ourselves with is $\nabla^{2N+3}\bar{\eta}$, and to simplify things we will only estimate it in terms of $\mathcal{D}_{N+2,2}$. This suffices since $\mathcal{D}_{N+2,2} \lesssim \mathcal{D}_{N+2,1}$.

Lemma 3.18. *We have the estimate*

$$\left\|D^{2N+4}\eta\right\|_{1/2}^2 + \left\|\nabla^{2N+5}\bar{\eta}\right\|_0^2 \lesssim (\mathcal{E}_{2N})^{2/(4N-7)}(\mathcal{D}_{N+2,2})^{(4N-9)/(4N-7)}. \tag{3.44}$$

Proof. According to Lemma A.5, with $q = 2N+5$, we may bound

$$\left\|\nabla^{2N+5}\bar{\eta}\right\|_0^2 \lesssim \|\eta\|^2_{\dot{H}^{2N+9/2}(\Sigma)} \lesssim \left\|D^{2N+4}\eta\right\|_{1/2}^2, \tag{3.45}$$

so it suffices to prove (3.44) with only the $D^{2N+4}\eta$ term on the left side. To prove this, we will use a standard Sobolev interpolation inequality:

$$\|f\|_s \lesssim \|f\|_{s-r}^{q/(r+q)} \|f\|_{s+q}^{r/(r+q)} \tag{3.46}$$

for $s, q > 0$ and $0 \le r \le s$. Applying this to $f = D^3\eta$ with $s = 2N+3/2$, $r=1$, and $q = 2N - 9/2$, we find that

$$\left\|D^{2N+4}\eta\right\|_{1/2} \le \left\|D^3\eta\right\|_{2N+3/2} \lesssim \left\|D^3\eta\right\|_{2N+1/2}^{(4N-9)/(4N-7)} \left\|D^3\eta\right\|_{4N-3}^{2/(4N-7)}. \tag{3.47}$$

The desired inequality then follows by squaring and using the definitions of $\mathcal{E}_{2N}$ and $\mathcal{D}_{N+2,2}$. □

Our next result utilizes Lemma 3.18 to estimate products such as $uD^{2N+4}\eta$.

Lemma 3.19. *Let $P = P(K, D\eta)$ be a polynomial in $K, D\eta$. Then there exists a $\theta > 0$ so that*

$$\left\|(D^{2N+4}\eta)u\right\|_{H^{1/2}(\Sigma)}^2 + \left\|(D^{2N+4}\eta)P\nabla u\right\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,2}. \tag{3.48}$$

Let $Q = Q(K, \tilde{b}, \nabla\bar{\eta})$ be a polynomial. Then exists a $\theta > 0$ so that

$$\left\|(\nabla^{2N+5}\bar{\eta})Q\nabla u\right\|_0^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,2}. \tag{3.49}$$

Proof. According to the bound (A.2) of Lemma A.1, we may bound

$$\begin{aligned} &\left\|(D^{2N+4}\eta)u\right\|_{H^{1/2}(\Sigma)}^2 + \left\|(D^{2N+4}\eta)P\nabla u\right\|_{H^{1/2}(\Sigma)}^2 \\ &\qquad \lesssim \left\|D^{2N+4}\eta\right\|_{H^{1/2}(\Sigma)}^2 \|u\|_{H^2(\Sigma)}^2 + \left\|D^{2N+4}\eta\right\|_{H^{1/2}(\Sigma)}^2 \|P\nabla u\|_{H^2(\Sigma)}^2. \end{aligned} \tag{3.50}$$

Trace theory implies that

$$\begin{aligned} \|u\|_{H^2(\Sigma)}^2 + \|\nabla u\|_{H^2(\Sigma)}^2 &\le \|u\|_{H^0(\Sigma)}^2 + \left\|D^2u\right\|_{H^0(\Sigma)}^2 + \|\nabla u\|_{H^0(\Sigma)}^2 + \left\|D^2\nabla u\right\|_{H^0(\Sigma)}^2 \\ &\lesssim \|\nabla u\|_0^2 + \left\|D^2\nabla u\right\|_0^2 + \left\|\nabla^2 u\right\|_0^2 + \left\|\nabla^2 D^2 u\right\|_0^2, \end{aligned} \tag{3.51}$$

but then an application of Theorem 3.14 to all the terms on the right side shows that

$$\|u\|_{H^2(\Sigma)}^2 + \|\nabla u\|_{H^2(\Sigma)}^2 \lesssim (\mathcal{D}_{N+2,2})^{(1+\lambda)/(3+\lambda)}. \tag{3.52}$$

It is easy to see, based on the terms controlled by $\mathcal{E}_{2N}$, that $\|P\|^2_{H^2(\Sigma)} \lesssim \mathcal{E}_{2N} \le 1$. We may then combine this with (3.52) and (A.1) of Lemma A.1 to deduce that

$$\|u\|_{H^2(\Sigma)}^2 + \|P\nabla u\|_{H^2(\Sigma)}^2 \lesssim (\mathcal{D}_{N+2,2})^{(1+\lambda)/(3+\lambda)}. \tag{3.53}$$

Then this bound, (3.50), and Lemma 3.18 imply that

$$\left\|(D^{2N+4}\eta)u\right\|_{H^{1/2}(\Sigma)}^2 + \left\|(D^{2N+4}\eta)P\nabla u\right\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,2}^\kappa \tag{3.54}$$

for some $\theta > 0$ and for

$$\kappa = \frac{4N-9}{4N-7} + \frac{\lambda+1}{\lambda+3} \ge \frac{4N-9}{4N-7} + \frac{1}{3} = \frac{16N-34}{12N-21} \ge 1 \tag{3.55}$$

since $N \ge 4$. Since $\mathcal{D}_{N+2,2} \le \mathcal{E}_{2N} \le 1$, we may bound $\mathcal{D}_{N+2,2}^\kappa \le \mathcal{D}_{N+2,2}$ in (3.54), which then yields (3.48).

To derive (3.49) we first bound

$$\left\|(\nabla^{2N+5}\bar{\eta})Q\nabla u\right\|_0^2 \le \left\|\nabla^{2N+5}\bar{\eta}\right\|_0^2 \|\nabla u\|_{L^\infty}^2 \|Q\|_{L^\infty}^2. \tag{3.56}$$

The first term on the right is controlled with Lemma 3.18. The second term satisfies

$$\|\nabla u\|_{L^\infty}^2 \lesssim (\mathcal{D}_{N+2,2})^{2/3} \tag{3.57}$$

by virtue of the L^∞ estimates of Proposition 3.9. The third term satisfies $\|Q\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N} \le 1$ by Sobolev embeddings and the definition of $\mathcal{E}_{2N}$. The estimate (3.49) follows by combining these bounds as above. □

4. Nonlinear estimates

4.1. **Estimates of G^i at the $N+2$ level.** We now provide estimates of G^i in terms of $\mathcal{E}_{N+2,m}$ and $\mathcal{D}_{N+2,m}$. Notice that our estimates are somewhat stronger than those stated in, say Theorem 3.15, since we include some power of $\mathcal{E}_{2N}$ multiplied by $\mathcal{E}_{N+2,m}$ or $\mathcal{D}_{N+2,m}$.

Theorem 4.1. *Let $m \in \{1,2\}$. Then there exists a $\theta > 0$ so that*

$$\left\|\bar{\nabla}_m^{2(N+2)-2} G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{2(N+2)-2} G^2\right\|_1^2 + \left\|\bar{D}_m^{2(N+2)-2} G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{2(N+2)-2} G^4\right\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{E}_{N+2,m} \tag{4.1}$$

and

$$\left\|\bar{\nabla}_m^{2(N+2)-1} G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{2(N+2)-1} G^2\right\|_1^2 + \left\|\bar{D}_m^{2(N+2)-1} G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{2(N+2)-1} G^4\right\|_{1/2}^2 + \left\|\bar{D}^{2(N+2)-2} \partial_t G^4\right\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m}. \tag{4.2}$$

Proof. The estimates of these nonlinearities are fairly routine to derive: we note that all terms are quadratic or of higher order; then we apply the differential operator and expand using the Leibniz rule; each term in the resulting sum is also at least quadratic, and we estimate one term in H^k ($k = 0, 1/2$, or 1 depending on G^i) and the other term in L^∞ or H^m for m depending on k, using Sobolev embeddings, trace theory, and Lemmas A.1, A.5, and A.6–A.8. The derivative count in the differential operators is chosen in order to allow estimation by $\mathcal{E}_{N+2,m}$ in (4.1) and by $\mathcal{D}_{N+2,m}$ in (4.2). There is only one difficulty that arises. Because $\mathcal{E}_{N+2,m}$ and $\mathcal{D}_{N+2,m}$ involve minimal derivative counts, there may be terms in the sum $\partial^\alpha G^i$ that cannot be directly estimated. To handle these terms, we invoke the interpolation results of Theorems 3.14 and 3.16 and Proposition 3.9, as well as the specialized interpolation results of Lemma 3.19. A detailed proof of the estimates is quite lengthy, so for the sake of brevity we present only a sketch.

Let $\alpha \in \mathbb{N}^{1+3}$ with $m \le |\alpha| \le 2(N+2)-2$ and consider $\partial^\alpha G^1$. Since G^1 involves ∇p and $\partial^\beta u$, $\partial^\beta \bar{\eta}$ with $|\beta| \le 2$, we find that $\partial^\alpha G^1$ involves at most (with parabolic counting) $2(N+2)-1$ derivatives of p, and at most $2(N+2)$ derivatives of u and $\bar{\eta}$. We have that G^1 is a linear combination of at least quadratic terms, and as such, so is $\partial^\alpha G^1$. Let us consider a generic term in the sum $\partial^\alpha G^1$, which we write as XY with X of the form $\partial^\beta u$ or $\partial^\beta \bar{\eta}$ with $|\beta| \le 2(N+2)$ or else $\partial^\beta p$ with $|\beta| \le 2(N+2)-1$, and Y a polynomial in lower-order derivatives. If $|\beta|$ is sufficiently large with respect to m, then the minimal derivative count is exceeded and we may estimate $\|X\|_0^2 \le \mathcal{E}_{N+2,m}$. It is easy to verify, using Sobolev embeddings and Lemmas A.1, A.5, and A.6–A.8, that we always have $\|Y\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N}^\theta$ for some $\theta > 0$. Then

$$\|XY\|_0^2 \lesssim \|X\|_0^2 \|Y\|_{L^\infty}^2 \lesssim \mathcal{E}_{N+2,m} \mathcal{E}_{2N}^\theta. \tag{4.3}$$

On the other hand, if $|\beta|$ is not large, then we must resort to interpolation, using Theorems 3.14 and 3.16 and Proposition 3.9. In this case, it can be verified that we always get estimates of the form $\|X\|_0^2 \lesssim (\mathcal{E}_{2N})^{1-\theta_1} (\mathcal{E}_{N+2,m})^{\theta_1}$ and $\|Y\|_{L^\infty}^2 \lesssim (\mathcal{E}_{2N})^{\theta_2} (\mathcal{E}_{N+2,m})^{\theta_3}$ with $\theta_1 \in (0,1]$, $\theta_2, \theta_3 \ge 0$, and $\theta_1 + \theta_3 \ge 1$ so that

$$\|XY\|_0^2 \lesssim \|X\|_0^2 \|Y\|_{L^\infty}^2 \lesssim \mathcal{E}_{N+2,m} \mathcal{E}_{2N}^\theta \tag{4.4}$$

for some $\theta > 0$. This analysis works for every XY appearing in $\partial^\alpha G^1$, so

$$\left\|\bar{\nabla}_m^{2(N+2)-2} G^2\right\|_0^2 \lesssim \mathcal{E}_{N+2,m} \mathcal{E}_{2N}^\theta \tag{4.5}$$

for some $\theta > 0$. It can then be verified, through a straightforward but lengthy analysis like that used above, that all of the estimates in (4.1) hold. We note though, that in order to estimate the G^3 terms, we must use Remark 2.2 to remove the appearance of $(p-\eta)$ in G^3.

Now we sketch the proof of the estimates in (4.2). We may argue as above to estimate all terms that arise in $\partial^\alpha G^i$ with two exceptions: terms involving $\nabla^{2N+5}\bar{\eta}$ on Ω or $D^{2N+4}\eta$ on Σ. These always have the form of the terms estimated in Lemma 3.19, so we may use it for estimates in terms of $\mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,2}$, which suffice for (4.2) since $\mathcal{D}_{N+2,2} \lesssim \mathcal{D}_{N+2,1}$. Then (4.2) follows by combining the estimates of the exceptional terms with the estimates of the terms as above. □

4.2. **Estimates of G^i at the $2N$ level.** Now we derive estimates for the nonlinear G^i terms at the $2N$ level.

Theorem 4.2. *Let $m \in \{1,2\}$. Then there exists a $\theta > 0$ so that*

$$\left\|\bar{\nabla}_0^{4N-2}G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{4N-2}G^2\right\|_1^2 + \left\|\bar{D}_0^{4N-2}G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{4N-2}G^4\right\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^{1+\theta}, \tag{4.6}$$

$$\begin{aligned} &\left\|\bar{\nabla}_0^{4N-2}G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{4N-2}G^2\right\|_1^2 + \left\|\bar{D}_0^{4N-2}G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{4N-2}G^4\right\|_{1/2}^2 \\ &\quad + \left\|\bar{\nabla}^{4N-3}\partial_t G^1\right\|_0^2 + \left\|\bar{\nabla}^{4N-3}\partial_t G^2\right\|_1^2 + \left\|\bar{D}^{4N-3}\partial_t G^3\right\|_{1/2}^2 + \left\|\bar{D}^{4N-2}\partial_t G^4\right\|_{1/2}^2 \\ &\qquad \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{2N}, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\left\|\nabla^{4N-1}G^1\right\|_0^2 + \left\|\nabla^{4N-1}G^2\right\|_1^2 + \left\|D^{4N-1}G^3\right\|_{1/2}^2 + \left\|D^{4N-1}G^4\right\|_{1/2}^2 \\ &\qquad \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \end{aligned} \tag{4.8}$$

Proof. As explained in Theorem 4.1, the estimates are routine and lengthy, so we present only a sketch. The estimates in (4.6) are straightforward since $\mathcal{E}_{2N}$ has no minimal derivative restrictions. They may be derived using Sobolev embeddings, trace theory, and Lemmas A.1, A.5, and the L^∞ estimates of A.6.

The only terms with minimal derivatives in $\mathcal{D}_{2N}$ are $D\eta$ and ∇p. The latter presents no problem since, owing to Remark 2.2, p itself never appears in any of the G^i terms. The former may be dealt with by using Lemmas A.6 and A.7 to produce interpolations estimates of $\bar{\eta}$ and η in terms of $D\eta$. Whenever interpolation is needed to estimate these terms, there are always other terms multiplying them that allow for the recovery of a power of 1 on $\mathcal{D}_{2N}$. Using these estimates with Sobolev embeddings, trace theory, and Lemmas A.1, A.5, and A.6 then yields (4.7).

We now turn to the derivation of (4.8). Consider $\partial^\alpha G^i$ with $|\alpha| = 4N-1$ and $\alpha_0 = 0$, i.e. purely spatial derivatives, and expand $\partial^\alpha G^i$ using the Leibniz rule. With two exceptions, we may argue as in the derivation of (4.7) to estimate the desired norms of all of the resulting terms by $\mathcal{E}_{2N}^\theta \mathcal{D}_{2N}$ for $\theta > 0$. The exceptional terms are ones involving either $\nabla^{4N+1}\bar{\eta}$ in Ω or $D^{4N}\eta$ on Σ. We will now show how to estimate the exceptional terms with $\mathcal{K}\mathcal{F}_{2N}$, as defined by (2.57) and (2.56).

In $\nabla^{4N-1}G^1$, there are terms of the form $\partial^\beta \bar{\eta} Q \partial^\gamma u$, with

$$Q = Q(A, B, J, K, \nabla A, \nabla B, \nabla J) \tag{4.9}$$

a polynomial and $\beta, \gamma \in \mathbb{N}^3$ with $|\beta| = 4N+1$ and $|\gamma| = 1$. To estimate such a term, we use Lemma A.5 to bound

$$\left\|\nabla^{4N+1}\bar{\eta}\right\|_0^2 \lesssim \left\|D^{4N+1/2}\eta\right\|_0^2 \lesssim \mathcal{F}_{2N}. \tag{4.10}$$

Sobolev embeddings imply that $\|Q\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N}^\theta \lesssim 1$ for some $\theta > 0$, so

$$\left\|\partial^\beta \bar{\eta} Q \partial^\gamma u\right\|_0^2 \lesssim \left\|\nabla^{4N+1}\bar{\eta}\right\|_0^2 \|\nabla u\|_{L^\infty}^2 \|Q\|_{L^\infty}^2 \lesssim \left\|D^{4N+1/2}\eta\right\|_0^2 \|\nabla u\|_{L^\infty}^2 \lesssim \mathcal{F}_{2N}\mathcal{K}. \tag{4.11}$$

This estimate then yields the G^1 estimate in (4.8).

In $\nabla^{4N-1}G^2$ there are terms of the form $\partial^\beta \bar{\eta} Q \partial^\gamma u$ with $Q = Q(A,B,K)$ a polynomial and $\beta, \gamma \in \mathbb{N}^3$ with $|\beta| = 4N$, $|\gamma| = 1$. Again, Sobolev embeddings imply that $\|Q\|_{C^1(\Omega)}^2 \lesssim \mathcal{E}_{2N}^\theta \lesssim 1$, so

$$
\begin{aligned}
(4.12)\quad \left\|\partial^\beta \bar{\eta} Q \partial^\gamma u\right\|_1^2 &\lesssim \|Q\|_{C^1(\Omega)}^2 \left\|\partial^\beta \bar{\eta} \partial^\gamma u\right\|_1^2 \lesssim \left\|\partial^\beta \bar{\eta} \partial^\gamma u\right\|_0^2 + \left\|\partial^\beta \bar{\eta} \nabla \partial^\gamma u\right\|_0^2 + \left\|\nabla \partial^\beta \bar{\eta} \partial^\gamma u\right\|_0^2 \\
&\lesssim \left\|\nabla^{4N} \bar{\eta}\right\|_0^2 \|\nabla u\|_{C^1(\Omega)}^2 + \left\|\nabla^{4N+1} \bar{\eta}\right\|_0^2 \|\nabla u\|_{L^\infty}^2 \lesssim \|\eta\|_{4N-1/2}^2 \|\nabla u\|_3^2 + \mathcal{K}\mathcal{F}_{2N} \\
&\lesssim \mathcal{E}_{2N}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N},
\end{aligned}
$$

where again we have used Lemma A.5 and Sobolev embeddings. This estimate yields the G^2 estimate in (4.8).

In $D^{4N-1}G^3$ there are terms of the form $\partial^\beta \eta Q \partial^\gamma u$, where $\beta \in \mathbb{N}^2$ with $|\beta| = 4N$, $\gamma \in \mathbb{N}^3$ with $|\gamma| = 1$, and Q is a term for which we can estimate $\|Q\|_{C^1(\Sigma)}^2 \lesssim \mathcal{E}_{2N}^\theta \lesssim 1$. Then Lemma A.2 implies that

$$
(4.13)\quad \left\|\partial^\beta \eta Q \partial^\gamma u\right\|_{1/2}^2 \lesssim \left\|\partial^\beta \eta\right\|_{1/2}^2 \|Q \partial^\gamma u\|_{C^1}^2 \lesssim \|\eta\|_{4N+1/2}^2 \|Q\|_{C^1}^2 \|\nabla u\|_{C^1(\Sigma)}^2 \lesssim \mathcal{F}_{2N}\mathcal{K},
$$

where in the last inequality we have used $\|\nabla u\|_{C^1(\Sigma)}^2 \lesssim \mathcal{K}$, which follows since ∇u and $\nabla^2 u$ are continuous on the closure of Ω. This estimate yields the G^3 estimate in (4.8).

In $D^{4N-1}G^4$ the exceptional terms are of the form $\partial^\beta u_i$, where $\beta \in \mathbb{N}^2$ with $|\beta| = 4N$ and $i = 1, 2$. Then Lemma A.1 implies that

$$
(4.14)\quad \left\|\partial^\beta \eta u_1\right\|_{1/2}^2 \lesssim \left\|\partial^\beta \eta\right\|_{1/2}^2 \|u_i\|_{H^2(\Sigma)}^2 \lesssim \mathcal{F}_{2N}\mathcal{K}.
$$

This estimate yields the G^4 estimate in (4.8). □

4.3. **Estimates of other nonlinearities.** The next result provides estimates for $\mathcal{I}_\lambda G^i$ and its derivatives.

Proposition 4.3. *We have that*

$$
(4.15)\quad \left\|\mathcal{I}_\lambda G^1\right\|_1^2 + \left\|\mathcal{I}_\lambda G^2\right\|_2^2 + \left\|\mathcal{I}_\lambda \partial_t G^2\right\|_0^2 \lesssim \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}
$$

and

$$
(4.16)\quad \left\|\mathcal{I}_\lambda G^3\right\|_1^2 + \left\|\mathcal{I}_\lambda G^4\right\|_1^2 \lesssim \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}.
$$

Also,

$$
(4.17)\quad \left\|\mathcal{I}_\lambda G^4\right\|_0^2 \lesssim \mathcal{D}_{2N}^2.
$$

Proof. For each $i = 1, 2$ and for $\alpha \in \mathbb{N}^{1+3}$ such that $|\alpha| \le 2$ we can write $\partial^\alpha G^i = P_\alpha^i Q_\alpha^i$, where P_α^i is polynomial in the terms $\partial^\beta \tilde{b}$, $\partial^\beta K$, $\partial^\beta \bar{\eta}$, and $\partial^\beta u$ for $\beta \in \mathbb{N}^{1+3}$ with $|\beta| \le 4$, and Q_α^i is linear in the terms $\partial^\beta \nabla u$, $\partial^\beta \nabla^2 u$, and $\partial^\beta \nabla p$ for $|\beta| \le 2$. Then we may employ the bound (A.9) of Lemma A.3 to see that

$$
(4.18)\quad \left\|\partial^\alpha \mathcal{I}_\lambda G^i\right\|_0^2 \lesssim \left\|P_\alpha^i\right\|_0^2 \left(\left\|Q_\alpha^i\right\|_1^2\right)^\lambda \left(\left\|D Q_\alpha^i\right\|_1^2\right)^{1-\lambda}.
$$

It is then easily verified, using the Sobolev embedding, Lemmas A.1, A.5, and A.6 and the fact that $\mathcal{E}_{2N} \le 1$, that

$$
(4.19)\quad \left\|P_\alpha^i\right\|_0^2 \lesssim \mathcal{E}_{2N} \text{ and } \left\|Q_\alpha^i\right\|_2^2 \lesssim \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\},
$$

which, together with (4.18), implies (4.15).

For $i = 3, 4$ and $\alpha \in \mathbb{N}^2$ so that $|\alpha| \le 1$, we may similarly decompose $\partial^\alpha G^i = P_\alpha^i Q_\alpha^i$. We then argue as above, employing the bound (A.10) of Lemma A.3 as well as trace estimates, to deduce (4.16). The bound (4.17) also follows from Lemma A.3 and trace estimate since

$$
(4.20)\quad \left\|\mathcal{I}_\lambda G^4\right\|_0^2 \lesssim \|u\|_{H^0(\Sigma)}^2 \left(\|D\eta\|_0^2\right)^\lambda \left(\left\|D^2\eta\right\|_0^2\right)^{1-\lambda} \le \mathcal{D}_{2N}\mathcal{D}_{2N}^\lambda \mathcal{D}_{2N}^{1-\lambda} = \mathcal{D}_{2N}^2.
$$

□

Now we provide some further estimates of product terms that will be useful later when we analyze the energy evolution for $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$.

Lemma 4.4. *It holds that*

$$\|\mathcal{I}_\lambda[(AK)\partial_3 u_1 + (BK)\partial_3 u_2]\|_0^2 + \sum_{i=1}^2 \|\mathcal{I}_\lambda[u\partial_i K]\|_0^2 \lesssim \mathcal{D}_{2N}^2 \tag{4.21}$$

and

$$\|\mathcal{I}_\lambda[(1-K)u]\|_0^2 \lesssim (\mathcal{E}_{2N})^{1/(1+\lambda)} (\mathcal{D}_{2N})^{(1+2\lambda)/(1+\lambda)}. \tag{4.22}$$

Also,

$$\left\|\mathcal{I}_\lambda[(1-K)G^2]\right\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{2N}^2. \tag{4.23}$$

Proof. We apply Lemma A.3, treating the $AK, BK, \partial_i K$ terms as f and the $u, \nabla u$ terms as g, to bound

$$\|\mathcal{I}_\lambda[(AK)\partial_3 u_1 + (BK)\partial_3 u_2]\|_0^2 + \sum_{i=1}^2 \|\mathcal{I}_\lambda[u\partial_i K]\|_0^2 \lesssim \left(\|AK\|_0^2 + \|BK\|_0^2 + \|DK\|_0^2\right)\|u\|_3^2. \tag{4.24}$$

From Lemma 2.4, the fact that $\partial_i K = -K^2\partial_i J$, and Lemma A.5, we know that

$$\|AK\|_0^2 + \|BK\|_0^2 + \|DK\|_0^2 \lesssim \|\nabla\bar\eta\|_1^2 \lesssim \|D\eta\|_1^2 \le \mathcal{D}_{2N}. \tag{4.25}$$

Then, since $\|u\|_3^2 \le \mathcal{D}_{2N}$, we know that (4.21) holds.

Now, since $1-K = K(1-J)$, we can again use Lemmas A.3 and 2.4 to see that

$$\|\mathcal{I}_\lambda[(1-K)u]\|_0^2 \lesssim \|K(1-J)\|_0^2\|u\|_2^2 \lesssim \|\bar\eta\|_1^2\|u\|_2^2. \tag{4.26}$$

To control $\bar\eta$ we use Lemmas A.5 and A.7 to bound

$$\begin{aligned}\|\bar\eta\|_1^2 &\lesssim \|\eta\|_0^2 + \|D\eta\|_0^2 \\ &\lesssim \left(\|\mathcal{I}_\lambda\eta\|_0^2\right)^{1/(1+\lambda)}\left(\|D\eta\|_0^2\right)^{\lambda/(1+\lambda)} + \left(\|D\eta\|_0^2\right)^{1/(1+\lambda)}\left(\|D\eta\|_0^2\right)^{\lambda/(1+\lambda)} \\ &\lesssim (\mathcal{E}_{2N})^{1/(1+\lambda)}(\mathcal{D}_{2N})^{\lambda/(1+\lambda)}.\end{aligned} \tag{4.27}$$

Then (4.22) follows from these two estimates and the fact that $\|u\|_2^2 \le \mathcal{D}_{2N}$.

For the estimate of the $(1-K)G^2$ term, we once more use Lemma A.3 to see that

$$\left\|\mathcal{I}_\lambda[(1-K)G^2]\right\|_0^2 \lesssim \left\|G^2\right\|_0^2\|1-K\|_2^2. \tag{4.28}$$

By differentiating the equation $JK = 1$, we may compute the derivatives of K in terms of the derivatives of J; this allows us to bound, by virtue of Lemmas 2.4 and A.5,

$$\|1-K\|_2^2 \lesssim \|\bar\eta\|_2^2 \lesssim \|\eta\|_2^2 \le \|\eta\|_0^2 + \|D\eta\|_1^2. \tag{4.29}$$

Then we may argue as in (4.27) to estimate the right side of this inequality, and we deduce that

$$\|1-K\|_2^2 \lesssim (\mathcal{E}_{2N})^{1/(1+\lambda)}(\mathcal{D}_{2N})^{\lambda/(1+\lambda)}. \tag{4.30}$$

On the other hand,

$$\left\|G^2\right\|_0^2 \lesssim \|\nabla u\|_0^2(\|\bar\eta\|_{L^\infty}^2 + \|\nabla\bar\eta\|_{L^\infty}^2). \tag{4.31}$$

We estimate the L^∞ norms by using (A.25) of Lemma A.6 first with $q=0$, $s=1$, $r=\lambda^2+\lambda$ and then with $q=1$, $s=1$, $r=\lambda^2+2\lambda$ to see that

$$\begin{aligned}\|\bar\eta\|_{L^\infty}^2 &+ \|\nabla\bar\eta\|_{L^\infty}^2 \\ &\lesssim \left(\|\mathcal{I}_\lambda\eta\|_0^2\right)^{\lambda/(\lambda+1)}\left(\|D\eta\|_0^2\right)^{1/(\lambda+1)} + \left(\|\mathcal{I}_\lambda\eta\|_0^2\right)^{\lambda/(\lambda+1)}\left(\left\|D^2\eta\right\|_0^2\right)^{1/(\lambda+1)} \\ &\le (\mathcal{E}_{2N})^{\lambda/(\lambda+1)}(\mathcal{D}_{2N})^{1/(\lambda+1)}.\end{aligned} \tag{4.32}$$

Then, since $\|\nabla u\|_0^2 \le \mathcal{D}_{2N}$, we have that

$$\left\|G^2\right\|_0^2 \lesssim (\mathcal{E}_{2N})^{\lambda/(\lambda+1)} (\mathcal{D}_{2N})^{1+1/(\lambda+1)}, \tag{4.33}$$

which yields (4.23) when combined with (4.28) and (4.30). □

Now we provide an estimate of $\partial_t^j \mathcal{A}$ when $j = 2N+1$ and when $j = N+3$.

Lemma 4.5. *We have that*

$$\left\|\partial_t^{2N+1}\mathcal{A}\right\|_0^2 \lesssim \mathcal{D}_{2N}, \tag{4.34}$$

while for $m = 1, 2$,

$$\left\|\partial_t^{N+3}\mathcal{A}\right\|_0^2 \lesssim \mathcal{D}_{N+2,m}. \tag{4.35}$$

Proof. We will only prove (4.34); the bound (4.35) follows from similar analysis. Since we have that $\left\|\partial_t^{2N+1}\eta\right\|_{1/2}^2 \le \mathcal{D}_{2N}$ and temporal derivatives commute with the Poisson integral, we may employ Lemma A.5 to bound

$$\left\|\partial_t^{2N+1}\bar{\eta}\right\|_1^2 = \left\|\partial_t^{2N+1}\bar{\eta}\right\|_0^2 + \left\|\nabla\partial_t^{2N+1}\bar{\eta}\right\|_0^2 \lesssim \left\|\partial_t^{2N+1}\eta\right\|_{1/2}^2 \le \mathcal{D}_{2N}. \tag{4.36}$$

From this we easily deduce that

$$\left\|\partial_t^{2N+1}J\right\|_0^2 + \left\|\partial_t^{2N+1}K\right\|_0^2 \lesssim \mathcal{D}_{2N}. \tag{4.37}$$

This, the previous bound, and the Sobolev embeddings then imply (4.34) since the components of $\mathcal{A}$ are either unity, K, $\partial_1\bar{\eta}\tilde{b}K$, or $\partial_2\bar{\eta}\tilde{b}K$. □

5. Energy evolution using the geometric form

5.1. **Estimates of the perturbations when $\partial^\alpha = \partial_t^{\alpha_0}$ is applied to** (1.9)**.** We now present estimates of the perturbations F^i, defined by (2.13)–(2.22) when $\partial^\alpha = \partial_t^{2N}$.

Theorem 5.1. *Let* $\partial^\alpha = \partial_t^{2N}$ *and let* F^1*,* F^2*,* F^3*,* F^4 *be defined by* (2.13)–(2.22)*. Then*

$$\left\|F^1\right\|_0^2 + \left\|\partial_t(JF^2)\right\|_0^2 + \left\|F^3\right\|_0^2 + \left\|F^4\right\|_0 \lesssim \mathcal{E}_{2N}\mathcal{D}_{2N}. \tag{5.1}$$

Proof. We first consider the F^1 estimate. Each term in the sums that define F^1 is at least quadratic. It is straightforward to see that each such term can be written in the form XY, where we X involves fewer temporal derivatives than Y, and we may use the usual Sobolev embeddings and Lemmas A.1 and A.5 along with the definitions of $\mathcal{E}_{2N}$ and $\mathcal{D}_{2N}$ to estimate

$$\|X\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N} \text{ and } \|Y\|_0^2 \lesssim \mathcal{D}_{2N}. \tag{5.2}$$

Then $\|XY\|_0^2 \le \|X\|_{L^\infty}^2 \|Y\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{2N}$, and the F^1 estimate in (5.1) follows by summing. A similar argument, also employing trace estimates, yields the F^3 and F^4 estimates in (5.1). Note though, that to estimate the $\beta = \alpha$ term in $F^{3,1}$, we use Remark 2.2 to replace $(p-\eta)$.

The same analysis also works for $\partial_t(JF^{2,1})$ and shows that $\left\|\partial_t(JF^{2,1})\right\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{2N}$. To handle $\partial_t(JF^{2,2})$ we must also be able to estimate $\left\|\partial_t^{2N+1}\mathcal{A}\right\|_0^2 \lesssim \mathcal{D}_{2N}$, but this is possible due to Lemma 4.5. Then a similar splitting into L^∞ and H^0 estimates shows that $\left\|\partial_t(JF^{2,2})\right\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{2N}$, and then the $\partial_t(JF^2)$ estimate in (5.1) follows since $F^2 = F^{2,1} + F^{2,2}$. □

We now present estimates for these perturbations when $\partial^\alpha = \partial_t^{N+2}$.

Theorem 5.2. *Let* $\partial^\alpha = \partial_t^{N+2}$ *and let* F^1*,* F^2*,* F^3*,* F^4 *be defined by* (2.13)–(2.22)*. Then for* $m = 1, 2$ *we have*

$$\left\|F^1\right\|_0^2 + \left\|\partial_t(JF^2)\right\|_0^2 + \left\|F^3\right\|_0^2 + \left\|F^4\right\|_0 \lesssim \mathcal{E}_{2N}\mathcal{D}_{N+2,m}. \tag{5.3}$$

Also, if $N \ge 3$, then there exists a $\theta > 0$ so that

$$\left\|F^2\right\|_0^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{E}_{N+2,m} \tag{5.4}$$

for $m = 1, 2$.

Proof. The proof of (5.3) is essentially the same as that of Theorem 5.1. For the F^1, F^3, and F^4 estimates we note that each term in their definition is of the form XY where X involves fewer temporal derivatives than Y, which involves at least two temporal derivatives. We estimate $\|X\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N}$ and $\|Y\|_0^2 \lesssim \mathcal{D}_{N+2,m}$ and then sum to get (5.3). Note that since Y involves at least two temporal derivatives, there is no problem estimating it in terms of $\mathcal{D}_{N+2,m}$. The $\partial_t(JF^2)$ estimate works similarly, except we must also use the bound (4.35) from Lemma 4.5. Note also that in estimating the $\beta = \alpha$ term in $F^{3,1}$, we must employ Remark 2.2 to remove $(p - \eta)$.

We now turn to the proof of (5.4). Recall that $F^2 = F^{2,1} + F^{2,2}$. Since the sum in $F^{2,1}$ runs over $1 \le \beta \le N+1$, we may bound

$$\begin{aligned} \left\|F^{2,1}\right\|_0^2 &\lesssim \sum_{1\le\beta\le N+1} \left\|\partial_t^\beta \mathcal{A}\right\|_{L^\infty}^2 \left\|\partial_t^{N+2-\beta} u\right\|_1^2 \\ &\lesssim \sum_{1\le\beta\le N+1} \mathcal{E}_{2N} \left\|\partial_t^{N+2-\beta} u\right\|_{2(N+2)-2(N+2-\beta)}^2 \lesssim \mathcal{E}_{2N}\mathcal{E}_{N+2,m}. \end{aligned} \tag{5.5}$$

For $F^{2,2}$, a calculation reveals that

$$\begin{aligned} F^{2,2} &= -\partial_t^{N+2}\mathcal{A}_{ij}\partial_j u_i = -\partial_t^{N+2}\mathcal{A}_{i3}\partial_3 u_i \\ &= \partial_t^{N+2}(\partial_1\bar{\eta}\tilde{b}K)\partial_3 u_1 + \partial_t^{N+2}(\partial_2\bar{\eta}\tilde{b}K)\partial_3 u_2 - \partial_t^{N+2}K\partial_3 u_3. \end{aligned} \tag{5.6}$$

We may use the L^∞ interpolation estimate of Proposition 3.9 to bound $\|\partial_3 u_i\|_{L^\infty}^2 \lesssim \mathcal{E}_{N+2,m}$ for $i = 1, 2$ and $m = 1, 2$, which then implies that

$$\left\|\partial_t^{N+2}(\partial_1\bar{\eta}\tilde{b}K)\partial_3 u_1 + \partial_t^{N+2}(\partial_2\bar{\eta}\tilde{b}K)\partial_3 u_2\right\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{E}_{N+2,m} \tag{5.7}$$

if we estimate $\partial_3 u_i$ in L^∞ and the ∂_t^{N+1} terms in H^0. On the other hand, the relation $JK = 1$, the Leibniz rule, and Lemma A.5 imply that

$$\begin{aligned} \left\|\partial_t^{N+2}K\right\|_0^2 &\lesssim \sum_{1\le\gamma\le N+2} \|\partial_t^\gamma J\|_0^2 \lesssim \sum_{1\le\gamma\le N+2} \|\partial_t^\gamma \bar{\eta}\|_1^2 \lesssim \sum_{1\le\gamma\le N+2} \|\partial_t^\gamma \eta\|_{1/2}^2 \\ &= \sum_{1\le\gamma\le N+1} \|\partial_t^\gamma \eta\|_{1/2}^2 + \left\|\partial_t^{N+2}\eta\right\|_{1/2}^2 \lesssim \mathcal{E}_{N+2,m} + \left\|\partial_t^{N+2}\eta\right\|_{1/2}^2. \end{aligned} \tag{5.8}$$

To handle the last term we must use the standard Sobolev interpolation (3.46) with $s = r = 1/2$ and $q = 2N - 9/2$:

$$\left\|\partial_t^{N+2}\eta\right\|_{1/2}^2 \lesssim \left(\left\|\partial_t^{N+2}\eta\right\|_0^2\right)^\kappa \left(\left\|\partial_t^{N+2}\eta\right\|_{2N-4}^2\right)^{1-\kappa} \lesssim (\mathcal{E}_{N+2,m})^\kappa (\mathcal{E}_{2N})^{1-\kappa} \tag{5.9}$$

for $\kappa = (4N-8)/(4N-9)$. Then

$$\begin{aligned} \left\|\partial_t^{N+2}K\partial_3 u_3\right\|_0^2 &\le \left\|\partial_t^{N+2}K\right\|_0^2 \|\partial_3 u_3\|_{L^\infty}^2 \\ &\lesssim \mathcal{E}_{N+2,m} \|\partial_3 u_3\|_{L^\infty}^2 + (\mathcal{E}_{N+2,m})^\kappa (\mathcal{E}_{2N})^{1-\kappa} \|\partial_3 u_3\|_{L^\infty}^2. \end{aligned} \tag{5.10}$$

For the first term on the right we bound $\|\partial_3 u_3\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N}$, and for the second we use the L^∞ interpolation bound of Proposition 3.9 with $r = 1/2$ so that $2/(2+r) = 5/4 \ge 1 - \kappa$ and $\|\partial_3 u_3\|_{L^\infty}^2 \lesssim \mathcal{E}_{N+2,m}^{2/(2+r)} \lesssim \mathcal{E}_{N+2,m}^{1-\kappa}$. Then these estimates and (5.10) imply that

$$\left\|\partial_t^{N+2}K\partial_3 u_3\right\|_0^2 \lesssim \mathcal{E}_{N+2,m}(\mathcal{E}_{2N})^{1-\kappa}. \tag{5.11}$$

We then combine (5.6), (5.7), and (5.11) to see that

$$\left\|F^{2,2}\right\|_0^2 \lesssim \mathcal{E}_{N+2,m}(\mathcal{E}_{2N})^{1-\kappa}. \tag{5.12}$$

Then the estimate (5.4) follows from (5.5) and (5.12). □

5.2. **Energy evolution with the highest and lowest count of temporal derivatives.** We now show the time-integrated evolution estimate for $2N$ temporal derivatives.

Proposition 5.3. *There exists a $\theta > 0$ so that*

$$\left\|\partial_t^{2N} u(t)\right\|_0^2 + \left\|\partial_t^{2N}\eta(t)\right\|_0^2 + \int_0^t \left\|\mathbb{D}\partial_t^{2N} u\right\|_0^2 \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t \mathcal{E}_{2N}^\theta \mathcal{D}_{2N}. \tag{5.13}$$

Proof. We apply $\partial^\alpha = \partial_t^{2N}$ to (1.9). Then $v = \partial_t^{2N} u$, $q = \partial_t^{2N} p$, and $\zeta = \partial_t^{2N}\eta$ solve (2.1) with F^i, $i = 1, 2, 3, 4$ given by (2.13)–(2.22). Applying Lemma 2.1 to these functions and then integrating in time from 0 to t gives

$$\begin{aligned} \frac{1}{2}\int_\Omega J\left|\partial_t^{2N} u(t)\right|^2 + \frac{1}{2}\int_\Sigma \left|\partial_t^{2N}\eta(t)\right|^2 + \frac{1}{2}\int_0^t\int_\Omega J\left|\mathbb{D}_{\mathcal{A}}\partial_t^{2N} u\right|^2 = \frac{1}{2}\int_\Omega J\left|\partial_t^{2N} u(0)\right|^2 \\ + \frac{1}{2}\int_\Sigma \left|\partial_t^{2N}\eta(0)\right|^2 + \int_0^t\int_\Omega J(\partial_t^{2N} u\cdot F^1 + \partial_t^{2N} pF^2) + \int_0^t\int_\Sigma -\partial_t^{2N} u\cdot F^3 + \partial_t^{2N}\eta F^4. \end{aligned} \tag{5.14}$$

We will estimate all of the terms involving F^i on the right side of this equation.

We begin with the F^1 term. According to Theorem 5.1 and Lemma 2.4, we may bound

$$\begin{aligned} \int_0^t\int_\Omega J\partial_t^{2N} u\cdot F^1 \le \int_0^t \left\|\partial_t^{2N} u\right\|_0 \left\|J\right\|_{L^\infty}\left\|F^1\right\|_0 \lesssim \int_0^t \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} \\ = \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \end{aligned} \tag{5.15}$$

Similarly, we use Theorem 5.1 and trace theory to handle the F^3 and F^4 terms:

$$\begin{aligned} \int_0^t\int_\Sigma -\partial_t^{2N} u\cdot F^3 + \partial_t^{2N}\eta F^4 \le \int_0^t \left\|\partial_t^{2N} u\right\|_{H^0(\Sigma)}\left\|F^3\right\|_0 + \left\|\partial_t^{2N}\eta\right\|_0\left\|F^4\right\|_0 \\ \lesssim \int_0^t \left(\left\|\partial_t^{2N} u\right\|_1 + \left\|\partial_t^{2N}\eta\right\|_0\right)\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} \lesssim \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \end{aligned} \tag{5.16}$$

For the term $\partial_t^{2N} pF^2$, there is one more time derivative on p than can be controlled by $\mathcal{D}_{2N}$. We are then forced to integrate by parts in time:

$$\int_0^t\int_\Omega \partial_t^{2N} pJF^2 = -\int_0^t\int_\Omega \partial_t^{2N-1} p\partial_t(JF^2) + \int_\Omega(\partial_t^{2N-1} pJF^2)(t) - \int_\Omega(\partial_t^{2N-1} pJF^2)(0). \tag{5.17}$$

Then according to Theorem 5.1 we may estimate

$$\begin{aligned} -\int_0^t\int_\Omega \partial_t^{2N-1} p\partial_t(JF^2) \lesssim \int_0^t \left\|\partial_t^{2N-1} p\right\|_0\left\|\partial_t(JF^2)\right\|_0 \lesssim \int_0^t \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} \\ = \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \end{aligned} \tag{5.18}$$

On the other hand, it is easy to verify using the Sobolev embeddings that

$$\int_\Omega(\partial_t^{2N-1} pJF^2)(t) - \int_\Omega(\partial_t^{2N-1} pJF^2)(0) \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2}. \tag{5.19}$$

Hence

$$\int_0^t\int_\Omega \partial_t^{2N} pJF^2 \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{5.20}$$

Now we combine (5.15), (5.16), and (5.20) to deduce that

$$\frac{1}{2}\int_\Omega J\left|\partial_t^{2N}u(t)\right|^2+\frac{1}{2}\int_\Sigma\left|\partial_t^{2N}\eta(t)\right|^2+\frac{1}{2}\int_0^t\int_\Omega J\left|\mathbb{D}_\mathcal{A}\partial_t^{2N}u\right|^2 \lesssim \mathcal{E}_{2N}(0)+(\mathcal{E}_{2N}(t))^{3/2}+\int_0^t\sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{5.21}$$

We now seek to replace $J\left|\mathbb{D}_\mathcal{A}\partial_t^{2N}u\right|^2$ with $\left|\mathbb{D}\partial_t^{2N}u\right|^2$ and $J\left|\partial_t^{2N}u(t)\right|^2$ with $\left|\partial_t^{2N}u(t)\right|^2$ in (5.21). To this end we write

$$J\left|\mathbb{D}_\mathcal{A}\partial_t^{2N}u\right|^2=\left|\mathbb{D}\partial_t^{2N}u\right|^2+(J-1)\left|\mathbb{D}\partial_t^{2N}u\right|^2+J\left(\mathbb{D}_\mathcal{A}\partial_t^{2N}u+\mathbb{D}\partial_t^{2N}u\right):\left(\mathbb{D}_\mathcal{A}\partial_t^{2N}u-\mathbb{D}\partial_t^{2N}u\right) \tag{5.22}$$

and estimate the last three terms on the right side. For the last term we note that

$$\mathbb{D}_\mathcal{A}\partial_t^{2N}u\pm\mathbb{D}\partial_t^{2N}u=(\mathcal{A}_{ik}\pm\delta_{ik})\partial_k\partial_t^{2N}u_j+(\mathcal{A}_{jk}\pm\delta_{jk})\partial_k\partial_t^{2N}u_i \tag{5.23}$$

so that Sobolev embeddings and Lemma A.5 provide the bounds

$$\left|\mathbb{D}_\mathcal{A}\partial_t^{2N}u-\mathbb{D}\partial_t^{2N}u\right|\lesssim\sqrt{\mathcal{E}_{2N}}\left|\nabla\partial_t^{2N}u\right| \text{ and } \left|\mathbb{D}_\mathcal{A}\partial_t^{2N}u+\mathbb{D}\partial_t^{2N}u\right|\lesssim(1+\sqrt{\mathcal{E}_{2N}})\left|\nabla\partial_t^{2N}u\right|. \tag{5.24}$$

We then get

$$\int_0^t\int_\Omega\left|J\left(\mathbb{D}_\mathcal{A}\partial_t^{2N}u+\mathbb{D}\partial_t^{2N}u\right):\left(\mathbb{D}_\mathcal{A}\partial_t^{2N}u-\mathbb{D}\partial_t^{2N}u\right)\right| \lesssim\int_0^t(\sqrt{\mathcal{E}_{2N}}+\mathcal{E}_{2N})\int_\Omega\left|\nabla\partial_t^{2N}u\right|^2\lesssim\int_0^t\sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{5.25}$$

Similarly,

$$\int_0^t\int_\Omega|J-1|\left|\mathbb{D}\partial_t^{2N}u\right|^2\lesssim\int_0^t\sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N} \text{ and } \int_\Omega|J-1|\left|\partial_t^{2N}u(t)\right|^2\lesssim(\mathcal{E}_{2N}(t))^{3/2}. \tag{5.26}$$

We may then use (5.22) and (5.25)–(5.26) to replace in (5.21) and derive the bound (5.13). □

Now we prove a similar result for when ∂_t^{N+2} is applied. This time, however, we do not want an inequality that is integrated in time, so we are forced to introduce an error term involving $\partial_t^{N+1}p$.

Proposition 5.4. *Let F^2 be given by* (2.19) *with $\partial^\alpha=\partial_t^{N+2}$. Then it holds that*

$$\partial_t\left(\left\|\sqrt{J}\partial_t^{N+2}u\right\|_0^2+\left\|\partial_t^{N+2}\eta\right\|_0^2-2\int_\Omega J\partial_t^{N+1}pF^2\right)+\left\|\mathbb{D}\partial_t^{N+2}u\right\|_0^2\lesssim\sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2,m}. \tag{5.27}$$

Proof. We apply $\partial^\alpha=\partial_t^{N+2}$ to (1.9). Then $v=\partial_t^{N+2}u$, $q=\partial_t^{N+2}p$, and $\zeta=\partial_t^{N+2}\eta$ solve (2.1) with F^i, $i=1,2,3,4$ given by (2.13)–(2.22). Applying Lemma 2.1 to these functions gives

$$\partial_t\left(\frac{1}{2}\int_\Omega J\left|\partial_t^{N+2}u\right|^2+\frac{1}{2}\int_\Sigma\left|\partial_t^{N+2}\eta\right|^2\right)+\frac{1}{2}\int_\Omega J\left|\mathbb{D}_\mathcal{A}\partial_t^{N+2}u\right|^2 =\int_\Omega J(\partial_t^{N+2}u\cdot F^1+\partial_t^{N+2}pF^2)+\int_\Sigma-\partial_t^{N+2}u\cdot F^3+\partial_t^{N+2}\eta F^4. \tag{5.28}$$

We will estimate all of the terms involving F^i on the right side of this equation as in Proposition 5.3.

We begin with the F^1 term. According to Theorem 5.2 and Lemma 2.4, we may bound

$$\int_\Omega J\partial_t^{N+2}u\cdot F^1\le\left\|\partial_t^{N+2}u\right\|_0\|J\|_{L^\infty}\left\|F^1\right\|_0\lesssim\sqrt{\mathcal{D}_{N+2,m}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{N+2,m}} =\sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2,m}. \tag{5.29}$$

Similarly, we use Theorem 5.2 and trace theory to handle the F^3 and F^4 terms:

$$\int_\Sigma -\partial_t^{N+2} u \cdot F^3 + \partial_t^{N+2}\eta F^4 \le \left\|\partial_t^{N+2} u\right\|_{H^0(\Sigma)} \left\|F^3\right\|_0 + \left\|\partial_t^{N+2}\eta\right\|_0 \left\|F^4\right\|_0 \lesssim \left(\left\|\partial_t^{N+2} u\right\|_1 + \left\|\partial_t^{N+2}\eta\right\|_0\right)\sqrt{\mathcal{E}_{2N}\mathcal{D}_{N+2,m}} \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2,m}. \tag{5.30}$$

For the term $\partial_t^{N+2} pF^2$, there is one more time derivative on p than can be controlled by $\mathcal{D}_{N+2,m}$. We are then forced to pull out a time derivative:

$$\int_\Omega \partial_t^{N+2} pJF^2 = \partial_t \int_\Omega \partial_t^{N+1} pJF^2 - \int_\Omega \partial_t^{N+1} p\partial_t(JF^2). \tag{5.31}$$

Then according to Theorem 5.2 we may estimate

$$-\int_\Omega \partial_t^{N+1} p\partial_t(JF^2) \lesssim \left\|\partial_t^{N+1} p\right\|_0 \left\|\partial_t(JF^2)\right\|_0 \lesssim \sqrt{\mathcal{D}_{N+2,m}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{N+2,m}} = \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2,m}. \tag{5.32}$$

Hence

$$\int_0^t \int_\Omega \partial_t^{2N} pJF^2 \lesssim \partial_t \int_\Omega \partial_t^{N+1} pJF^2 + \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2,m}. \tag{5.33}$$

Now we combine (5.28)–(5.30) and (5.33) to deduce that

$$\partial_t \left(\frac{1}{2}\int_\Omega J\left|\partial_t^{N+2} u\right|^2 + \frac{1}{2}\int_\Sigma \left|\partial_t^{N+2}\eta\right|^2 - \int_\Omega \partial_t^{N+1} pJF^2\right) + \frac{1}{2}\int_\Omega J\left|\mathbb{D}_{\mathcal{A}}\partial_t^{N+2} u\right|^2 \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2,m}. \tag{5.34}$$

We may argue as in (5.22)–(5.26) of Theorem 5.3 to show that

$$\frac{1}{2}\int_\Omega \left|\mathbb{D}\partial_t^{N+2} u\right|^2 \lesssim \frac{1}{2}\int_\Omega J\left|\mathbb{D}_{\mathcal{A}}\partial_t^{N+2} u\right|^2 + \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2,m}. \tag{5.35}$$

Then (5.27) follows from (5.34) and (5.35). □

Finally, we record the basic energy estimate when no derivatives are applied.

Proposition 5.5. *It holds that*

$$\partial_t \left(\frac{1}{2}\int_\Omega J|u|^2 + \frac{1}{2}\int_\Sigma |\eta|^2\right) + \frac{1}{2}\int_\Omega J|\mathbb{D}_{\mathcal{A}} u|^2 = 0. \tag{5.36}$$

In particular

$$\|u(t)\|_0^2 + \|\eta(t)\|_0^2 + \int_0^t \|\mathbb{D}u\|_0^2 \lesssim \mathcal{E}_{2N}(0) + \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{5.37}$$

Proof. Setting $F^i = 0$ in Lemma 2.1 for $i = 1,2,3,4$ yields (5.36). We may argue as in (5.22)–(5.26) of Theorem 5.3 to estimate

$$\frac{1}{2}\int_\Omega |\mathbb{D}u|^2 \lesssim \frac{1}{2}\int_\Omega J|\mathbb{D}_{\mathcal{A}} u|^2 + \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{5.38}$$

Similarly, Lemma 2.4 allows us to estimate

$$\frac{1}{4}\int_\Omega |u|^2 \le \frac{1}{2}\int_\Omega J|u|^2. \tag{5.39}$$

Now we may integrate (5.36) in time from 0 to t and use these two estimates to derive (5.37). □

6. Energy evolution in the perturbed linear form

6.1. **Energy evolution for horizontal derivatives.** We now estimate how the evolution of the horizontal energy is coupled to the horizontal dissipation and the full energy and dissipation.

Lemma 6.1. *Let $\alpha \in \mathbb{N}^2$ be such that $|\alpha| = 4N$, i.e. let ∂^α be $4N$ spatial derivatives in the x_1, x_2 directions. Then*

$$\left| \int_\Sigma \partial^\alpha \eta \partial^\alpha G^4 \right| \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N} \mathcal{K} \mathcal{F}_{2N}}. \tag{6.1}$$

Proof. Throughout the proof β will always denote an element of $\mathbb{N}^2$, and we will write $Df \cdot \partial^\beta u = \partial_1 f \partial^\beta u_1 + \partial_2 f \partial^\beta u_2$ for a function f defined on Σ. Then by the Leibniz rule, we have that

$$\partial^\alpha G^4 = \partial^\alpha (D\eta \cdot u) = D\partial^\alpha \eta \cdot u + \sum_{\substack{0<\beta\le\alpha \\ |\beta|=1}} C_{\alpha,\beta} D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u + \sum_{\substack{0<\beta\le\alpha \\ |\beta|\ge 2}} C_{\alpha,\beta} D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \tag{6.2}$$

for constants $C_{\alpha,\beta}$ depending on α and β. We will analyze each of the three terms on the right separately.

For the first term, we integrate by parts to see that.

$$\int_\Sigma \partial^\alpha \eta D \partial^\alpha \eta \cdot u = \frac{1}{2} \int_\Sigma D \left| \partial^\alpha \eta \right|^2 \cdot u = -\frac{1}{2} \int_\Sigma \partial^\alpha \eta \partial^\alpha \eta (\partial_1 u_1 + \partial_2 u_2). \tag{6.3}$$

This then allows us to use (A.3) of Lemma A.1 to bound

$$\begin{aligned} \left| \int_\Sigma \partial^\alpha \eta D \partial^\alpha \eta \cdot u \right| &\lesssim \left\| \partial^\alpha \eta \right\|_{1/2} \left\| \partial^\alpha \eta (\partial_1 u_1 + \partial_2 u_2) \right\|_{H^{-1/2}(\Sigma)} \\ &\le \left\| \eta \right\|_{4N+1/2} \left\| \partial^\alpha \eta \right\|_{-1/2} \left\| \partial_1 u_1 + \partial_2 u_2 \right\|_{H^2(\Sigma)} \\ &\le \left\| \eta \right\|_{4N+1/2} \left\| D\eta \right\|_{4N-3/2} \left\| \partial_1 u_1 + \partial_2 u_2 \right\|_{H^2(\Sigma)} \le \sqrt{\mathcal{F}_{2N} \mathcal{D}_{2N} \mathcal{K}}. \end{aligned} \tag{6.4}$$

Similarly, for the second term we estimate

$$\begin{aligned} \left| \int_\Sigma \partial^\alpha \eta \sum_{\substack{0<\beta\le\alpha \\ |\beta|=1}} C_{\alpha,\beta} D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right| &\lesssim \left\| D^{4N} \eta \right\|_{1/2} \left\| D^{4N} \eta \right\|_{-1/2} \sum_{i=1}^2 \left\| D u_i \right\|_{H^2(\Sigma)} \\ &\le \left\| \eta \right\|_{4N+1/2} \left\| D\eta \right\|_{4N-3/2} \sum_{i=1}^2 \left\| D u_i \right\|_{H^2(\Sigma)} \le \sqrt{\mathcal{F}_{2N} \mathcal{D}_{2N} \mathcal{K}}. \end{aligned} \tag{6.5}$$

For the third term we first note that $\left\| \partial^\alpha \eta \right\|_{-1/2} \le \left\| D\eta \right\|_{4N-3/2} \le \sqrt{\mathcal{D}_{2N}}$, which allows us to bound

$$\begin{aligned} \left| \int_\Sigma \partial^\alpha \eta D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right| &\le \left\| \partial^\alpha \eta \right\|_{-1/2} \left\| D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right\|_{H^{1/2}(\Sigma)} \\ &\le \sqrt{\mathcal{D}_{2N}} \left\| D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right\|_{H^{1/2}(\Sigma)}. \end{aligned} \tag{6.6}$$

We estimate the last term on the right using Lemma A.1, but in different ways depending on $|\beta|$:

$$\begin{aligned} \left\| D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right\|_{H^{1/2}(\Sigma)} &\lesssim \begin{cases} \left\| D\partial^{\alpha-\beta}\eta \right\|_{1/2} \left\| \partial^\beta u \right\|_{H^2(\Sigma)} & \text{for } 2 \le |\beta| \le 2N \\ \left\| D\partial^{\alpha-\beta}\eta \right\|_2 \left\| \partial^\beta u \right\|_{H^{1/2}(\Sigma)} & \text{for } 2N+1 \le |\beta| \le 4N \end{cases} \\ &\lesssim \begin{cases} \left\| D\eta \right\|_{4N-3/2} \left\| u \right\|_{2N+3} & \text{for } 2 \le |\beta| \le 2N \\ \left\| D\eta \right\|_{2N+1} \left\| u \right\|_{4N+1} & \text{for } 2N+1 \le |\beta| \le 4N \end{cases}, \end{aligned} \tag{6.7}$$

so that $\left\| D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right\|_{H^{1/2}(\Sigma)} \lesssim \sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}}$ for all $0 < \beta \le \alpha$ with $|\beta| \ge 2$. Hence

$$\left| \int_\Sigma \partial^\alpha \eta \sum_{\substack{0<\beta\le\alpha \\ |\beta|\ge 2}} C_{\alpha,\beta} D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right| \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} = \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{6.8}$$

The estimate (6.1) then follows from (6.4), (6.5), and (6.8). □

Now we prove an estimate for horizontal derivatives up to order $2N$, excluding $\partial^\alpha = \partial_t^{2N}$ and no derivatives.

Proposition 6.2. *Suppose that $\alpha \in \mathbb{N}^{1+2}$ is such that $\alpha_0 \le 2N-1$ and $1 \le |\alpha| \le 4N$. Then there exists a $\theta > 0$ so that*

$$\partial_t \left(\frac{1}{2}\int_\Omega |\partial^\alpha u|^2 + \frac{1}{2}\int_\Sigma |\partial^\alpha \eta|^2 \right) + \frac{1}{2}\int_\Omega |\mathbb{D}\partial^\alpha u|^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}, \tag{6.9}$$

and in particular,

$$\begin{aligned} &\left\| \bar{D}_1^{4N-1} u \right\|_0^2 + \left\| D\bar{D}^{4N-1} u \right\|_0^2 + \left\| \bar{D}_1^{4N-1} \eta \right\|_0^2 + \left\| D\bar{D}^{4N-1} \eta \right\|_0^2 \\ &\quad + \int_0^t \left\| \bar{D}_1^{4N-1}\mathbb{D}u \right\|_0^2 + \left\| D\bar{D}^{4N-1}\mathbb{D}u \right\|_0^2 \lesssim \bar{\mathcal{E}}_{2N}(0) + \int_0^t \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \end{aligned} \tag{6.10}$$

Proof. Let $\alpha \in \mathbb{N}^{1+2}$ satisfy $\alpha_0 \le 2N-1$ and $1 \le |\alpha| \le 4N$. Note that the constraint on α_0 implies that we do not exceed the number of temporal derivatives of p that we can control. An application of Lemma 2.3 to $v = \partial^\alpha u$, $q = \partial^\alpha p$, $\zeta = \partial^\alpha \eta$ with $\Phi^1 = \partial^\alpha G^1$, $\Phi^2 = \partial^\alpha G^2$, $\Phi^3 = \partial^\alpha G^3$, $\Phi^4 = \partial^\alpha G^4$, and $a = 1$ reveals that

$$\begin{aligned} \partial_t \left(\frac{1}{2}\int_\Omega |\partial^\alpha u|^2 + \frac{1}{2}\int_\Sigma |\partial^\alpha \eta|^2 \right) + \frac{1}{2}\int_\Omega |\mathbb{D}\partial^\alpha u|^2 &= \int_\Omega \partial^\alpha u \cdot \partial^\alpha G^1 + \partial^\alpha p \partial^\alpha G^2 \\ &\quad + \int_\Sigma -\partial^\alpha u \cdot \partial^\alpha G^3 + \partial^\alpha \eta \partial^\alpha G^4. \end{aligned} \tag{6.11}$$

Assume initially that $1 \le |\alpha| \le 4N-1$. Then according to the estimates (4.7)–(4.8) of Theorem 4.2 and the definition of $\mathcal{D}_{2N}$, we have

$$\begin{aligned} \left| \int_\Omega \partial^\alpha u \cdot \partial^\alpha G^1 + \partial^\alpha p \partial^\alpha G^2 \right| &\le \|\partial^\alpha u\|_0 \left\|\partial^\alpha G^1\right\|_0 + \|\partial^\alpha p\|_0 \left\|\partial^\alpha G^2\right\|_0 \\ &\lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}, \end{aligned} \tag{6.12}$$

where in the last equality we have written $\kappa = \theta/2$ for $\theta > 0$ the number provided by Theorem 4.2. Similarly, we may use Theorem 4.2 along with the trace estimate $\|\partial^\alpha u\|_{H^0(\Sigma)} \lesssim \|\partial^\alpha u\|_1 \le \sqrt{\mathcal{D}_{2N}}$ to find that

$$\begin{aligned} \left| \int_\Sigma -\partial^\alpha u \cdot \partial^\alpha G^3 + \partial^\alpha \eta \partial^\alpha G^4 \right| &\le \|\partial^\alpha u\|_{H^0(\Sigma)} \left\|\partial^\alpha G^3\right\|_0 + \|\partial^\alpha \eta\|_0 \left\|\partial^\alpha G^4\right\|_0 \\ &\lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \end{aligned} \tag{6.13}$$

Now assume that $|\alpha| = 4N$. Since $\alpha_0 \le 2N-1$, we may write $\alpha = \beta + (\alpha - \beta)$ for some $\beta \in \mathbb{N}^2$ with $|\beta| = 1$, i.e. ∂^α involves at least one spatial derivative. Since $|\alpha - \beta| = 4N-1$, we can then integrate by parts and use (4.8) of Theorem 4.2 to see that

$$\begin{aligned} \left| \int_\Omega \partial^\alpha u \cdot \partial^\alpha G^1 \right| &= \left| \int_\Omega \partial^{\alpha+\beta} u \cdot \partial^{\alpha-\beta} G^1 \right| \le \left\| \partial^{\alpha+\beta} u \right\|_0 \left\| \partial^{\alpha-\beta} G^1 \right\|_0 \\ &\le \|\partial^\alpha u\|_1 \left\| \bar{\nabla}^{4N-1} G^1 \right\|_0 \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \end{aligned} \tag{6.14}$$

For the pressure term we do not need to integrate by parts:

$$(6.15)\quad \left|\int_\Omega \partial^\alpha p \partial^\alpha G^2\right| \le \|\partial^\alpha p\|_0 \left\|\partial^{\alpha-\beta}\partial^\beta G^1\right\|_0 \le \|\partial^\alpha p\|_0 \left\|\bar{\nabla}^{4N-1} G^1\right\|_1 \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}.$$

We integrate by parts and use the trace estimate $H^1(\Omega) \hookrightarrow H^{1/2}(\Sigma)$ to see that

$$(6.16)\quad \left|\int_\Sigma \partial^\alpha u \cdot \partial^\alpha G^3\right| = \left|\int_\Sigma \partial^{\alpha+\beta} u \cdot \partial^{\alpha-\beta} G^3\right| \le \left\|\partial^{\alpha+\beta} u\right\|_{H^{-1/2}(\Sigma)} \left\|\partial^{\alpha-\beta} G^3\right\|_{1/2} \le \|\partial^\alpha u\|_{H^{1/2}(\Sigma)} \left\|\bar{D}^{4N-1} G^3\right\|_{1/2} \le \|\partial^\alpha u\|_1 \left\|\bar{D}^{4N-1} G^3\right\|_{1/2} \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}.$$

For the term $\partial^\alpha \eta \partial^\alpha G^4$ we must split to two cases: $\alpha_0 \ge 1$ and $\alpha_0 = 0$. In the former case, there is at least one temporal derivative in ∂^α, so $\|\partial^\alpha \eta\|_{1/2} \le \sqrt{\mathcal{D}_{2N}}$, and hence

$$(6.17)\quad \left|\int_\Sigma \partial^\alpha \eta \partial^\alpha G^4\right| = \left|\int_\Sigma \partial^{\alpha+\beta} \eta \partial^{\alpha-\beta} G^4\right| \le \left\|\partial^{\alpha+\beta}\eta\right\|_{-1/2} \left\|\partial^{\alpha-\beta} G^4\right\|_{1/2} \le \|\partial^\alpha \eta\|_{1/2} \left\|\bar{D}^{4N-1} G^3\right\|_{1/2} \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}.$$

In the latter case, $\alpha_0 = 0$, so that ∂^α involves only spatial derivatives; in this case we use Lemma 6.1 to bound

$$(6.18)\quad \left|\int_\Sigma \partial^\alpha \eta \partial^\alpha G^4\right| \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}.$$

Now, in light of (6.11)–(6.18) we know that (6.9) holds. The bound (6.10) follows by applying (6.9) to all $1 \le |\alpha| \le 4N$ with $\alpha_0 \le 2N-1$, summing, and integrating in time from 0 to t. □

Our next result provides some preliminary interpolation estimates for G^2 and G^4 in terms of the dissipation at the $N+2$ level, but with a power greater than 1.

Lemma 6.3. *We have the estimate*

$$(6.19)\quad \left\|D^{2N+3} G^4\right\|_{1/2} \lesssim \left(\mathcal{D}_{N+2,2}\right)^{1+2/(4N-7)}.$$

Also, there exists a $\theta > 0$ so that

$$(6.20)\quad \left\|D G^4\right\|_0^2 \lesssim \mathcal{E}_{2N}^\theta \left(\mathcal{D}_{N+2,1}\right)^{1+1/(\lambda+2)}, \text{ and } \left\|\bar{D}^2 G^4\right\|_0^2 \lesssim \mathcal{E}_{2N}^\theta \left(\mathcal{D}_{N+2,2}\right)^{1+1/(\lambda+3)}.$$

Finally,

$$(6.21)\quad \left\|D G^2\right\|_{L^1}^2 \lesssim \mathcal{E}_{2N}^\theta \left(\mathcal{D}_{N+2,1}\right)^{1+\lambda/(\lambda+2)}, \text{ and } \left\|\bar{D}^2 G^2\right\|_{L^1}^2 \lesssim \mathcal{E}_{2N}^\theta \left(\mathcal{D}_{N+2,2}\right)^{1+\lambda/(\lambda+3)}.$$

Proof. Let $\alpha \in \mathbb{N}^2$ be such that $|\alpha| = 2(N+2)-1$. The Leibniz rule, Lemma A.1, and trace theory imply that

$$(6.22)\quad \left\|\partial^\alpha G^4\right\|_{1/2} \lesssim \sum_{\substack{\beta\le\alpha \\ |\beta|\le N+2}} \left\|D\partial^\beta \eta\right\|_2 \left\|\partial^{\alpha-\beta} u\right\|_{H^{1/2}(\Sigma)} + \sum_{\substack{\beta\le\alpha \\ N+3\le|\beta|\le 2N+3}} \left\|D\partial^\beta\eta\right\|_{1/2} \left\|\partial^{\alpha-\beta}u\right\|_{H^2(\Sigma)} \lesssim \|D\eta\|_{N+4} \left\|D_{N+1}^{2N+3} u\right\|_1 + \left\|D^3\eta\right\|_{2(N+2)-5/2} \|u\|_{H^{N+2}(\Sigma)}.$$

Trace theory, Poincaré's inequality, and the $H^0(\Omega)$ interpolation result for ∇u of Lemma 3.14 imply that

$$(6.23)\quad \|u\|_{H^{N+2}(\Sigma)}^2 \lesssim \|u\|_{H^0(\Sigma)}^2 + \left\|D^{N+2}u\right\|_{H^0(\Sigma)}^2 \lesssim \|\nabla u\|_0^2 + \left\|D^{N+2}u\right\|_1^2 \le \mathcal{D}_{N+2,2}^{(\lambda+1)/(\lambda+3)} + \left(\mathcal{E}_{2N}\right)^{(\lambda+2)/(\lambda+3)} \left(\mathcal{D}_{N+2,2}\right)^{(\lambda+1)/(\lambda+3)} \lesssim \mathcal{D}_{N+2,2}^{(\lambda+1)/(\lambda+3)}.$$

Since $N \geq 5$ and $\lambda \in (0,1)$, we may define

$$q = \frac{8N + 2\lambda - 8}{4N(1+\lambda) - 9\lambda - 13} \in \left[\frac{8N-6}{8N-22}, \frac{8N-8}{4N-13} \right] \subset [1, 2N - 9/2]. \tag{6.24}$$

Using this q, $r = 1$ and $s = 2(N+2) - 5/2$ in the standard Sobolev interpolation inequality (3.46), we find that

$$\begin{aligned} \left\| D^3 \eta \right\|_{2(N+2)-5/2}^2 &\lesssim \left(\left\| D^3 \eta \right\|_{2(N+2)-7/2}^2 \right)^{q/(1+q)} \left(\left\| D^3 \eta \right\|_{2(N+2)-5/2+q}^2 \right)^{1/(1+q)} \\ &\lesssim (\mathcal{D}_{N+2,2})^{q/(1+q)} (\mathcal{E}_{2N})^{1/(1+q)} \lesssim (\mathcal{D}_{N+2,2})^{q/(1+q)} . \end{aligned} \tag{6.25}$$

Our choice of q implies that

$$\frac{\lambda+1}{\lambda+3} + \frac{q}{q+1} = 1 + \frac{2}{4N-7}, \tag{6.26}$$

so that (6.23) and (6.25) then imply that

$$\left\| D^3 \eta \right\|_{2(N+2)-5/2}^2 \left\| u \right\|_{H^{N+2}(\Sigma)}^2 \lesssim (\mathcal{D}_{N+2,2})^{1+2/(4N-7)} . \tag{6.27}$$

The $H^0(\Sigma)$ interpolation result for $D\eta$ of Lemma 3.1 implies that

$$\begin{aligned} \left\| D\eta \right\|_{N+4}^2 &\lesssim \left\| D\eta \right\|_0^2 + \left\| D^3 \eta \right\|_{N+2}^2 \\ &\lesssim \mathcal{D}_{N+2,2}^{(\lambda+1)/(\lambda+3)} + \left(\left\| D^3 \eta \right\|_{N+2}^2 \right)^{(\lambda+2)/(\lambda+3)} \left(\left\| D^3 \eta \right\|_{N+2}^2 \right)^{(\lambda+1)/(\lambda+3)} \\ &\leq \mathcal{D}_{N+2,2}^{(\lambda+1)/(\lambda+3)} + (\mathcal{E}_{2N})^{(\lambda+2)/(\lambda+3)} (\mathcal{D}_{N+2,2})^{(\lambda+1)/(\lambda+3)} \lesssim \mathcal{D}_{N+2,2}^{(\lambda+1)/(\lambda+3)} . \end{aligned} \tag{6.28}$$

On the other hand, using the same q as above and Lemma A.12, we have

$$\begin{aligned} \left\| D_{N+1}^{2N+3} u \right\|_1 &= \left(\left\| D_{N+1}^{2N+3} u \right\|_1 \right)^{q/(q+1)} \left(\left\| D_{N+1}^{2N+3} u \right\|_1 \right)^{1/(q+1)} \\ &\lesssim (\mathcal{D}_{N+2,2})^{q/(1+q)} (\mathcal{E}_{2N})^{1/(1+q)} \leq (\mathcal{D}_{N+2,2})^{q/(1+q)} . \end{aligned} \tag{6.29}$$

Then (6.28) and (6.29) imply that

$$\left\| D\eta \right\|_{N+4}^2 \left\| D_{N+1}^{2N+3} u \right\|_1 \lesssim (\mathcal{D}_{N+2,2})^{1+2/(4N-7)} . \tag{6.30}$$

We then combine (6.22), (6.27), and (6.30) to deduce (6.19).

We now turn to the proof of the bounds (6.20) and (6.21). The bounds (6.20) may be deduced by applying an operator ∂^α with $\alpha \in \mathbb{N}^{1+2}$ satisfying either $|\alpha| = 1$ or $|\alpha| = 2$ to G^4, and then estimating the resulting products with one norm taken in H^0 and the others in L^∞, employing the H^0 and L^∞ interpolation estimates for η, u and their derivatives recorded in Lemma 3.1, Proposition 3.9, and Theorem 3.14. The bounds (6.21) may be deduced similarly except that at least two terms in the resulting products must be estimated in H^0 to deduce the resulting L^1 bounds. This presents no problem since G^2 is a linear combination of products of two or more terms. □

With this lemma in place, we may record the estimates for the evolution of the energy at the $N+2$ level.

Proposition 6.4. *Suppose that $m \in \{1,2\}$ and $\alpha \in \mathbb{N}^{1+2}$ is such that $\alpha_0 \leq N+1$ and $m \leq |\alpha| \leq 2(N+2)$. Then there exists a $\theta > 0$ so that*

$$\partial_t \left(\left\| \partial^\alpha u \right\|_0^2 + \left\| \partial^\alpha \eta \right\|_0^2 \right) + \left\| \mathbb{D} \partial^\alpha u \right\|_0^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m} . \tag{6.31}$$

In particular,

$$\begin{aligned} \partial_t \Big(\left\| \bar{D}_m^{2N+3} u \right\|_0^2 + \left\| D \bar{D}^{2N+3} u \right\|_0^2 + \left\| \bar{D}_m^{2N+3} \eta \right\|_0^2 + \left\| D \bar{D}^{2N+3} \eta \right\|_0^2 \Big) \\ + \left\| \bar{D}_m^{2N+3} \mathbb{D} u \right\|_0^2 + \left\| D \bar{D}^{2N+3} \mathbb{D} u \right\|_0^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m} . \end{aligned} \tag{6.32}$$

Proof. For $m \in \{1,2\}$ and $\alpha \in \mathbb{N}^{1+2}$ such that $\alpha_0 \le N+1$ and $m \le |\alpha| \le 2(N+2)$, we argue as in Proposition 6.2 to deduce that (6.11) holds. Let X_α denote the right hand side of (6.11) for our range of α. To bound X_α, we break to three cases.

If $m+1 \le |\alpha| \le 2(N+2)-1$ or $|\alpha| = 2(N+2)$ with $1 \le \alpha_0 \le N+1$, then we know from trace theory and the definitions of $\mathcal{D}_{N+2,m}$ that

$$\|\partial^\alpha u\|_0^2 + \|\partial^\alpha p\|_0^2 + \|\partial^\alpha u\|_{H^{1/2}(\Sigma)}^2 + \|\partial^\alpha \eta\|_{1/2}^2 \lesssim \mathcal{D}_{N+2,m}. \tag{6.33}$$

This allows us to argue as in Proposition 6.2, employing Theorem 4.1 in place of Theorem 4.2, to bound

$$|X_\alpha| \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m} \tag{6.34}$$

for some $\theta > 0$.

Now consider $|\alpha| = 2(N+2)$ with $\alpha_0 = 0$. In this case we still know that

$$\|\partial^\alpha u\|_1^2 + \|\partial^\alpha p\|_0^2 + \|\partial^\alpha u\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{D}_{N+2,m}, \tag{6.35}$$

so we may argue as in Proposition 6.2, integrating by parts and using these bounds as well as those from Theorem 4.1 to show that the first, second, and third integrals in the definition of X_α are bounded by $\mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m}$. For the fourth integral, we control $\|\partial^\alpha \eta\|_{1/2}^2$ through the interpolation estimate of Lemma 3.18:

$$\|\partial^\alpha \eta\|_{1/2}^2 \le \left\|D^{2N+4}\eta\right\|_{1/2}^2 \lesssim (\mathcal{E}_{2N})^{2/(4N-7)} (\mathcal{D}_{N+2,2})^{(4N-9)/(4N-7)}. \tag{6.36}$$

Then we may integrate by parts with $\alpha = \beta + (\alpha - \beta)$, $|\beta| = 1$ and employ this estimate along with (6.19) of Lemma 6.3 to see that

$$\begin{aligned} \left| \int_\Sigma \partial^\alpha \eta \partial^\alpha G^4 \right| &= \left| \int_\Sigma \partial^{\alpha+\beta} \eta \partial^{\alpha-\beta} G^4 \right| \le \left\| \partial^{\alpha+\beta}\eta \right\|_{-1/2} \left\| \partial^{\alpha-\beta} G^4 \right\|_{1/2} \\ &\le \|\partial^\alpha \eta\|_{1/2} \left\|D^{2N+3} G^4\right\|_{1/2} \lesssim \sqrt{(\mathcal{E}_{2N})^{2/(4N-7)} (\mathcal{D}_{N+2,2})^{(4N-9)/(4N-7)}} \sqrt{(\mathcal{D}_{N+2,2})^{1+2/(4N-7)}} \\ &= (\mathcal{E}_{2N})^{1/(4N-7)} \mathcal{D}_{N+2,2} \le (\mathcal{E}_{2N})^{1/(4N-7)} \mathcal{D}_{N+2,m}. \end{aligned} \tag{6.37}$$

Hence, when $|\alpha| = 2(N+2)$ with $\alpha_0 = 0$ we also have that there is a $\theta > 0$ so that

$$|X_\alpha| \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m}. \tag{6.38}$$

Finally, we consider the case of $|\alpha| = m$ for $m = 1,2$. In this case we only know that

$$\|\partial^\alpha u\|_1^2 + \|\partial^\alpha u\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{D}_{N+2,m}, \tag{6.39}$$

so only the first and third integrals of X_α may be handled directly as above to be bounded by $\mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m}$. For the fourth term we first use the $H^0(\Sigma)$ interpolation results of Lemma 3.1 and Theorem 3.16 to bound

$$\|D\eta\|_0^2 \lesssim (\mathcal{D}_{N+2,1})^{(\lambda+1)/(\lambda+2)} \text{ and } \left\|D^2\eta\right\|_0^2 + \|\partial_t \eta\|_0^2 \lesssim (\mathcal{D}_{N+2,2})^{(\lambda+2)/(\lambda+3)}. \tag{6.40}$$

Then by (6.20) of Lemma 6.3, we know that

$$\begin{aligned} \left| \int_\Sigma \partial^\alpha \eta \partial^\alpha G^4 \right| &\le \|\partial^\alpha \eta\|_0 \left\|\partial^\alpha G^4\right\|_0 \\ &\lesssim \begin{cases} \sqrt{(\mathcal{D}_{N+2,1})^{(\lambda+1)/(\lambda+2)}} \sqrt{\mathcal{E}_{2N}^\theta (\mathcal{D}_{N+2,1})^{1+1/(\lambda+2)}} & \text{for } m = 1 \\ \sqrt{(\mathcal{D}_{N+2,2})^{(\lambda+2)/(\lambda+3)}} \sqrt{\mathcal{E}_{2N}^\theta (\mathcal{D}_{N+2,2})^{1+1/(\lambda+3)}} & \text{for } m = 2 \end{cases} \\ &\le \mathcal{E}_{2N}^{\theta/2} \mathcal{D}_{N+2,m}. \end{aligned} \tag{6.41}$$

For the third term we first use Lemma A.8 to bound

$$\|Dp\|_{L^\infty}^2 \lesssim (\mathcal{D}_{N+2,1})^{2/(\lambda+2)} \text{ and } \left\|D^2\eta\right\|_0^2 + \|\partial_t \eta\|_0^2 \lesssim (\mathcal{D}_{N+2,2})^{3/(\lambda+3)}. \tag{6.42}$$

Then by (6.21) of Lemma 6.3, we know that

$$\left|\int_\Omega \partial^\alpha p \partial^\alpha G^2\right| \le \|\partial^\alpha p\|_{L^\infty} \left\|\partial^\alpha G^2\right\|_{L^1} \lesssim \begin{cases} \sqrt{(\mathcal{D}_{N+2,1})^{2/(\lambda+2)}}\sqrt{\mathcal{E}_{2N}^\theta (\mathcal{D}_{N+2,1})^{1+\lambda/(\lambda+2)}} & \text{for } m=1 \\ \sqrt{(\mathcal{D}_{N+2,2})^{3/(\lambda+3)}}\sqrt{\mathcal{E}_{2N}^\theta (\mathcal{D}_{N+2,2})^{1+\lambda/(\lambda+3)}} & \text{for } m=2 \end{cases} \le \mathcal{E}_{2N}^{\theta/2}\mathcal{D}_{N+2,m}. \tag{6.43}$$

Hence when $|\alpha| = m$ for $m = 1, 2$ it also holds that

$$|X_\alpha| \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m}. \tag{6.44}$$

Now, by (6.34), (6.38), and (6.44) we know that (6.31) holds. The bound (6.32) follows by summing (6.31) over the specified range of α. □

6.2. **Energy evolution for $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$.** Before we can analyze the energy evolution for $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$ we must first prove a lemma that provides control of $\mathcal{I}_\lambda p$.

Lemma 6.5. *It holds that*

$$\|\mathcal{I}_\lambda p\|_0^2 \lesssim \mathcal{E}_{2N}, \text{ and} \tag{6.45}$$

$$\|\mathcal{I}_\lambda D p\|_0^2 \lesssim (\mathcal{E}_{2N})^{\lambda/(1+\lambda)} (\mathcal{D}_{2N})^{1/(1+\lambda)}. \tag{6.46}$$

Proof. Let $\alpha \in \mathbb{N}^2$ be such that $|\alpha| \in \{0, 1\}$. We may apply Lemma A.10 to see that

$$\|\partial^\alpha \mathcal{I}_\lambda p\|_0^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda p\|_{H^0(\Sigma)}^2 + \|\partial_3 \partial^\alpha \mathcal{I}_\lambda p\|_0^2. \tag{6.47}$$

In order to estimate each term on the right we will use the structure of the equation (2.23). Indeed, using the boundary condition, we find that

$$\|\partial^\alpha \mathcal{I}_\lambda p\|_{H^0(\Sigma)}^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \eta\|_0^2 + \|\partial^\alpha \mathcal{I}_\lambda \partial_3 u_3\|_{H^0(\Sigma)}^2 + \left\|\partial^\alpha \mathcal{I}_\lambda G^3\right\|_0^2. \tag{6.48}$$

Trace theory and the divergence equation in (2.23) allow us to bound

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_3 u_3\|_{H^0(\Sigma)}^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \partial_3 u_3\|_1^2 \lesssim \left\|\partial^\alpha \mathcal{I}_\lambda G^2\right\|_1^2 + \|\partial^\alpha \mathcal{I}_\lambda D u\|_1^2 \lesssim \|\mathcal{I}_\lambda D u\|_2^2 + \left\|\mathcal{I}_\lambda G^2\right\|_2^2, \tag{6.49}$$

regardless of whether $|\alpha| = 0$ or 1. To estimate this $\mathcal{I}_\lambda D u$ term we apply Lemmas A.4 and A.13 to see that

$$\|\mathcal{I}_\lambda D u\|_2^2 \lesssim \sum_{k=1}^2 \left\|\mathcal{I}_\lambda D \nabla^k u\right\|_0^2 \lesssim \sum_{k=1}^2 \left(\left\|\nabla^k u\right\|_0^2\right)^\lambda \left(\left\|D\nabla^k u\right\|_0^2\right)^{1-\lambda} \lesssim \|u\|_3^2. \tag{6.50}$$

By chaining together the bounds (6.48)–(6.50) and employing the G^i estimates of Proposition 4.3, we deduce that

$$\|\partial^\alpha \mathcal{I}_\lambda p\|_{H^0(\Sigma)}^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \eta\|_0^2 + \|u\|_3^2 + \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}. \tag{6.51}$$

Now we estimate $\partial_3 \partial^\alpha \mathcal{I}_\lambda p$ by using the first equation in (2.23) to bound

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_3 p\|_0^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \partial_t u_3\|_0^2 + \left\|\partial^\alpha \mathcal{I}_\lambda D^2 u\right\|_0^2 + \left\|\partial^\alpha \mathcal{I}_\lambda \partial_3^2 u_3\right\|_0^2 + \left\|\partial^\alpha \mathcal{I}_\lambda G^1\right\|_0^2. \tag{6.52}$$

When $|\alpha| = 1$ we can use Lemma A.4 to see that

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_t u_3\|_0^2 \lesssim \|\mathcal{I}_\lambda D \partial_t u_3\|_0^2 \lesssim \left(\|\partial_t u_3\|_0^2\right)^\lambda \left(\|D \partial_t u_3\|_0^2\right)^{1-\lambda} \le \|\partial_t u\|_1^2. \tag{6.53}$$

When $|\alpha| = 0$ we cannot use Lemma A.4 directly, so we first use Poincaré's inequality and the divergence equation in (2.23), and then use Lemma A.4:

$$\|\mathcal{I}_\lambda \partial_t u_3\|_0^2 \lesssim \|\partial_3 \mathcal{I}_\lambda \partial_t u_3\|_0^2 = \|\mathcal{I}_\lambda \partial_t \partial_3 u_3\|_0^2 \lesssim \left\|\mathcal{I}_\lambda \partial_t G^2\right\|_0^2 + \|\mathcal{I}_\lambda D \partial_t u\|_0^2 \lesssim \left\|\mathcal{I}_\lambda \partial_t G^2\right\|_0^2 + \|\partial_t u\|_1^2. \tag{6.54}$$

Then (6.53) and (6.54) imply that, regardless of whether $|\alpha| = 0$ or 1, we may bound

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_t u_3\|_0^2 \lesssim \left\|\mathcal{I}_\lambda \partial_t G^2\right\|_0^2 + \|\partial_t u\|_1^2 . \tag{6.55}$$

The term $\partial^\alpha \mathcal{I}_\lambda D^2 u$ may be estimated as in (6.50):

$$\left\|\partial^\alpha \mathcal{I}_\lambda D^2 u\right\|_0^2 \lesssim \|u\|_3^2 . \tag{6.56}$$

To estimate the term $\partial^\alpha \mathcal{I}_\lambda \partial_3^2 u_3$, we again use the divergence equation to bound

$$\left\|\partial^\alpha \mathcal{I}_\lambda \partial_3^2 u_3\right\|_0^2 \lesssim \left\|\partial^\alpha \mathcal{I}_\lambda \partial_3 G^2\right\|_0^2 + \|\partial^\alpha \mathcal{I}_\lambda \partial_3 D u\|_0^2 \lesssim \left\|\partial^\alpha \mathcal{I}_\lambda \partial_3 G^2\right\|_0^2 + \|u\|_3^2 , \tag{6.57}$$

where in the second inequality we have again argued as in (6.50). Then (6.52) and (6.55)–(6.57), together with Proposition 4.3, imply that

$$\|\partial^\alpha \mathcal{I}_\lambda \partial_3 p\|_0^2 \lesssim \|u\|_3^2 + \|\partial_t u\|_1^2 + \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}. \tag{6.58}$$

The estimates (6.51) and (6.58) may be combined with (6.47) to show that

$$\|\partial^\alpha \mathcal{I}_\lambda p\|_0^2 \lesssim \|\partial^\alpha \mathcal{I}_\lambda \eta\|_0^2 + \|u\|_3^2 + \|\partial_t u\|_1^2 + \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}. \tag{6.59}$$

When $|\alpha| = 0$ we bound the first three terms on the right side of (6.59) by $\mathcal{E}_{2N}$ and use the fact that $\mathcal{E}_{2N}^2 \le \mathcal{E}_{2N} \le 1$ to deduce (6.45). When $|\alpha| = 1$, we first use Lemma A.7 to bound

$$\|\partial^\alpha \mathcal{I}_\lambda \eta\|_0^2 \le \|D \mathcal{I}_\lambda \eta\|_0^2 \lesssim \left(\|\mathcal{I}_\lambda \eta\|_0^2\right)^{\lambda/(1+\lambda)} \left(\|D\eta\|_0^2\right)^{1/(1+\lambda)} \lesssim (\mathcal{E}_{2N})^{\lambda/(1+\lambda)} (\mathcal{D}_{2N})^{1/(1+\lambda)} . \tag{6.60}$$

Then we use the fact that $\mathcal{E}_{2N} \le 1$ to bound

$$\begin{aligned} \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\} &\le (\min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\})^{\lambda/(1+\lambda)} (\min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\})^{1/(1+\lambda)} \\ &\le (\mathcal{E}_{2N})^{\lambda/(1+\lambda)} (\mathcal{D}_{2N})^{1/(1+\lambda)} . \end{aligned} \tag{6.61}$$

Similarly, since $\|u\|_3^2 + \|\partial_t u\|_1^2 \le \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}$, we have

$$\|u\|_3^2 + \|\partial_t u\|_1^2 \le (\mathcal{E}_{2N})^{\lambda/(1+\lambda)} (\mathcal{D}_{2N})^{1/(1+\lambda)} . \tag{6.62}$$

We then combine (6.59) with (6.60)–(6.62) to deduce (6.46). □

Our next lemma provides a bound for the integral of the product $\mathcal{I}_\lambda p \mathcal{I}_\lambda G^2$. The estimate is essential to analyzing the energy evolution of $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$.

Lemma 6.6. *It holds that*

$$\left| \int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda G^2 \right| \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{2N}. \tag{6.63}$$

Proof. We begin by writing

$$\int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda G^2 = I + II \tag{6.64}$$

for

$$I := \int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda [(AK)\partial_3 u_1 + (BK)\partial_3 u_2], \text{ and } II := \int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda (1-K)\partial_3 u_3. \tag{6.65}$$

The term I is straightforward to estimate because of the bounds (4.21) of Lemma 4.4 and (6.45) of Lemma 6.5:

$$|I| \le \|\mathcal{I}_\lambda p\|_0 \|\mathcal{I}_\lambda [(AK)\partial_3 u_1 + (BK)\partial_3 u_2]\| \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{2N}. \tag{6.66}$$

To estimate the term II, we must first use the divergence equation in (2.23) to rewrite

$$(1-K)\partial_3 u_3 = (1-K)[G^2 - \partial_1 u_1 - \partial_2 u_2] \tag{6.67}$$

so that

$$II = \int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda [(1-K)G^2] - \int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda [(1-K)(\partial_1 u_1 + \partial_2 u_2)] := II_1 + II_2. \tag{6.68}$$

For the term II_1 we use the estimates (6.45) of Lemma 6.5 and (4.23) of Lemma 4.4 to bound

$$|II_1| \le \|\mathcal{I}_\lambda p\|_0 \left\|\mathcal{I}_\lambda[(1-K)G^2]\right\|_0 \lesssim \sqrt{\mathcal{E}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}^2} = \mathcal{E}_{2N}\mathcal{D}_{2N}. \tag{6.69}$$

In order to control the term II_2 we first integrate by parts:

$$II_2 = \int_\Omega \mathcal{I}_\lambda \partial_1 p \mathcal{I}_\lambda[(1-K)u_1] + \mathcal{I}_\lambda \partial_2 p \mathcal{I}_\lambda[(1-K)u_2] - \mathcal{I}_\lambda p \mathcal{I}_\lambda[u_1\partial_1 K + u_2 \partial_2 K]. \tag{6.70}$$

Then we use Lemmas 6.5 and 4.4 to estimate

$$\begin{aligned} |II_2| &\le \|\mathcal{I}_\lambda Dp\|_0 \|\mathcal{I}_\lambda[(1-K)u]\|_0 + \|\mathcal{I}_\lambda p\|_0 \sum_{i=1}^2 \|\mathcal{I}_\lambda[u\partial_i K]\|_0^2 \\ &\lesssim \sqrt{(\mathcal{E}_{2N})^{\lambda/(1+\lambda)}(\mathcal{D}_{2N})^{1/(1+\lambda)}}\sqrt{(\mathcal{E}_{2N})^{1/(1+\lambda)}(\mathcal{D}_{2N})^{(1+2\lambda)/(1+\lambda)}} + \sqrt{\mathcal{E}_{2N}}\sqrt{\mathcal{D}_{2N}^2} \\ &= \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \end{aligned} \tag{6.71}$$

Since $\mathcal{E}_{2N} \le 1$, we can combine (6.69) and (6.71) to find that $|II| \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}$, which yields (6.63) when combined with (6.66). □

With these two lemmas in hand, we can now estimate how the energies of $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$ evolve.

Proposition 6.7. *It holds that*

$$\partial_t \left(\frac{1}{2}\int_\Omega |\mathcal{I}_\lambda u|^2 + \frac{1}{2}\int_\Sigma |\mathcal{I}_\lambda \eta|^2 \right) + \frac{1}{2}\int_\Omega |\mathbb{D}\mathcal{I}_\lambda u|^2 \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{6.72}$$

In particular,

$$\frac{1}{2}\int_\Omega |\mathcal{I}_\lambda u|^2 + \frac{1}{2}\int_\Sigma |\mathcal{I}_\lambda \eta|^2 + \frac{1}{2}\int_0^t\int_\Omega |\mathbb{D}\mathcal{I}_\lambda u|^2 \lesssim \mathcal{E}_{2N}(0) + \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{6.73}$$

Proof. We apply $\mathcal{I}_\lambda$ to the equations (2.23) and then use Lemma 2.3 to see that

$$\begin{aligned} \partial_t \left(\frac{1}{2}\int_\Omega |\mathcal{I}_\lambda u|^2 + \frac{1}{2}\int_\Sigma |\mathcal{I}_\lambda \eta|^2 \right) + \frac{1}{2}\int_\Omega |\mathbb{D}\mathcal{I}_\lambda u|^2 &= \int_\Omega \mathcal{I}_\lambda u \cdot \mathcal{I}_\lambda G^1 + \mathcal{I}_\lambda p \mathcal{I}_\lambda G^2 \\ &+ \int_\Sigma -\mathcal{I}_\lambda u \cdot \mathcal{I}_\lambda G^3 + \mathcal{I}_\lambda \eta \mathcal{I}_\lambda G^4. \end{aligned} \tag{6.74}$$

We will estimate each term on the right side of the equation. First we use trace theory and (4.15) and (4.16) of Lemma 4.3 to bound the first and third terms:

$$\begin{aligned} \left|\int_\Omega \mathcal{I}_\lambda u \cdot \mathcal{I}_\lambda G^1\right| + \left|\int_\Sigma \mathcal{I}_\lambda u \cdot \mathcal{I}_\lambda G^3\right| &\le \|\mathcal{I}_\lambda u\|_0 \left\|\mathcal{I}_\lambda G^1\right\|_0 + \|\mathcal{I}_\lambda u\|_1 \left\|\mathcal{I}_\lambda G^3\right\|_0 \\ &\lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} = \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \end{aligned} \tag{6.75}$$

For the third term we use Lemma 6.6 for

$$\left|\int_\Omega \mathcal{I}_\lambda p \mathcal{I}_\lambda G^2\right| \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{6.76}$$

Finally, for the fourth term we use (4.17) of Lemma 4.3:

$$\int_\Sigma \mathcal{I}_\lambda \eta \mathcal{I}_\lambda G^4 \le \|\mathcal{I}_\lambda \eta\|_0 \left\|\mathcal{I}_\lambda G^4\right\|_0 \lesssim \sqrt{\mathcal{E}_{2N}}\sqrt{\mathcal{D}_{2N}^2} = \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{6.77}$$

The bound (6.72) follows by combining (6.74)–(6.77), and then (6.73) follows from (6.72) by integrating in time from 0 to t. □

7. Energy evolution estimates

We now assemble the estimates of the previous two sections into an estimate for the evolution of $\bar{\mathcal{E}}_{2N}$ and $\bar{\mathcal{D}}_{2N}$.

Theorem 7.1. *There exists a $\theta > 0$ so that*

$$\bar{\mathcal{E}}_{2N}(t) + \int_0^t \bar{\mathcal{D}}_{2N}(r)dr \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t (\mathcal{E}_{2N}(r))^{\theta} \mathcal{D}_{2N}(r)dr + \int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}dr. \tag{7.1}$$

Proof. The result follows by summing the estimates of Propositions 5.3, 5.5, 6.2, and 6.7 and recalling the definition of $\bar{\mathcal{E}}_{2N}$ and $\bar{\mathcal{D}}_{2N}$ given by (2.48) and (2.49), respectively. □

We can also assemble the estimates of the previous two sections into a similar estimate for the evolution of $\bar{\mathcal{E}}_{N+2,m}$ and $\bar{\mathcal{D}}_{N+2,m}$.

Theorem 7.2. *Let F^2 be given by (2.19) with $\partial^\alpha = \partial_t^{N+2}$. There exists a $\theta > 0$ so that*

$$\partial_t \left(\bar{\mathcal{E}}_{N+2,m} - 2\int_\Omega J \partial_t^{N+1} p F^2 \right) + \bar{\mathcal{D}}_{N+2,m} \lesssim \mathcal{E}_{2N}^{\theta} \mathcal{D}_{N+2,m}. \tag{7.2}$$

Proof. The result follows by summing the estimates of Propositions 5.4 and 6.4 and recalling the definition of $\bar{\mathcal{E}}_{N+2,m}$ and $\bar{\mathcal{D}}_{N+2,m}$ given by (2.45) and (2.47), respectively. □

8. Comparison results

We now prove a pair of estimates that compare the full dissipation and energy to the horizontal dissipation and energy. We will show that, up to some error terms, the instantaneous energy $\mathcal{E}_{2N}$ is comparable to the horizontal energy $\bar{\mathcal{E}}_{2N}$ and that the dissipation rate $\mathcal{D}_{2N}$ is comparable to the horizontal dissipation rate $\bar{\mathcal{D}}_{2N}$. We will also prove similar results for $\bar{\mathcal{E}}_{N+2,m}$ and $\bar{\mathcal{D}}_{N+2,m}$. To prove results for both $2N$ and $N+2$, we will first prove general estimates involving $\mathcal{D}_n$ and $\mathcal{E}_n$, and then we will specialize to the cases $n = N+2$ and $n = 2N$. The dissipation estimates are more involved, so we begin with them.

8.1. Dissipation.

We first consider the dissipation rate.

Theorem 8.1. *Let $m \in \{1, 2\}$ and*

$$\mathcal{Y}_{n,m} := \left\|\bar{\nabla}_m^{2n-1} G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{2n-1} G^2\right\|_1^2 + \left\|\bar{D}_m^{2n-1} G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{2n-1} G^4\right\|_{1/2}^2 + \left\|\bar{D}_0^{2n-2} \partial_t G^4\right\|_{1/2}^2. \tag{8.1}$$

If $m = 1$, then

$$\left\|\nabla^3 u\right\|_{2n-2}^2 + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|_{2n-2j+1}^2 + \left\|\nabla^2 p\right\|_{2n-2}^2 + \sum_{j=1}^{n-1} \left\|\partial_t^j p\right\|_{2n-2j} + \left\|D^2 \eta\right\|_{2n-5/2}^2 + \left\|\partial_t \eta\right\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \left\|\partial_t^j \eta\right\|_{2n-2j+5/2}^2 \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m}. \tag{8.2}$$

If $m = 2$, then

$$\left\|\nabla^4 u\right\|_{2n-3}^2 + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|_{2n-2j+1}^2 + \left\|\nabla^3 p\right\|_{2n-3}^2 + \left\|\partial_t \nabla p\right\|_{2n-3}^2 + \sum_{j=2}^{n-1} \left\|\partial_t^j p\right\|_{2n-2j}^2 + \left\|D^3 \eta\right\|_{2n-7/2}^2 + \left\|D\partial_t \eta\right\|_{2n-3/2}^2 + \sum_{j=2}^{n+1} \left\|\partial_t^j \eta\right\|_{2n-2j+5/2}^2 \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m}. \tag{8.3}$$

Proof. In this proof we must use a separate counting for spatial and temporal derivatives, so unlike elsewhere in the paper, we now only use $\alpha \in \mathbb{N}^2$ to refer to spatial derivatives. In order to compactly write our estimates, throughout the proof we write

$$\mathcal{Z} := \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m}. \tag{8.4}$$

The proof is divided into several steps.

Step 1 – Application of Korn's inequality

Since any horizontal or temporal derivative of u vanishes on the lower boundary Σ_b, we may apply Lemma A.12 to derive the bound

$$\left\| \bar{D}_m^{2n} u \right\|_1^2 \lesssim \left\| \bar{D}_m^{2n} \mathbb{D} u \right\|_0^2 = \bar{\mathcal{D}}_{n,m}. \tag{8.5}$$

This $H^1(\Omega)$ bound will be more useful in what follows than an $H^0(\Omega)$ estimate of the symmetric gradient.

Step 2 – Initial estimates of the pressure and improvement of u estimates

Let $0 \le j \le n-1$ and $\alpha \in \mathbb{N}^2$ be such that

$$m \le 2j + |\alpha| \le 2n - 1. \tag{8.6}$$

Note that if $2j + |\alpha| = 2n-1$, then the condition $j \le n-1$ implies that $|\alpha| \ge 1$. This means that we are free to use (8.5) to bound

$$\left\| \partial^\alpha \partial_t^{j+1} u \right\|_0^2 \le \left\| \bar{D}_m^{2n} u \right\|_1^2 \lesssim \mathcal{Z}. \tag{8.7}$$

In order to extract further information, we apply the operator $\partial_t^j \partial^\alpha$ to the first two equations in (2.23) to find that

$$\partial^\alpha \partial_t^{j+1} u - \Delta \partial^\alpha \partial_t^j u + \nabla \partial^\alpha \partial_t^j p = \partial^\alpha \partial_t^j G^1 \tag{8.8}$$

$$\operatorname{div} \partial^\alpha \partial_t^j u = \partial^\alpha \partial_t^j G^2. \tag{8.9}$$

Because of the constraints on j, α given by (8.6) we may control

$$\left\| \partial^\alpha \partial_t^j G^1 \right\|_0^2 + \left\| \partial^\alpha \partial_t^j G^2 \right\|_1^2 \le \left\| \bar{D}_m^{2n-1} G^1 \right\|_0^2 + \left\| \bar{D}_m^{2n-1} G^2 \right\|_1^2 \le \mathcal{Z}. \tag{8.10}$$

We will utilize the structure of (8.8)–(8.9) in conjunction with (8.7) and (8.10) in order to improve our estimates.

We begin by utilizing (8.9) to control one of the terms in the third component of (8.8). We have

$$\partial^\alpha \partial_t^j (\partial_3 u_3) = \partial^\alpha \partial_t^j (-\partial_1 u_1 - \partial_2 u_2 + G^2) \tag{8.11}$$

so that (8.5) and (8.10) imply

$$\left\| \partial_3^2 \partial^\alpha \partial_t^j u_3 \right\|_0^2 \lesssim \left\| \bar{D}_m^{2n} u \right\|_1^2 + \left\| \bar{D}_m^{2n-1} G^2 \right\|_1^2 \lesssim \mathcal{Z}. \tag{8.12}$$

A further application of (8.5) to control $(\partial_1^2 + \partial_2^2) \partial^\alpha \partial_t^j u_3$ then provides the estimate

$$\left\| \Delta \partial^\alpha \partial_t^j u_3 \right\|_0^2 \lesssim \mathcal{Z}. \tag{8.13}$$

Applying the bounds (8.7), (8.10), and (8.13) to the third component of (8.8), we arrive at a partial bound for the pressure:

$$\left\| \partial_3 \partial^\alpha \partial_t^j p \right\|_0^2 \lesssim \mathcal{Z}. \tag{8.14}$$

It remains to control the terms $\partial_i \partial^\alpha \partial_t^j p$ and $\partial_3^2 \partial^\alpha \partial_t^j u_i$ for $i = 1, 2$. To accomplish this, we employ an elliptic estimate of $\operatorname{curl} u := \omega$. Taking the curl of (8.8) eliminates the pressure gradient and yields

$$\partial^\alpha \partial_t^{j+1} \omega = \Delta \partial^\alpha \partial_t^j \omega + \operatorname{curl}(\partial^\alpha \partial_t^j G^1). \tag{8.15}$$

We only need the first two components $\omega_1 = \partial_2 u_3 - \partial_3 u_2$, $\omega_2 = \partial_3 u_1 - \partial_1 u_3$, for which we use the Σ boundary condition (2.23)

$$\partial_i u_3 + \partial_3 u_i = \mathbb{D} u e_3 \cdot e_i = -G^3 \cdot e_i \text{ for } i = 1, 2 \tag{8.16}$$

to derive the boundary conditions

$$\begin{cases} \omega_1 = 2\partial_2 u_3 + G^3 \cdot e_2 & \text{on } \Sigma \\ \omega_2 = -2\partial_1 u_3 - G^3 \cdot e_1 & \text{on } \Sigma. \end{cases} \tag{8.17}$$

No similar boundary condition is available on Σ_b, so we must resort to a localization using a cutoff function $\chi = \chi(x_3)$ given by $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(x_3) = 1$ for $x_3 \in \Omega_1 := [-2b/3, 0]$ and $\chi(x_3) = 0$ for $x_3 \notin (-3b/4, 1/2)$.

The functions $\chi\omega_i$, $i = 1, 2$, satisfy

$$\Delta \partial^\alpha \partial_t^j (\chi\omega_i) = \chi(\partial^\alpha \partial_t^{j+1} \omega_i) + 2(\partial_3 \chi)(\partial_3 \partial^\alpha \partial_t^j \omega_i) + (\partial_3^2 \chi)(\partial^\alpha \partial_t^j \omega_i) - \chi \operatorname{curl}(\partial^\alpha \partial_t^j G^1) \tag{8.18}$$

in Ω as well as the boundary conditions

$$\begin{cases} \partial^\alpha \partial_t^j (\chi\omega_1) = 2\partial_2 \partial^\alpha \partial_t^j u_3 + \partial^\alpha \partial_t^j G^3 \cdot e_2 & \text{on } \Sigma \\ \partial^\alpha \partial_t^j (\chi\omega_2) = -2\partial_1 \partial^\alpha \partial_t^j u_3 - \partial^\alpha \partial_t^j G^3 \cdot e_1 & \text{on } \Sigma \\ \partial^\alpha \partial_t^j (\chi\omega_1) = \partial^\alpha \partial_t^j (\chi\omega_2) = 0 & \text{on } \Sigma_b. \end{cases} \tag{8.19}$$

In order to employ an elliptic estimate of $\partial^\alpha \partial_t^j (\chi\omega_i)$ we must first prove two auxiliary estimates.

First we derive an estimate of the $H^{-1}(\Omega) = (H_0^1(\Omega))^*$ norm of each term on the right side of equation (8.18). Let $\varphi \in H_0^1(\Omega)$. When $\alpha \neq 0$ we may write $\alpha = \beta + (\alpha - \beta)$ with $|\beta| = 1$ and integrate by parts to bound

$$\left| \int_\Omega \varphi \chi \partial^\alpha \partial_t^{j+1} \omega_i \right| = \left| \int_\Omega \partial^\beta \varphi \chi \partial^{\alpha-\beta} \partial_t^{j+1} \omega_i \right| \le \|\varphi\|_1 \left\| \chi \bar{D}_m^{2n} \omega_i \right\|_0 \tag{8.20}$$

since $2(j+1) + |\alpha - \beta| = 2j + |\alpha| + 1 \in [m+1, 2n]$. We may use (8.5) for

$$\left\| \chi \bar{D}_m^{2n} \omega_i \right\|_0^2 \lesssim \left\| \bar{D}_m^{2n} u \right\|_1^2 \lesssim \mathcal{Z}. \tag{8.21}$$

Chaining these inequalities together when $\alpha \neq 0$ and taking the supremum over all φ such that $\|\varphi\|_1 \le 1$, we get

$$\left\| \partial^\alpha \partial_t^{j+1} \omega_i \right\|_{H^{-1}}^2 \lesssim \mathcal{Z}. \tag{8.22}$$

A similar argument without an integration by parts shows that (8.22) is also true when $\alpha = 0$ since in this case the condition $j \le n - 1$ implies that $m + 2 \le 2(j+1) \le 2n$. Similarly integrating by parts with ∂_3 in the dual-pairing, we may estimate the second term on the right side of (8.18):

$$\left\| 2(\partial_3 \chi)(\partial_3 \partial^\alpha \partial_t^j \omega_i) \right\|_{H^{-1}}^2 \lesssim (\|\partial_3 \chi\|_{L^\infty}^2 + \left\|\partial_3^2 \chi\right\|_{L^\infty}^2) \left\| \bar{D}_m^{2n} \omega_i \right\|_0^2 \lesssim \left\| \bar{D}_m^{2n} u \right\|_1^2 \lesssim \mathcal{Z}. \tag{8.23}$$

The third term may be estimated without integration by parts in the dual-pairing:

$$\left\| (\partial_3^2 \chi)(\partial^\alpha \partial_t^j \omega_i) \right\|_{H^{-1}}^2 \lesssim \left\| \partial_3^2 \chi \right\|_{L^\infty}^2 \left\| \bar{D}_m^{2n} \omega_i \right\|_0^2 \lesssim \left\| \bar{D}_m^{2n} u \right\|_1^2 \lesssim \mathcal{Z}. \tag{8.24}$$

The fourth term is estimated by integrating by parts with the curl operator and using (8.10):

$$\left\| \chi \operatorname{curl}(\partial^\alpha \partial_t^j G^1) \right\|_{H^{-1}}^2 \lesssim (\|\chi\|_{L^\infty}^2 + \|\partial_3 \chi\|_{L^\infty}^2) \left\| \bar{D}_m^{2n-1} G^1 \right\|_0^2 \lesssim \mathcal{Z}. \tag{8.25}$$

Combining these four estimates of the right hand side of (8.18) yields

$$\left\| \Delta \partial^\alpha \partial_t^j (\chi\omega_i) \right\|_{H^{-1}}^2 \lesssim \mathcal{Z} \text{ for } i = 1, 2. \tag{8.26}$$

Next, to complete the elliptic estimate of $\partial^\alpha \partial_t^j(\chi\omega_i)$, we also need $H^{1/2}(\Sigma)$ estimates for the boundary terms on the right side of the first two equations in (8.19). We may estimate the $\partial_i u_3$, $i = 1, 2$, terms with the embedding $H^1(\Omega) \hookrightarrow H^{1/2}(\Sigma)$:

$$\left\| \partial^\alpha \partial_t^j \partial_1 u_3 \right\|^2_{H^{1/2}(\Sigma)} + \left\| \partial^\alpha \partial_t^j \partial_2 u_3 \right\|^2_{H^{1/2}(\Sigma)} \lesssim \left\| \bar{D}^{2n}_m u \right\|_1^2 \lesssim \mathcal{Z}. \tag{8.27}$$

On the other hand, estimates of G^3 are already built into $\mathcal{Z}$:

$$\left\| \partial^\alpha \partial_t^j G^3 \right\|^2_{1/2} \le \left\| \bar{D}^{2n-1}_m G^3 \right\|^2_{1/2} \le \mathcal{Y}_{n,m} \le \mathcal{Z}. \tag{8.28}$$

Since $\chi\omega_i = 0$ on Σ_b for $i = 1, 2$ we then deduce that

$$\left\| \partial^\alpha \partial_t^j (\chi\omega_i) \right\|^2_{H^{1/2}(\partial\Omega)} \lesssim \mathcal{Z} \text{ for } i = 1, 2. \tag{8.29}$$

Now according to (8.26), (8.29), standard elliptic estimates, and the fact that $\chi = 1$ on $\Omega_1 = [-2b/3, 0]$ we have

$$\left\| \partial^\alpha \partial_t^j \omega_i \right\|^2_{H^1(\Omega_1)} \lesssim \left\| \partial^\alpha \partial_t^j (\chi\omega_i) \right\|^2_1 \lesssim \mathcal{Z} \text{ for } i = 1, 2. \tag{8.30}$$

We may then rewrite

$$\partial_3^2 \partial^\alpha \partial_t^j u_1 = \partial_3 \partial^\alpha \partial_t^j (\omega_2 + \partial_1 u_3) \text{ and } \partial_3^2 \partial^\alpha \partial_t^j u_2 = \partial_3 \partial^\alpha \partial_t^j (\partial_2 u_3 - \omega_1) \tag{8.31}$$

and deduce from (8.30) and (8.5) that for $i = 1, 2$ we have

$$\left\| \partial_3^2 \partial^\alpha \partial_t^j u_i \right\|^2_{H^0(\Omega_1)} \lesssim \left\| \bar{D}^{2n}_m u_3 \right\|^2_1 + \sum_{k=1}^2 \left\| \partial^\alpha \partial_t^j \omega_k \right\|^2_{H^1(\Omega_1)} \lesssim \mathcal{Z}. \tag{8.32}$$

We then apply this estimate along with (8.5) and (8.10) to the first two components of equation (8.8) to find that

$$\left\| \partial_i \partial^\alpha \partial_t^j p \right\|^2_{H^0(\Omega_1)} \lesssim \mathcal{Z} \text{ for } i = 1, 2. \tag{8.33}$$

Now we sum the estimates (8.5), (8.12), (8.14), (8.32), and (8.33) over all $j \le n-1$ and $\alpha \in \mathbb{N}^2$ with $m \le 2j + |\alpha| \le 2n-1$ to deduce that

$$\left\| \bar{D}^{2n-1}_m u \right\|^2_{H^2(\Omega_1)} + \left\| \bar{D}^{2n-1}_m \nabla p \right\|^2_{H^0(\Omega_1)} \lesssim \mathcal{Z}. \tag{8.34}$$

Step 3 – Bootstrapping, η estimates, and improved pressure estimates

Now we make use of Lemma 8.2 to bootstrap from (8.34) to

$$\begin{aligned} \left\| \nabla^{2+m} u \right\|^2_{H^{2n-m-1}(\Omega_1)} + \left\| D^m u \right\|^2_{H^{2n-m+1}(\Omega_1)} + \sum_{j=1}^n \left\| \partial_t^j u \right\|^2_{H^{2n-2j+1}(\Omega_1)} \\ + \left\| \nabla^{1+m} p \right\|^2_{H^{2n-m-1}(\Omega_1)} + \sum_{j=1}^{n-1} \left\| \partial_t^j \nabla p \right\|^2_{H^{2n-2j-1}(\Omega_1)} \lesssim \mathcal{Z}. \end{aligned} \tag{8.35}$$

With this estimate in hand, we may derive some estimates for η on Σ by employing the boundary conditions of (2.23):

$$\eta = p - 2\partial_3 u_3 - G^3_3, \tag{8.36}$$

$$\partial_t \eta = u_3 + G^4. \tag{8.37}$$

Then (8.35) allows us to differentiate (8.36) to find that

$$\begin{aligned} \left\| D^{1+m} \eta \right\|^2_{2n-m-3/2} \lesssim \left\| D^{1+m} p \right\|^2_{H^{2n-m-3/2}(\Sigma)} + \left\| D^{1+m} \partial_3 u_3 \right\|^2_{H^{2n-m-3/2}(\Sigma)} \\ + \left\| D^{1+m} G^3 \right\|^2_{2n-m-3/2} \lesssim \left\| \nabla^{1+m} p \right\|^2_{H^{2n-m-1}(\Omega_1)} + \left\| \nabla^{2+m} u \right\|^2_{H^{2n-m-1}(\Omega_1)} \\ + \left\| G^3 \right\|^2_{2n-1/2} \lesssim \mathcal{Z}. \end{aligned} \tag{8.38}$$

Similarly, for $j = 2, \dots, n+1$ we may apply ∂_t^{j-1} to (8.37) and estimate

$$\left\|\partial_t^j \eta\right\|_{2n-2j+5/2}^2 \lesssim \left\|\partial_t^{j-1} u_3\right\|_{H^{2n-2j+5/2}(\Sigma)}^2 + \left\|\partial_t^{j-1} G^4\right\|_{2n-2j+5/2}^2 \lesssim \left\|\partial_t^{j-1} u\right\|_{H^{2n-2(j-1)+1}(\Omega_1)}^2 + \left\|\partial_t^{j-1} G^4\right\|_{2n-2(j-1)+1/2}^2 \lesssim \mathcal{Z}. \tag{8.39}$$

It remains only to consider $\partial_t \eta$; in this case we must consider $m = 1$ and $m = 2$ separately. For $m = 1$, we again use (8.37) to see that

$$\|\partial_t \eta\|_{2n-1/2}^2 \lesssim \|u_3\|_{H^{2n-1/2}(\Sigma)}^2 + \left\|G^4\right\|_{2n-1/2}^2 \lesssim \|u_3\|_{H^{2n-1/2}(\Sigma)}^2 + \mathcal{Z}, \tag{8.40}$$

but now we use Lemma A.11, trace theory, and the second equation in (2.23) for the estimate

$$\|u_3\|_{H^{2n-1/2}(\Sigma)}^2 \lesssim \|u_3\|_{H^0(\Sigma)}^2 + \|Du_3\|_{H^{2n-3/2}(\Sigma)}^2 \lesssim \|\partial_3 u_3\|_{H^0(\Omega)}^2 + \|Du_3\|_{H^{2n-1}(\Omega_1)}^2 \lesssim \left\|G^2\right\|_0^2 + \|Du\|_0^2 + \|Du\|_{H^{2n-1}(\Omega_1)}^2 \lesssim \mathcal{Z} \tag{8.41}$$

by (8.10) and (8.35). Chaining (8.40)–(8.41) together implies that

$$\|\partial_t \eta\|_{2n-1/2}^2 \lesssim \mathcal{Z} \text{ when } m = 1. \tag{8.42}$$

For $m = 2$, we differentiate (8.37) for the bound

$$\|D\partial_t \eta\|_{2n-3/2}^2 \lesssim \|Du_3\|_{H^{2n-3/2}(\Sigma)}^2 + \left\|DG^4\right\|_{2n-3/2}^2 \lesssim \|Du_3\|_{H^{2n-3/2}(\Sigma)}^2 + \mathcal{Z}, \tag{8.43}$$

but then the analog of (8.41) is

$$\|Du_3\|_{H^{2n-3/2}(\Sigma)}^2 \lesssim \left\|DG^2\right\|_0^2 + \left\|D^2 u\right\|_0^2 + \left\|D^2 u\right\|_{H^{2n-2}(\Omega_1)}^2 \lesssim \mathcal{Z}. \tag{8.44}$$

Hence

$$\|D\partial_t \eta\|_{2n-3/2}^2 \lesssim \mathcal{Z} \text{ when } m = 2. \tag{8.45}$$

Summing estimates (8.38), (8.39), (8.42), and (8.45) over $j = 0, \dots, n+1$ yields

$$\left\|D^2 \eta\right\|_{2n-5/2}^2 + \|\partial_t \eta\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \left\|\partial_t^j \eta\right\|_{2n-2j+5/2}^2 \lesssim \mathcal{Z} \text{ for } m = 1, \text{ and} \tag{8.46}$$

$$\left\|D^3 \eta\right\|_{2n-7/2}^2 + \|D\partial_t \eta\|_{2n-3/2}^2 + \sum_{j=2}^{n+1} \left\|\partial_t^j \eta\right\|_{n-2j+5/2}^2 \lesssim \mathcal{Z} \text{ for } m = 2. \tag{8.47}$$

The η estimates (8.46)–(8.47) now allow us to further improve the estimates for the pressure. Indeed, for $j = 2, \dots, n-1$ we may use Lemma A.10 and (8.36) to bound

$$\left\|\partial_t^j p\right\|_{H^0(\Omega_1)}^2 \lesssim \left\|\partial_t^j \eta\right\|_{H^0(\Sigma)}^2 + \left\|\partial_3 \partial_t^j u_3\right\|_{H^0(\Sigma)}^2 + \left\|\partial_t^j G^3\right\|_0^2 + \left\|\partial_t^j \nabla p\right\|_{H^0(\Omega_1)}^2 \lesssim \left\|\partial_t^j u_3\right\|_{H^2(\Omega_1)}^2 + \mathcal{Z} \lesssim \mathcal{Z}. \tag{8.48}$$

This, (8.35), and (8.46)–(8.47) allow us to improve (8.35); when $m = 1$ we find that

$$\begin{aligned} &\left\|\nabla^3 u\right\|_{H^{2n-2}(\Omega_1)}^2 + \|Du\|_{H^{2n}(\Omega_1)}^2 + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|_{H^{2n-2j+1}(\Omega_1)}^2 + \left\|\nabla^2 p\right\|_{H^{2n-2}(\Omega_1)}^2 \\ &+ \sum_{j=1}^{n-1} \left\|\partial_t^j p\right\|_{H^{2n-2j}(\Omega_1)}^2 + \left\|D^2 \eta\right\|_{2n-5/2}^2 + \|\partial_t \eta\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \left\|\partial_t^j \eta\right\|_{2n-2j+5/2}^2 \lesssim \mathcal{Z}, \end{aligned} \tag{8.49}$$

and when $m = 2$ we get the estimate

$$\left\|\nabla^4 u\right\|^2_{H^{2n-3}(\Omega_1)} + \left\|D^2 u\right\|^2_{H^{2n-1}(\Omega_1)} + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|^2_{H^{2n-2j+1}(\Omega_1)} + \left\|\nabla^3 p\right\|^2_{H^{2n-3}(\Omega_1)} + \left\|\partial_t \nabla p\right\|^2_{H^{2n-3}(\Omega_1)} + \sum_{j=2}^{n-1} \left\|\partial_t^j p\right\|^2_{H^{2n-2j}(\Omega_1)} + \left\|D^3 \eta\right\|^2_{2n-7/2} + \left\|D\partial_t \eta\right\|^2_{2n-3/2} + \sum_{j=2}^{n+1} \left\|\partial_t^j \eta\right\|^2_{n-2j+5/2} \lesssim \mathcal{Z}. \tag{8.50}$$

Step 4 – Estimates in Ω_2

We now extend our estimates to the lower part of the domain, i.e. $\Omega_2 := [-b, -b/3]$, by applying Lemma 8.3 to deduce that (8.96) holds when $m = 1$ and (8.97) holds when $m = 2$. We will now show that $\mathcal{X}_{n,m}$, defined by (8.95), can be controlled by $\mathcal{Z}$. The key to this is that, by construction, $\operatorname{supp}(\nabla\chi_2) \subset \Omega_1$, which implies that the H^1 and H^2 defined in the lemma satisfy $\operatorname{supp}(H^1) \cup \operatorname{supp}(H^2) \subset \Omega_1$. This allows us to use the estimates (8.49) in the case $m = 1$ and (8.50) in the case $m = 2$ to bound

$$\left\|\bar{D}^{2n-1}_{m+1} H^1\right\|^2_0 + \left\|\bar{D}^{2n-1}_{m+1} H^2\right\|^2_0 \lesssim \mathcal{Z}. \tag{8.51}$$

In order to estimate $\partial_t H^1 \cdot e_i$ for $i = 1, 2$, we note that it does not involve the pressure:

$$\partial_t H^1 \cdot e_i = -(\partial_3 \chi_2)\partial_3 \partial_t u_i - (\partial_3^2 \chi_2)\partial_t u_i. \tag{8.52}$$

Then we may again use (8.49)–(8.50) to see that

$$\sum_{i=1}^{2} \left\|\partial_t H^1 \cdot e_i\right\|^2_{2n-3} \lesssim \mathcal{Z}, \tag{8.53}$$

so that $\mathcal{X}_{n,m} \lesssim \mathcal{Z}$. Replacing in (8.96) and (8.97), we then find that

$$\left\|\nabla^3 u\right\|^2_{H^{2n-2}(\Omega_2)} + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|^2_{H^{2n-2j+1}(\Omega_2)} + \left\|\nabla^2 p\right\|^2_{H^{2n-2}(\Omega_2)} + \sum_{j=1}^{n-1} \left\|\partial_t^j p\right\|^2_{H^{2n-2j}(\Omega_2)} \lesssim \mathcal{Z} \tag{8.54}$$

for $m = 1$, while for $m = 2$

$$\left\|\nabla^4 u\right\|^2_{H^{2n-3}(\Omega_2)} + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|^2_{H^{2n-2j+1}(\Omega_2)} + \left\|\nabla^3 p\right\|^2_{H^{2n-3}(\Omega_2)} + \left\|\partial_t \nabla p\right\|^2_{H^{2n-3}(\Omega_2)} + \sum_{j=2}^{n-1} \left\|\partial_t^j p\right\|^2_{H^{2n-2j}(\Omega_2)} \lesssim \mathcal{Z}. \tag{8.55}$$

Step 5 – Synthesis and conclusion

To conclude, we note that $\Omega = \Omega_1 \cup \Omega_2$, which allows us to add the localized estimates (8.49) and (8.54) to deduce (8.2), and to add (8.50) to (8.55) to deduce (8.3). □

We now present the key bootstrap estimate used in the proof of Theorem 8.1.

Lemma 8.2. *Let $\mathcal{Y}_{n,m}$ and Ω_1 be as defined in Theorem 8.1. Suppose that*

$$\left\|\bar{D}^{2n-2r+1}_m u\right\|^2_{H^{2r}(\Omega_1)} + \left\|\bar{D}^{2n-2r+1}_m \nabla p\right\|^2_{H^{2r-2}(\Omega_1)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} \tag{8.56}$$

for an integer $r \in [1, \dots, n-(m+1)/2]$. *Then*

$$\begin{aligned} \|\partial_t^{n-r} u\|_{H^{2r+1}(\Omega_1)}^2 + \|\partial_t^{n-r} \nabla p\|_{H^{2r-1}(\Omega_1)}^2 \\ + \left\| \bar{D}_m^{2n-2(r+1)+1} u \right\|_{H^{2r+2}(\Omega_1)}^2 + \left\| \bar{D}_m^{2n-2(r+1)+1} \nabla p \right\|_{H^{2r}(\Omega_1)}^2 \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m}. \end{aligned} \tag{8.57}$$

Moreover, if (8.56) *holds with* $r = 1$*, then for* $m = 1, 2$ *we have that*

$$\begin{aligned} \|\nabla^{2+m} u\|_{H^{2n-m-1}(\Omega_1)}^2 + \|D^m u\|_{H^{2n-m+1}(\Omega_1)}^2 + \sum_{j=1}^n \left\| \partial_t^j u \right\|_{H^{2n-2j+1}(\Omega_1)}^2 \\ + \|\nabla^{1+m} p\|_{H^{2n-m-1}(\Omega_1)}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j \nabla p \right\|_{H^{2n-2j-1}(\Omega_1)}^2 \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m}. \end{aligned} \tag{8.58}$$

Proof. Throughout the proof we will write $\mathcal{Z} := \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m}$. Let $\ell \in \{1, 2\}$ and take $0 \le j \le n-1-r$ and $\alpha \in \mathbb{N}^2$ so that $m \le 2j + |\alpha| \le 2n - 2r + 1 - \ell$. We apply the differential operator $\partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j$ to the first equation in (2.23) and split into separate equations for its third and first two components; after some rearrangement, these read

$$\partial_3^{2r-1+\ell} \partial^\alpha \partial_t^j p = -\partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u_3 + \Delta \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j u_3 + \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G_3^1 \tag{8.59}$$

and

$$\Delta \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j u_i = \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u_i + \partial_i \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j p - \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G_i^1 \tag{8.60}$$

for $i = 1, 2$. Notice that the constraints on $r, j, |\alpha|$ imply that $m \le |\alpha| + (2r-2+\ell) + 2j \le 2n-1$, so we may estimate

$$\left\| \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G^1 \right\|_0^2 + \left\| \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G^2 \right\|_1^2 \le \mathcal{Y}_{n,m} \le \mathcal{Z}. \tag{8.61}$$

Since $2r - 2 + \ell \ge 0$, we know that

$$\left\| \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u \right\|_{H^0(\Omega_1)}^2 \le \left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r-2+\ell}(\Omega_1)}^2. \tag{8.62}$$

If $\ell = 2$ then $|\alpha| + 2(j+1) \le 2n - 2r + 1$ so that

$$\left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r-2+\ell}(\Omega_1)}^2 = \left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r}(\Omega_1)}^2 \le \|\bar{D}_m^{2n-2r+1} u\|_{H^{2r}(\Omega_1)}^2 \le \mathcal{Z}. \tag{8.63}$$

On the other hand, if $\ell = 1$, then either $\alpha = 0$, in which case the bound on j implies that $2(j+1) \le 2n - 2r$, and hence

$$\left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r-2+\ell}(\Omega_1)}^2 = \left\| \partial_t^{j+1} u \right\|_{H^{2r-1}(\Omega_1)}^2 \le \|\bar{D}_m^{2n-2r+1} u\|_{H^{2r}(\Omega_1)}^2 \le \mathcal{Z}, \tag{8.64}$$

or else $|\alpha| \ge 1$, and so $\alpha = \beta + (\alpha - \beta)$ for $|\beta| = 1$, which implies that

$$\begin{aligned} \left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r-2+\ell}(\Omega_1)}^2 = \left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r-1}(\Omega_1)}^2 \le \left\| \partial^{\alpha-\beta} \partial_t^{j+1} u \right\|_{H^{2r}(\Omega_1)}^2 \\ \le \|\bar{D}_m^{2n-2r+1} u\|_{H^{2r}(\Omega_1)}^2 \le \mathcal{Z}. \end{aligned} \tag{8.65}$$

Then in either case,

$$\left\| \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u \right\|_{H^0(\Omega_1)}^2 \le \mathcal{Z}. \tag{8.66}$$

We have written the equations (8.59)–(8.60) in this form so as to be able to employ the estimates (8.56), (8.61), (8.66) to derive (8.57). We must consider the case of $\ell = 1$ and $\ell = 2$ separately, starting with $\ell = 1$.

Let $\ell = 1$. According to the equation $\operatorname{div} u = G^2$ (the second of (2.23)) and the bounds (8.56) and (8.61) we may estimate

$$\begin{aligned}(8.67) \quad \left\|\partial_3^{2r+1}\partial^\alpha\partial_t^j u_3\right\|^2_{H^0(\Omega_1)} &= \left\|\partial_3^{2r}\partial^\alpha\partial_t^j (G^2 - \partial_1 u_1 - \partial_2 u_2)\right\|^2_{H^0(\Omega_1)} \\ &\lesssim \left\|\partial_3^{2r-1}\partial^\alpha\partial_t^j G^2\right\|^2_1 + \left\|\partial^\alpha\partial_t^j(\partial_1 u_1 + \partial_2 u_2)\right\|^2_{H^{2r}(\Omega_1)} \lesssim \mathcal{Z},\end{aligned}$$

and hence
(8.68)

$$\left\|\Delta(\partial_3^{2r-1}\partial^\alpha\partial_t^j u_3)\right\|^2_{H^0(\Omega_1)} \lesssim \left\|\partial_3^{2r+1}\partial^\alpha\partial_t^j u_3\right\|^2_{H^0(\Omega_1)} + \left\|\partial_3^{2r-1}(\partial_1^2+\partial_2^2)\partial^\alpha\partial_t^j u_3\right\|^2_{H^0(\Omega_1)} \lesssim \mathcal{Z}.$$

We may then use (8.61), (8.66), and (8.68) in (8.59) for the pressure estimate

$$(8.69) \qquad \left\|\partial_3^{2r}\partial^\alpha\partial_t^j p\right\|^2_{H^0(\Omega_1)} \lesssim \mathcal{Z}.$$

Turning now to the $i = 1, 2$ components, we note that by (8.56)

$$\begin{aligned}(8.70) \quad \left\|\partial_i\partial_3^{2r-1}\partial^\alpha\partial_t^j p\right\|^2_{H^0(\Omega_1)} &+ \left\|(\partial_1^2+\partial_2^2)\partial_3^{2r-1}\partial^\alpha\partial_t^j u_i\right\|^2_{H^0(\Omega_1)} \\ &\lesssim \left\|\bar{D}_m^{2n-2r+1}\nabla p\right\|^2_{H^{2r-2}(\Omega_1)} + \left\|\bar{D}_m^{2n-2r+1} u\right\|^2_{H^{2r}(\Omega_1)} \lesssim \mathcal{Z}\end{aligned}$$

for $i = 1, 2$. Plugging this, (8.61), and (8.66) into (8.60) then shows that

$$(8.71) \qquad \left\|\partial_3^{2r+1}\partial^\alpha\partial_t^j u_i\right\|^2_{H^0(\Omega_1)} \lesssim \mathcal{Z} \text{ for } i = 1, 2.$$

Upon summing (8.67), (8.69), and (8.71) over $0 \le j \le 2n-r-1$ and α satisfying $m \le 2j+|\alpha| \le 2n-2r$, we deduce, in light of (8.56), that

$$(8.72) \qquad \left\|\bar{D}_m^{2n-2r} u\right\|^2_{H^{2r+1}(\Omega_1)} + \left\|\bar{D}_m^{2n-2r}\nabla p\right\|^2_{H^{2r-1}(\Omega_1)} \lesssim \mathcal{Z}.$$

In the case $\ell = 2$ we may argue as in the case $\ell = 1$, utilizing both (8.56) and (8.72) to derive the bound

$$(8.73) \qquad \left\|\bar{D}_m^{2n-2r-1} u\right\|^2_{H^{2r+2}(\Omega_1)} + \left\|\bar{D}_m^{2n-2r-1}\nabla p\right\|^2_{H^{2r}(\Omega_1)} \lesssim \mathcal{Z}.$$

Then we may add (8.72) to (8.73) to deduce (8.57).

Now we turn to the proof of (8.58), assuming that (8.56) holds with $r = 1$. By (8.57) we may iterate with $r = 2$, $r = 3$, etc, until

$$(8.74) \qquad r = \begin{cases} n-1 & \text{if } m = 1 \\ n-2 & \text{if } m = 2 \end{cases} \text{ so that } 2n - 2(r+2) + 1 = \begin{cases} 1 & \text{if } m = 1 \\ 3 & \text{if } m = 2. \end{cases}$$

Summing the resulting bounds yields the estimates

$$(8.75) \quad \left\|D_1^1 u\right\|^2_{H^{2n}(\Omega_1)} + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|^2_{H^{2n-2j+1}(\Omega_1)} + \left\|D_1^1\nabla p\right\|^2_{H^{2n-2}(\Omega_1)} + \sum_{j=1}^{n-1} \left\|\partial_t^j \nabla p\right\|^2_{H^{2n-2j-1}(\Omega_1)} \lesssim \mathcal{Z}$$

in the case $m = 1$ and

$$\begin{aligned}(8.76) \quad \left\|D_2^3 u\right\|^2_{H^{2n-2}(\Omega_1)} &+ \left\|D_0^1\partial_t u\right\|^2_{H^{2n-2}(\Omega_1)} + \sum_{j=2}^{n} \left\|\partial_t^j u\right\|^2_{H^{2n-2j+1}(\Omega_1)} \\ &+ \left\|D_2^3\nabla p\right\|^2_{H^{2n-4}(\Omega_1)} + \left\|D_0^1\partial_t\nabla p\right\|^2_{H^{2n-4}(\Omega_1)} + \sum_{j=2}^{n-1} \left\|\partial_t^j\nabla p\right\|^2_{H^{2n-2j-1}(\Omega_1)} \lesssim \mathcal{Z}\end{aligned}$$

in the case $m = 2$.

As a first step, we improve the estimate (8.76). Let $0 \le j$ and $\alpha \in \mathbb{N}^2$ be such that $2j+|\alpha| = 2$ and apply the operator $\partial_3^{2n-3}\partial^\alpha \partial_t^j$ to the first equation of (2.23) and split into components as above to get

$$\partial_3^{2n-2}\partial^\alpha\partial_t^j p = -\partial_3^{2n-3}\partial^\alpha\partial_t^{j+1}u_3 + \Delta\partial_3^{2n-3}\partial^\alpha\partial_t^j u_3 + \partial_3^{2n-3}\partial^\alpha\partial_t^j G_3^1 \tag{8.77}$$

and

$$\Delta\partial_3^{2n-3}\partial^\alpha\partial_t^j u_i = \partial_3^{2n-3}\partial^\alpha\partial_t^{j+1}u_i + \partial_i\partial_3^{2n-3}\partial^\alpha\partial_t^j p - \partial_3^{2n-3}\partial^\alpha\partial_t^j G_i^1 \tag{8.78}$$

for $i = 1, 2$. We may then argue as above, utilizing (8.76) and (8.56), to deduce the bounds

$$\left\|\partial_3^{2n-1}\partial^\alpha\partial_t^j u_3\right\|^2_{H^0(\Omega_1)} + \left\|\partial_3^{2n-3}\partial^\alpha\partial_t^j u\right\|^2_{H^0(\Omega_1)} \lesssim \mathcal{Z}, \tag{8.79}$$

which, in turn, implies that

$$\left\|\partial_3^{2n-2}\partial^\alpha\partial_t^j p\right\|^2_{H^0(\Omega_1)} + \left\|\partial_3^{2n-1}\partial^\alpha\partial_t^j u_i\right\|^2_{H^0(\Omega_1)} \lesssim \mathcal{Z} \tag{8.80}$$

for $i = 1, 2$. We may then use (8.79)–(8.80) with (8.76) to deduce that

$$\left\|D_2^2 u\right\|^2_{H^{2n-1}(\Omega_1)} + \sum_{j=1}^{n}\left\|\partial_t^j u\right\|^2_{H^{2n-2j+1}(\Omega_1)} + \left\|D_2^2\nabla p\right\|^2_{H^{2n-3}(\Omega_1)} + \sum_{j=1}^{n-1}\left\|\partial_t^j\nabla p\right\|^2_{H^{2n-2j-1}(\Omega_1)} \lesssim \mathcal{Z} \tag{8.81}$$

in the case $m = 2$.

Now we claim that if for $m = 1, 2$ we have the inequality

$$\begin{aligned} \left\|D_m^m u\right\|^2_{H^{2n-m+1}(\Omega_1)} &+ \sum_{j=1}^{n}\left\|\partial_t^j u\right\|^2_{H^{2n-2j+1}(\Omega_1)} \\ &+ \left\|D_m^m\nabla p\right\|^2_{H^{2n-m-1}(\Omega_1)} + \sum_{j=1}^{n-1}\left\|\partial_t^j\nabla p\right\|^2_{H^{2n-2j-1}(\Omega_1)} \lesssim \mathcal{Z}, \end{aligned} \tag{8.82}$$

then the inequality

$$\left\|\nabla^{2+m}u\right\|^2_{H^{2n-m-1}(\Omega_1)} + \left\|\nabla^{1+m}p\right\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathcal{Z} \tag{8.83}$$

also holds, which establishes the desired bound, (8.58), because of our inequalities (8.75) in the case $m = 1$ and (8.81) in the case $m = 2$. We begin the proof of the claim by noting that since $2 \ge m$ we may use (8.82) to bound

$$\left\|\partial_3^m D^2 u\right\|^2_{H^{2n-m-1}(\Omega_1)} + \left\|\partial_3^{m-1}D^2 p\right\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathcal{Z}. \tag{8.84}$$

Now we let $|\alpha| = 1$ and apply $\partial_3^m\partial^\alpha$ to the second equation of (2.23) to find that

$$\left\|\partial_3^{m+1}\partial^\alpha u_3\right\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \left\|\partial_3^m DG^2\right\|^2_{H^{2n-m-1}(\Omega_1)} + \left\|\partial_3^m D^2 u\right\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathcal{Z}. \tag{8.85}$$

Then we apply $\partial_3^{m-1}\partial^\alpha$ to the first equation of (2.23) to bound

$$\begin{aligned} \left\|\partial_3^m\partial^\alpha p\right\|^2_{H^{2n-m-1}(\Omega_1)} &\lesssim \left\|\partial_3^{m+1}\partial^\alpha u_3\right\|^2_{H^{2n-m-1}(\Omega_1)} \\ &+ \left\|\partial_3^{m-1}\partial^\alpha\bar{D}_2^2 u_3\right\|^2_{H^{2n-m-1}(\Omega_1)} + \left\|\partial_3^{m-1}\partial^\alpha G^2\right\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathcal{Z} \end{aligned} \tag{8.86}$$

and

$$\begin{aligned} \left\|\partial_3^{m+1}\partial^\alpha u_i\right\|^2_{H^{2n-m-1}(\Omega_1)} &\lesssim \left\|\partial_3^{m-1}\partial^\alpha\bar{D}_2^2 u\right\|^2_{H^{2n-m-1}(\Omega_1)} \\ &+ \left\|\partial_3^{m-1}\partial^\alpha Dp\right\|^2_{H^{2n-m-1}(\Omega_1)} + \left\|\partial_3^{m-1}\partial^\alpha G^2\right\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathcal{Z} \end{aligned} \tag{8.87}$$

for $i = 1, 2$. Summing (8.85)–(8.87) over all $|\alpha| = 1$ then yields the inequality

$$\left\|\partial_3^{m+1}Du\right\|^2_{H^{2n-m-1}(\Omega_1)} + \left\|\partial_3^m Dp\right\|^2_{H^{2n-m-1}(\Omega_1)} \lesssim \mathcal{Z}. \tag{8.88}$$

Now we use (8.88) to improve to one more ∂_3 and one fewer horizontal derivative. We apply ∂_3^{m+1} to the second equation of (2.23) to find that

$$\left\|\partial_3^{m+2} u_3\right\|_{H^{2n-m-1}(\Omega_1)}^2 \lesssim \left\|\partial_3^{m+1} G^2\right\|_{H^{2n-m-1}(\Omega_1)}^2 + \left\|\partial_3^{m+1} D u\right\|_{H^{2n-m-1}(\Omega_1)}^2 \lesssim \mathcal{Z}. \tag{8.89}$$

Then we apply ∂_3^m to the first equation of (2.23) to bound

$$\begin{aligned}\left\|\partial_3^{m+1} p\right\|_{H^{2n-m-1}(\Omega_1)}^2 \lesssim \left\|\partial_3^{m+2} u_3\right\|_{H^{2n-m-1}(\Omega_1)}^2 \\ + \left\|\partial_3^m \bar{D}_2^2 u_3\right\|_{H^{2n-m-1}(\Omega_1)}^2 + \left\|\partial_3^m G^2\right\|_{H^{2n-m-1}(\Omega_1)}^2 \lesssim \mathcal{Z}\end{aligned} \tag{8.90}$$

and

$$\begin{aligned}\left\|\partial_3^{m+2} u_i\right\|_{H^{2n-m-1}(\Omega_1)}^2 \lesssim \left\|\partial_3^m \bar{D}_2^2 u\right\|_{H^{2n-m-1}(\Omega_1)}^2 \\ + \left\|\partial_3^m D p\right\|_{H^{2n-m-1}(\Omega_1)}^2 + \left\|\partial_3^m G^2\right\|_{H^{2n-m-1}(\Omega_1)}^2 \lesssim \mathcal{Z}\end{aligned} \tag{8.91}$$

for $i = 1, 2$. Summing (8.89)–(8.91) then yields the inequality

$$\left\|\partial_3^{m+2} u\right\|_{H^{2n-m-1}(\Omega_1)}^2 + \left\|\partial_3^{m+1} p\right\|_{H^{2n-m-1}(\Omega_1)}^2 \lesssim \mathcal{Z}. \tag{8.92}$$

Finally, to complete the proof of the claim, we note that

(8.93)
$$\begin{aligned}\left\|\nabla^{2+m} u\right\|_{H^{2n-m-1}(\Omega_1)}^2 + \left\|\nabla^{1+m} p\right\|_{H^{2n-m-1}(\Omega_1)}^2 \lesssim \left\|D_m^m u\right\|_{H^{2n-m+1}(\Omega_1)}^2 + \left\|D_m^m \nabla p\right\|_{H^{2n-m-1}(\Omega_1)}^2 \\ + \sum_{l=1}^{2} \left\|\partial_3^{m+2-\ell} D^\ell u\right\|_{H^{2n-m-1}(\Omega_1)}^2 + \left\|\partial_3^{m+1-\ell} D^\ell p\right\|_{H^{2n-m-1}(\Omega_1)}^2 .\end{aligned}$$

This and the bounds (8.82), (8.84), (8.88), and (8.92) prove the claim. □

The following result allows for control of the dissipation rate in the lower domain.

Lemma 8.3. *Let $\chi_2 \in C_c^\infty(\mathbb{R})$ be such that $\chi_2(x_3) = 1$ for $x_3 \in \Omega_2 := [-b, -b/3]$ and $\chi_2(x_3) = 0$ for $x_3 \notin (-2b, -b/6)$. Let*

$$H^1 = \partial_3 \chi_2 (p e_3 - 2\partial_3 u) - (\partial_3^2 \chi_2) u \text{ and } H^2 = \partial_3 \chi_2 u_3. \tag{8.94}$$

Define

$$\mathcal{X}_{n,m} = \left\|\bar{D}_{m+1}^{2n-1} H^1\right\|_0^2 + \left\|\bar{D}_{m+1}^{2n-1} H^2\right\|_0^2 + \sum_{i=1}^{2} \left\|\partial_t H^1 \cdot e_i\right\|_{2n-3}^2 , \tag{8.95}$$

and let $\mathcal{Y}_{n,m}$ be as defined in Theorem 8.1. If $m = 1$, then

$$\begin{aligned}\left\|\nabla^3 u\right\|_{H^{2n-2}(\Omega_2)}^2 + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|_{H^{2n-2j+1}(\Omega_2)}^2 \\ + \left\|\nabla^2 p\right\|_{H^{2n-2}(\Omega_2)}^2 + \sum_{j=1}^{n-1} \left\|\partial_t^j p\right\|_{H^{2n-2j}(\Omega_2)}^2 \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}.\end{aligned} \tag{8.96}$$

If $m = 2$, then

$$\begin{aligned}\left\|\nabla^4 u\right\|_{H^{2n-3}(\Omega_2)}^2 + \sum_{j=1}^{n} \left\|\partial_t^j u\right\|_{H^{2n-2j+1}(\Omega_2)}^2 + \left\|\nabla^3 p\right\|_{H^{2n-3}(\Omega_2)}^2 \\ + \left\|\partial_t \nabla p\right\|_{H^{2n-3}(\Omega_2)}^2 + \sum_{j=2}^{n-1} \left\|\partial_t^j p\right\|_{H^{2n-2j}(\Omega_2)}^2 \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}.\end{aligned} \tag{8.97}$$

Proof. When we localize with χ_2 we find that $\chi_2 u$ and $\chi_2 p$ solve

$$\begin{cases} -\Delta(\chi_2 u) + \nabla(\chi_2 p) = -\partial_t(\chi_2 u) + \chi_2 G^1 + H^1 & \text{in } \Omega \\ \operatorname{div}(\chi_2 u) = \chi_2 G^2 + H^2 & \text{in } \Omega \\ ((\chi_2 p)I - \mathbb{D}(\chi_2 u))e_3 = 0 & \text{on } \Sigma \\ \chi_2 u = 0 & \text{on } \Sigma_b. \end{cases} \tag{8.98}$$

Let $0 \le j \le n-1$ and $\alpha \in \mathbb{N}^2$ be so that $m+1 \le |\alpha| + 2j \le 2n-1$. Then we may apply Lemma A.14 to see that

$$\begin{aligned} \left\| \partial^\alpha \partial_t^j (\chi_2 u) \right\|_{2n-|\alpha|-2j+1}^2 + \left\| \partial^\alpha \partial_t^j (\chi_2 p) \right\|_{2n-|\alpha|-2j}^2 \lesssim \left\| \partial^\alpha \partial_t^{j+1} (\chi_2 u) \right\|_{2n-|\alpha|-2(j+1)+1}^2 \\ + \left\| \partial^\alpha \partial_t^j (\chi_2 G^1 + H^1) \right\|_{2n-|\alpha|-2j-1}^2 + \left\| \partial^\alpha \partial_t^j (\chi_2 G^2 + H^2) \right\|_{2n-|\alpha|-2j}^2 \\ \lesssim \left\| \partial^\alpha \partial_t^{j+1} (\chi_2 u) \right\|_{2n-|\alpha|-2(j+1)+1}^2 + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \end{aligned} \tag{8.99}$$

We first use estimate (8.99) and a finite induction to arrive at initial estimates for $\chi_2 u$ and $\chi_2 p$; we will then use the structure of the equations (2.23) to improve these estimates.

Our finite induction will be performed on $\ell \in [1, 2n-m-1]$, starting with the first two initial values, $\ell = 1$ and $\ell = 2$. We use the definition of $\bar{\mathcal{D}}_{n,m}$ and Lemma A.12 in conjunction with the bounds on $j, |\alpha|$ to see that

$$\left\| \partial^\alpha \partial_t^{j+1} (\chi_2 u) \right\|_0^2 \lesssim \left\| \partial^\alpha \partial_t^{j+1} u \right\|_0^2 \lesssim \bar{\mathcal{D}}_{n,m}. \tag{8.100}$$

Then (8.99) with $|\alpha| + 2j = 2n-1 = 2n-\ell$ implies that

$$\left\| \partial^\alpha \partial_t^j (\chi_2 u) \right\|_2^2 + \left\| \partial^\alpha \partial_t^j (\chi_2 p) \right\|_1^2 \lesssim \left\| \partial^\alpha \partial_t^{j+1} (\chi_2 u) \right\|_0^2 + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.101}$$

Applying this bound for all α and j satisfying $|\alpha| + 2j = 2n-1$ and summing, we find

$$\left\| \bar{D}_{2n-1}^{2n-1} (\chi_2 u) \right\|_2^2 + \left\| \bar{D}_{2n-1}^{2n-1} (\chi_2 p) \right\|_1^2 \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.102}$$

When $\ell = 2$ and $|\alpha| + 2j = 2n-\ell = 2n-2$, a similar application of Lemma A.12 implies

$$\left\| \partial^\alpha \partial_t^{j+1} (\chi_2 u) \right\|_1^2 \lesssim \bar{\mathcal{D}}_{n,m} \tag{8.103}$$

so that

$$\left\| \partial^\alpha \partial_t^j (\chi_2 u) \right\|_3^2 + \left\| \partial^\alpha \partial_t^j (\chi_2 p) \right\|_3^2 \lesssim \left\| \partial^\alpha \partial_t^{j+1} (\chi_2 u) \right\|_1^2 + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.104}$$

This may be summed over $2j + |\alpha| = 2n-2$ for the estimate

$$\left\| \bar{D}_{2n-2}^{2n-2} (\chi_2 u) \right\|_3^2 + \left\| \bar{D}_{2n-2}^{2n-2} (\chi_2 p) \right\|_2^2 \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.105}$$

Then (8.102) and (8.105) imply that

$$\left\| \bar{D}_{2n-1}^{2n-1} (\chi_2 u) \right\|_2^2 + \left\| \bar{D}_{2n-2}^{2n-2} (\chi_2 u) \right\|_3^2 + \left\| \bar{D}_{2n-1}^{2n-1} (\chi_2 p) \right\|_1^2 + \left\| \bar{D}_{2n-2}^{2n-2} (\chi_2 p) \right\|_2^2 \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.106}$$

Now suppose that the inequality

$$\sum_{\ell=1}^{\ell_0} \left\| \bar{D}_{2n-\ell}^{2n-\ell} (\chi_2 u) \right\|_{\ell+1}^2 + \left\| \bar{D}_{2n-\ell}^{2n-\ell} (\chi_2 p) \right\|_\ell^2 \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m} \tag{8.107}$$

holds for $2 \le \ell_0 < 2n-m-1$. We claim that (8.107) holds with ℓ_0 replaced by $\ell_0 + 1$. Suppose $|\alpha| + 2j = 2n - (\ell_0 + 1)$ and apply (8.99) to see that

$$\begin{aligned} \left\| \partial^\alpha \partial_t^j (\chi_2 u) \right\|_{\ell_0+2}^2 + \left\| \partial^\alpha \partial_t^j (\chi_2 p) \right\|_{\ell_0+1}^2 \lesssim \left\| \partial^\alpha \partial_t^{j+1} (\chi_2 u) \right\|_{\ell_0}^2 + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m} \\ \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}, \end{aligned} \tag{8.108}$$

where in the last inequality we have invoked (8.107) with $|\alpha| + 2(j+1) = 2n - (\ell_0 + 1) + 2 = 2n - (\ell_0 - 1)$. This proves the claim, so by finite induction the bound (8.107) holds for all $\ell_0 = 2, \dots, 2n - m - 1$. Choosing $\ell_0 = 2n - m - 1$ yields the estimate

$$\sum_{\ell=1}^{2n-m-1} \left\| \bar{D}^{2n-\ell}_{2n-\ell}(\chi_2 u) \right\|^2_{\ell+1} + \left\| \bar{D}^{2n-\ell}_{2n-\ell}(\chi_2 p) \right\|^2_{\ell} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}, \tag{8.109}$$

which implies, by virtue of the fact that $\chi_2 = 1$ on Ω_2, that

$$\sum_{\ell=1}^{2n-m-1} \left\| \bar{D}^{2n-\ell}_{2n-\ell} u \right\|^2_{H^{\ell+1}(\Omega_2)} + \left\| \bar{D}^{2n-\ell}_{2n-\ell} p \right\|^2_{H^{\ell}(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.110}$$

Now we will improve the estimate (8.110) by using the equations (2.23), considering the cases $m = 1, 2$ separately. Let $m = 1$. Since $m+1 = 2$, the bound (8.110) already covers all temporal derivatives, so we must only improve spatial derivatives. First note that (8.110) implies that

$$\left\| \partial_3 D^2 u \right\|^2_{H^{2n-2}(\Omega_2)} + \left\| D^2 p \right\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.111}$$

Then we may apply the operator $\partial_3 D$ to the divergence equation in (2.23) to bound

$$\left\| \partial_3^2 D u_3 \right\|^2_{H^{2n-2}(\Omega_2)} \lesssim \left\| \partial_3 D G^2 \right\|^2_{H^{2n-2}(\Omega_2)} + \left\| \partial_3 D^2 u \right\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.112}$$

Then applying the operator D to the first equation in (2.23) implies that

$$\begin{aligned} \left\| \partial_3 D p \right\|^2_{H^{2n-2}(\Omega_2)} + \left\| \partial_3^2 D u_i \right\|^2_{H^{2n-2}(\Omega_2)} &\lesssim \left\| D G^1 \right\|^2_{H^{2n-2}(\Omega_2)} + \left\| D^2 p \right\|^2_{H^{2n-2}(\Omega_2)} \\ &+ \left\| D \bar{D}^2_2 u \right\|^2_{H^{2n-2}(\Omega_2)} + \left\| \partial_3^2 D u_3 \right\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m} \end{aligned} \tag{8.113}$$

for $i = 1, 2$. We can then iterate this process, applying ∂_3^2 to the divergence equation, then ∂_3 to the first equation in (2.23), and using all of the bounds derived from the previous step, to deduce that

$$\left\| \partial_3^2 p \right\|^2_{H^{2n-2}(\Omega_2)} + \left\| \partial_3^3 u \right\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.114}$$

Combining (8.111)–(8.114) yields the estimate

$$\left\| \nabla^3 u \right\|^2_{H^{2n-2}(\Omega_2)} + \left\| \nabla^2 p \right\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}, \tag{8.115}$$

which together with (8.110) implies (8.96).

In the case $m = 2$, we can argue as in the case $m = 1$ to control the spatial derivatives. That is, we first control $\partial_3 D^3 u, D^3 p$, then iteratively apply operators with an increasing number of ∂_3 powers to arrive at the bound

$$\left\| \nabla^4 u \right\|^2_{H^{2n-3}(\Omega_2)} + \left\| \nabla^3 p \right\|^2_{H^{2n-3}(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.116}$$

It remains to control $\partial_t u$ and $\partial_t \nabla p$. For the latter we apply $\partial_3 \partial_t$ to the divergence equation to bound

$$\left\| \partial_3^2 \partial_t u_3 \right\|^2_{H^{2n-3}(\Omega_2)} \lesssim \left\| \partial_3 \partial_t G^2 \right\|^2_{H^{2n-3}(\Omega_2)} + \left\| \partial_3 \partial_t D u \right\|^2_{H^{2n-3}(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.117}$$

Then applying ∂_t to the third component of the first equation in (2.23) shows that

$$\begin{aligned} \left\| \partial_3 \partial_t p \right\|^2_{H^{2n-3}(\Omega_2)} &\lesssim \left\| \partial_t G^1 \right\|^2_{H^{2n-3}(\Omega_2)} + \left\| \partial_t \bar{D}^2_2 u \right\|^2_{H^{2n-3}(\Omega_2)} + \left\| \partial_3^2 \partial_t u_3 \right\|^2_{H^{2n-3}(\Omega_2)} \\ &\lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}, \end{aligned} \tag{8.118}$$

which in turn implies that

$$\left\| \nabla \partial_t p \right\|^2_{H^{2n-3}(\Omega_2)} \lesssim \left\| \partial_3 \partial_t p \right\|^2_{H^{2n-3}(\Omega_2)} + \left\| D \partial_t p \right\|^2_{H^{2n-3}(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \tag{8.119}$$

We may control $\partial_t u_3$ by applying ∂_t to the divergence equation in (2.23) to find that

$$\left\| \partial_3 \partial_t u_3 \right\|^2_{H^{2n-2}(\Omega_2)} \lesssim \left\| \partial_t G^2 \right\|^2_{H^{2n-2}(\Omega_2)} + \left\| \bar{D}^3_3 u \right\|^2_{H^{2n-2}(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}, \tag{8.120}$$

but then since $\partial_t u_3 = 0$ on Σ we can use Poincaré's inequality (Lemma A.13) to bound

$$\begin{aligned} \|\partial_t u_3\|^2_{H^{2n-1}(\Omega_2)} &\lesssim \|\partial_3\partial_t u_3\|^2_{H^{2n-2}(\Omega_2)} + \|\partial_t u_3\|^2_{H^0(\Omega_2)} + \left\|D_1^{2n-1}\partial_t u_3\right\|^2_{H^0(\Omega_2)} \\ &\lesssim \|\partial_3\partial_t u_3\|^2_{H^{2n-2}(\Omega_2)} + \left\|D_0^{2n-1}\partial_t u_3\right\|^2_{H^1(\Omega_2)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \end{aligned} \tag{8.121}$$

Control of the terms $\partial_t u_i$, $i = 1, 2$ is slightly more delicate; for it we appeal to the first of the localized equations (8.98) rather than (2.23). The reason for this is that using (8.98) will allow us to control $\partial_3^2\partial_t(\chi_2 u_i)$ in all of Ω, which will give us control of $\partial_t(\chi_2 u_i)$ in all of Ω via Poincaré and hence control of $\partial_t u_i$ in Ω_2. If instead we used (2.23), then control of $\partial_3^2\partial_t u_i$ in Ω_2 would not yield the desired control of $\partial_t u_i$ in Ω_2 because we could not apply Poincaré's inequality. We apply ∂_t to the $i = 1, 2$ components of the first localized equation in (8.98) and use (8.109) to see that

$$\begin{aligned} \left\|\partial_3^2\partial_t(\chi_2 u_i)\right\|^2_{H^{2n-3}(\Omega)} &\lesssim \left\|\partial_t H^1 \cdot e_i\right\|^2_{H^{2n-3}(\Omega)} + \left\|\chi_2\partial_t G^1\right\|^2_{H^{2n-3}(\Omega)} \\ &+ \|\partial_t D(\chi_2 p)\|^2_{H^{2n-3}(\Omega)} + \left\|\partial_t\bar{D}_2^2(\chi_2 u)\right\|^2_{H^{2n-3}(\Omega)} \lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m}. \end{aligned} \tag{8.122}$$

Now, since $\partial_t(\chi_2 u_i)$ and $\partial_3\partial_t(\chi_2 u_i)$ both vanish in an open set near Σ, we may apply Poincaré's inequality twice and use (8.122) to find that

$$\begin{aligned} \|\partial_t u_i\|^2_{H^{2n-1}(\Omega_2)} &\lesssim \|\partial_t(\chi_2 u_i)\|^2_{H^{2n-1}(\Omega)} \lesssim \left\|\partial_3^2\partial_t(\chi_2 u_i)\right\|^2_{H^{2n-3}(\Omega)} \\ &\lesssim \bar{\mathcal{D}}_{n,m} + \mathcal{Y}_{n,m} + \mathcal{X}_{n,m} + \|\partial_t u\|^2_{H^{2n-2}(\Omega_1)}. \end{aligned} \tag{8.123}$$

To conclude the analysis for $m = 2$ we sum (8.116), (8.119), (8.121), and (8.123) to derive (8.97). □

8.2. **Instantaneous energy.** Now we estimate the instantaneous energy. The proof is based on an argument very similar to the one used in the proof of Lemma 8.3.

Theorem 8.4. *Define*

$$\mathcal{W}_{n,m} = \left\|\bar{\nabla}_m^{2n-2}G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{2n-2}G^2\right\|_1^2 + \left\|\bar{D}_m^{2n-2}G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{2n-2}G^4\right\|_{1/2}^2. \tag{8.124}$$

If $m = 1$, then

$$\begin{aligned} \left\|\nabla^2 u\right\|^2_{2n-2} + \sum_{j=1}^{n}\left\|\partial_t^j u\right\|^2_{2n-2j} + \|\nabla p\|^2_{2n-2} + \sum_{j=1}^{n-1}\left\|\partial_t^j p\right\|^2_{2n-2j-1} \\ + \|D\eta\|^2_{2n-1} + \sum_{j=1}^{n}\left\|\partial_t^j \eta\right\|^2_{2n-2j} \lesssim \bar{\mathcal{E}}_{n,m} + \mathcal{W}_{n,m}. \end{aligned} \tag{8.125}$$

If $m = 2$, then

$$\begin{aligned} \left\|\nabla^3 u\right\|^2_{2n-3} + \sum_{j=1}^{n}\left\|\partial_t^j u\right\|^2_{2n-2j} + \left\|\nabla^2 p\right\|^2_{2n-3} + \sum_{j=1}^{n-1}\left\|\partial_t^j p\right\|^2_{2n-2j-1} \\ + \left\|D^2\eta\right\|^2_{2n-2} + \sum_{j=1}^{n}\left\|\partial_t^j \eta\right\|^2_{2n-2j} \lesssim \bar{\mathcal{E}}_{n,m} + \mathcal{W}_{n,m}. \end{aligned} \tag{8.126}$$

Proof. The proof is quite similar to that of Lemma 8.3, so we will not fill in all of the details. Throughout the proof we will employ the notation $\mathcal{Z} := \bar{\mathcal{E}}_{n,m} + \mathcal{W}_{n,m}$.

Let $0 \le j \le n-1$ and $\alpha \in \mathbb{N}^2$ satisfy $m \le |\alpha| + 2j \le 2n-2$. To begin, we utilize the equations (2.23) with the elliptic estimate Lemma A.14 to bound

(8.127)
$$\left\| \partial^\alpha \partial_t^j u \right\|_{2n-|\alpha|-2j}^2 + \left\| \partial^\alpha \partial_t^j p \right\|_{2n-|\alpha|-2j-1}^2 \lesssim \left\| \partial^\alpha \partial_t^{j+1} u \right\|_{2n-|\alpha|-2j-2}^2 + \left\| \partial^\alpha \partial_t^j G^1 \right\|_{2n-|\alpha|-2j-2}^2 + \left\| \partial^\alpha \partial_t^j G^2 \right\|_{2n-|\alpha|-2j-1}^2 + \left\| \partial^\alpha \partial_t^j \eta \right\|_{2n-|\alpha|-2j-3/2}^2 + \left\| \partial^\alpha \partial_t^j G^3 \right\|_{2n-|\alpha|-2j-3/2}^2 .$$

The constraints on j, α allow us to bound

$$\left\| \partial^\alpha \partial_t^j G^1 \right\|_{2n-|\alpha|-2j-2}^2 + \left\| \partial^\alpha \partial_t^j G^2 \right\|_{2n-|\alpha|-2j-1}^2 + \left\| \partial^\alpha \partial_t^j G^3 \right\|_{2n-|\alpha|-2j-3/2}^2 \lesssim \mathcal{W}_{n,m}, \tag{8.128}$$

and similarly

$$\left\| \partial^\alpha \partial_t^j \eta \right\|_{2n-|\alpha|-2j-3/2}^2 \lesssim \bar{\mathcal{E}}_{n,m}, \tag{8.129}$$

so that (8.127)–(8.129) imply that

$$\left\| \partial^\alpha \partial_t^j u \right\|_{2n-|\alpha|-2j}^2 + \left\| \partial^\alpha \partial_t^j p \right\|_{2n-|\alpha|-2j-1}^2 \lesssim \mathcal{Z} + \left\| \partial^\alpha \partial_t^{j+1} u \right\|_{2n-|\alpha|-2j-2}^2 . \tag{8.130}$$

As in Lemma 8.3, we argue with a finite induction on $\ell \in [2, 2n-m]$, beginning with $\ell = 2, 3$. When $\ell = 2$ and $|\alpha| + 2j = 2n-2 = 2n-\ell$, the definition of $\bar{\mathcal{E}}_{n,m}$ implies that

$$\left\| \partial^\alpha \partial_t^{j+1} u \right\|_0^2 \lesssim \bar{\mathcal{E}}_{n,m}, \tag{8.131}$$

which may be inserted into (8.130) for

$$\left\| \partial^\alpha \partial_t^j u \right\|_2^2 + \left\| \partial^\alpha \partial_t^j p \right\|_1^2 \lesssim \mathcal{Z}. \tag{8.132}$$

Summing over all α and j satisfying $|\alpha| + 2j = 2n-2$ shows that

$$\| \bar{D}_{2n-2}^{2n-2} u \|_2^2 + \| \bar{D}_{2n-2}^{2n-2} p \|_1^2 \lesssim \mathcal{Z}. \tag{8.133}$$

For $\ell = 3$ we note that $|\alpha| + 2j = 2n-3$ implies that $j \le n-2$, so that $|\alpha| \ge 1$. This allows us to write $\alpha = (\alpha - \beta) + \beta$ for $|\beta| = 1$ and to use (8.133) to see that

$$\left\| \partial^\alpha \partial_t^{j+1} u \right\|_1^2 \le \left\| \partial^{\alpha-\beta} \partial_t^{j+1} u \right\|_2^2 \le \| \bar{D}_{2n-2}^{2n-2} u \|_2^2 \lesssim \bar{\mathcal{E}}_{n,m}. \tag{8.134}$$

Then we can plug this into (8.130) for each $|\alpha| + 2j = 2n-3$ and sum to arrive at the bound

$$\| \bar{D}_{2n-3}^{2n-3} u \|_3^2 + \| \bar{D}_{2n-3}^{2n-3} p \|_2^2 \lesssim \mathcal{Z}. \tag{8.135}$$

Now we may use finite induction as in (8.107)–(8.110) of Lemma 8.3 to ultimately deduce the estimate

$$\sum_{\ell=2}^{2n-m} \left\| \bar{D}_{2n-\ell}^{2n-\ell} u \right\|_\ell^2 + \left\| \bar{D}_{2n-\ell}^{2n-\ell} p \right\|_{\ell-1}^2 \lesssim \mathcal{Z}. \tag{8.136}$$

Now we improve the estimate (8.136) by utilizing the structure of the equations (2.23), again arguing as in Lemma 8.3. The energy bound (8.136) in the case $m = 2$ is structurally similar to the bound (8.110) for the dissipation in the case $m = 1$, so we may argue as in (8.111)–(8.114), differentiating the equations (2.23) (with obvious modifications to the Sobolev indices and number of derivatives applied) and bootstrapping until we arrive at the bound

$$\| \nabla^3 u \|_{2n-3}^2 + \| \nabla^2 p \|_{2n-3}^2 \lesssim \mathcal{Z}. \tag{8.137}$$

Then (8.136) and (8.137) imply the bound (8.125).

In the case $m = 1$ we apply ∂_3 to the divergence equation in (2.23) to see that

$$\| \partial_3^2 u_3 \|_{2n-2}^2 \lesssim \| \partial_3 G^2 \|_{2n-2}^2 + \| \partial_3 D u \|_{2n-2}^2 \lesssim \mathcal{Z}. \tag{8.138}$$

We then use the first equation in (2.23) to bound

$$\|\partial_3 p\|_{2n-2}^2 + \sum_{i=1}^{2} \left\|\partial_3^2 u_i\right\|_{2n-2}^2 \lesssim \left\|G^1\right\|_{2n-2}^2 + \|\partial_3 Du\|_{2n-2}^2 + \|Dp\|_{2n-2}^2 \lesssim \mathcal{Z}. \tag{8.139}$$

Then (8.136), (8.138), and (8.139) imply that

$$\left\|\nabla^2 u\right\|_{2n-2}^2 + \|\nabla p\|_{2n-2}^2 \lesssim \mathcal{Z}, \tag{8.140}$$

and hence that (8.126) holds. □

8.3. **Specialization: estimates at the $2N$ and $N+2$ levels.** We now specialize the general results contained in Theorems 8.1 and 8.4 to the specific cases of $n = 2N$ with no minimal derivative restriction, and to the case $n = N+2$ with minimal derivative count $m = 1, 2$.

Theorem 8.5. *There exists a $\theta > 0$ so that*

$$\mathcal{D}_{2N} \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{8.141}$$

Proof. We apply Theorem 8.1 with $n = 2N$ and $m = 1$ to see that (8.2) holds. Theorem 4.2 provides the estimate

$$\mathcal{Y}_{2N,1} \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N} \tag{8.142}$$

for some $\theta > 0$. We may then use this in (8.2) to find that

$$\begin{aligned} &\left\|\nabla^3 u\right\|_{4N-2}^2 + \sum_{j=1}^{2N} \left\|\partial_t^j u\right\|_{4N-2j+1}^2 + \left\|\nabla^2 p\right\|_{4N-2}^2 + \sum_{j=1}^{2N-1} \left\|\partial_t^j p\right\|_{4N-2j}^2 \\ &+ \left\|D^2\eta\right\|_{4N-5/2}^2 + \|\partial_t\eta\|_{4N-1/2}^2 + \sum_{j=2}^{2N+1} \left\|\partial_t^j \eta\right\|_{4N-2j+5/2}^2 \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \end{aligned} \tag{8.143}$$

We can improve the estimate for u in (8.143) by using the fact that $\bar{\mathcal{D}}_{2N}$ does not have a minimal derivative count. Indeed, by definition, we know that

$$\|\mathcal{I}_\lambda u\|_1^2 + \|u\|_1^2 \lesssim \bar{\mathcal{D}}_{2N}. \tag{8.144}$$

Now, since Ω satisfies the uniform cone property, we can apply Corollary 4.16 of [2] to bound

$$\|u\|_{4N+1}^2 \lesssim \|u\|_0^2 + \left\|\nabla^{4N+1} u\right\|_0^2 \lesssim \|u\|_1^2 + \left\|\nabla^3 u\right\|_{4N-2}^2. \tag{8.145}$$

Then (8.143)–(8.145) imply that

$$\|\mathcal{I}_\lambda u\|_1^2 + \|u\|_{4N+1}^2 \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{8.146}$$

We can use this improved estimate of u to improve the estimate of p by employing the first equation of (2.23) to bound

$$\|\nabla p\|_{4N-1}^2 \lesssim \|\partial_t u\|_{4N-1}^2 + \|\Delta u\|_{4N-1}^2 + \left\|G^1\right\|_{4N-1}^2. \tag{8.147}$$

The bounds (8.143) and (8.146) imply that

$$\|\partial_t u\|_{4N-1}^2 + \|\Delta u\|_{4N-1}^2 \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}, \tag{8.148}$$

while (4.7)–(4.8) of Theorem 4.2 imply that

$$\left\|G^1\right\|_{4N-1}^2 \lesssim \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{8.149}$$

Hence (8.146)–(8.149) combine to show that

$$\|\nabla p\|_{4N-1}^2 \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^{\theta}\mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{8.150}$$

Finally, we improve the estimate for η. We use the boundary condition on Σ of (2.23) to bound

$$\|D\eta\|_{4N-3/2}^2 \lesssim \|Dp\|_{H^{4N-3/2}(\Sigma)}^2 + \|D\partial_3 u_3\|_{H^{4N-3/2}(\Sigma)}^2 + \left\|DG^3\right\|_{4N-3/2}^2 \\ \lesssim \|Dp\|_{4N-1}^2 + \|D\partial_3 u_3\|_{4N-1}^2 + \left\|DG^3\right\|_{4N-3/2}^2 \le \bar{\mathcal{D}}_{2N} + \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{8.151}$$

In the last inequality we have used (8.146), (8.150), and Theorem 4.2. Now (8.141) follows from (8.143), (8.146), (8.150), and (8.151). □

Now we perform a similar analysis for the energy at the $2N$ level.

Theorem 8.6. *There exists a $\theta > 0$ so that*

$$\mathcal{E}_{2N} \lesssim \bar{\mathcal{E}}_{2N} + \mathcal{E}_{2N}^{1+\theta}. \tag{8.152}$$

Proof. We apply Theorem 8.4 with $n = 2N$ and $m = 1$ to see that (8.125) holds. Theorem 4.2 provides the estimate

$$\mathcal{W}_{2N,1} \lesssim \mathcal{E}_{2N}^{1+\theta} \tag{8.153}$$

for some $\theta > 0$. Replacing in (8.125) shows that

$$\left\|\nabla^2 u\right\|_{4N-2}^2 + \sum_{j=1}^{2N} \left\|\partial_t^j u\right\|_{4N-2j}^2 + \|\nabla p\|_{4N-2}^2 + \sum_{j=1}^{2N-1} \left\|\partial_t^j p\right\|_{4N-2j-1}^2 \\ + \|D\eta\|_{4N-1}^2 + \sum_{j=1}^{2N} \left\|\partial_t^j \eta\right\|_{4N-2j}^2 \lesssim \bar{\mathcal{E}}_{2N} + \mathcal{E}_{2N}^{1+\theta}. \tag{8.154}$$

The definition of $\bar{\mathcal{E}}_{2N}$ implies that

$$\|\mathcal{I}_\lambda u\|_0^2 + \|u\|_0^2 + \|\mathcal{I}_\lambda \eta\|_0^2 + \|\eta\|_0^2 \le \bar{\mathcal{E}}_{2N}. \tag{8.155}$$

We may then sum the previous two bounds and employ Corollary 4.16 of [2] as in the proof of Theorem 8.5 to find that

$$\|\mathcal{I}_\lambda u\|_0^2 + \sum_{j=0}^{2N} \left\|\partial_t^j u\right\|_{4N-2j}^2 + \|\nabla p\|_{4N-2}^2 + \sum_{j=1}^{2N-1} \left\|\partial_t^j p\right\|_{4N-2j-1}^2 \\ + \|\mathcal{I}_\lambda \eta\|_0^2 + \sum_{j=0}^{2N} \left\|\partial_t^j \eta\right\|_{4N-2j}^2 \lesssim \bar{\mathcal{E}}_{2N} + \mathcal{E}_{2N}^{1+\theta}. \tag{8.156}$$

It remains only to estimate $\|p\|_{4N-1}^2$; since Lemma A.10 implies that

$$\|p\|_{4N-1}^2 \lesssim \|p\|_0^2 + \|\nabla p\|_{4N-2}^2 \lesssim \|p\|_{H^0(\Sigma)}^2 + \|\nabla p\|_{4N-2}^2, \tag{8.157}$$

it suffices to estimate $\|p\|_{H^0(\Sigma)}^2$. We do this by using the boundary condition in (2.23), trace theory, and estimate (4.6) of Theorem 4.2:

$$\|p\|_{H^0(\Sigma)}^2 \lesssim \|\eta\|_0^2 + \left\|G^3\right\|_0^2 + \|\partial_3 u_3\|_{H^0(\Sigma)}^2 \lesssim \|\eta\|_0^2 + \|u\|_{4N}^2 + \mathcal{E}_{2N}^{1+\theta}. \tag{8.158}$$

Then the estimate (8.152) easily follows from (8.156)–(8.158). □

We now consider the dissipation at the $N+2$ level.

Theorem 8.7. *For $m = 1, 2$ there exists a $\theta > 0$ so that*

$$\mathcal{D}_{N+2,m} \lesssim \bar{\mathcal{D}}_{N+2,m} + \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m}. \tag{8.159}$$

Proof. We apply Theorem 8.1 with $n = N+2$ to see that (8.2) holds for $m = 1$ and (8.3) holds for $m = 2$. Theorem 4.1 provides the estimate

$$\mathcal{Y}_{N+2,m} \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m} \tag{8.160}$$

for some $\theta > 0$. The bound (8.159) follows from using this in (8.2)–(8.3). □

We now consider the energy at the $N+2$ level.

Theorem 8.8. *For $m = 1, 2$ there exists a $\theta > 0$ so that*

$$\mathcal{E}_{N+2,m} \lesssim \bar{\mathcal{E}}_{N+2,m} + \mathcal{E}_{2N}^{\theta} \mathcal{E}_{N+2,m}. \tag{8.161}$$

Proof. We apply Theorem 8.4 with $n = N+2$ to see that (8.125) holds when $m = 1$ and (8.126) holds when $m = 2$. Theorem 4.1 provides the estimate

$$\mathcal{W}_{N+2,m} \lesssim \mathcal{E}_{2N}^{\theta} \mathcal{E}_{N+2,m} \tag{8.162}$$

for some $\theta > 0$. The bound (8.161) follows from using this in (8.125)–(8.126). □

9. A priori estimates

In this section we will combine the energy evolution estimates and the comparison estimates to derive a priori estimates for the total energy, $\mathcal{G}_{2N}$, defined by 2.58.

9.1. Estimates involving $\mathcal{F}_{2N}$ and $\mathcal{K}$.

We begin with an estimate for $\mathcal{F}_{2N}$.

Lemma 9.1. *There exists a $C > 0$ so that*

$$\sup_{0 \le r \le t} \mathcal{F}_{2N}(r) \lesssim \exp\left(C \int_0^t \sqrt{\mathcal{K}(r)} dr \right) \times \left[\mathcal{F}_{2N}(0) + t \int_0^t (1 + \mathcal{E}_{2N}(r)) \mathcal{D}_{2N}(r) dr + \left(\int_0^t \sqrt{\mathcal{K}(r) \mathcal{F}_{2N}(r)} dr \right)^2 \right]. \tag{9.1}$$

Proof. Throughout this proof we will write $u = \tilde{u} + u_3 e_3$, i.e. we write $\tilde{u}$ for the part of u parallel to Σ. Then η solves the transport equation $\partial_t \eta + \tilde{u} \cdot D\eta = u_3$ on Σ. We may then use Lemma A.9 with $s = 1/2$ to estimate

$$\sup_{0 \le r \le t} \|\eta(r)\|_{1/2} \le \exp\left(C \int_0^t \|D\tilde{u}(r)\|_{H^{3/2}(\Sigma)} dr \right) \left[\|\eta_0\|_{1/2} + \int_0^t \|u_3(r)\|_{H^{1/2}(\Sigma)} dr \right]. \tag{9.2}$$

By the definition of $\mathcal{K}$, (2.57), we may bound $\|D\tilde{u}(r)\|_{H^{3/2}(\Sigma)} \le \sqrt{\mathcal{K}(r)}$, but we may also use trace theory to bound $\|u_3(r)\|_{H^{1/2}(\Sigma)} \lesssim \mathcal{D}_{2N}(r)$. This allows us to square both sides of (9.2) and utilize Cauchy-Schwarz to deduce that

$$\sup_{0 \le r \le t} \|\eta(r)\|_{1/2}^2 \lesssim \exp\left(2C \int_0^t \sqrt{K(r)} dr \right) \left[\|\eta_0\|_{1/2}^2 + t \int_0^t \mathcal{D}_{2N}(r) dr \right]. \tag{9.3}$$

To go to higher regularity we let $\alpha \in \mathbb{N}^2$ with $|\alpha| = 4N$. Then we apply the operator ∂^α to the equation $\partial_t \eta + \tilde{u} \cdot D\eta = u_3$ to see that $\partial^\alpha \eta$ solves the transport equation

$$\partial_t(\partial^\alpha \eta) + \tilde{u} \cdot D(\partial^\alpha \eta) = \partial^\alpha u_3 - \sum_{0 < \beta \le \alpha} C_{\alpha,\beta} \partial^\beta \tilde{u} \cdot D\partial^{\alpha-\beta} \eta := G^\alpha \tag{9.4}$$

with the initial condition $\partial^\alpha \eta_0$. We may then apply Lemma A.9 with $s = 1/2$ to find that

$$\sup_{0 \le r \le t} \|\partial^\alpha \eta(r)\|_{1/2} \le \exp\left(C \int_0^t \|D\tilde{u}(r)\|_{H^{3/2}(\Sigma)} dr \right) \left[\|\partial^\alpha \eta_0\|_{1/2} + \int_0^t \|G^\alpha(r)\|_{1/2} dr \right]. \tag{9.5}$$

We will now estimate $\|G^\alpha\|_{H^{1/2}}$.

For $\beta \in \mathbb{N}^2$ satisfying $2N+1 \le |\beta| \le 4N$ we may apply Lemma A.1 with $s_1 = r = 1/2$ and $s_2 = 2$ to bound

$$\left\| \partial^\beta \tilde{u} D\partial^{\alpha-\beta} \eta \right\|_{1/2} \lesssim \left\| \partial^\beta \tilde{u} \right\|_{H^{1/2}(\Sigma)} \left\| D\partial^{\alpha-\beta} \eta \right\|_2. \tag{9.6}$$

This and trace theory then imply that

$$\sum_{\substack{0 < \beta \le \alpha \\ 2N+1 \le |\beta| \le 4N}} \left\| C_{\alpha,\beta} \partial^\beta \tilde{u} \cdot D\partial^{\alpha-\beta} \eta \right\|_{1/2} \lesssim \left\| D_{2N+1}^{4N} u \right\|_1 \left\| D_1^{2N} \eta \right\|_2 \lesssim \sqrt{\mathcal{D}_{2N} \mathcal{E}_{2N}}. \tag{9.7}$$

On the other hand, if β satisfies $1 \le |\beta| \le 2N$ then we use Lemma A.1 to bound

$$\left\| \partial^\beta \tilde{u} D \partial^{\alpha-\beta} \eta \right\|_{1/2} \lesssim \left\| \partial^\beta \tilde{u} \right\|_{H^2(\Sigma)} \left\| D \partial^{\alpha-\beta} \eta \right\|_{1/2} \tag{9.8}$$

so that

$$\begin{aligned} \sum_{\substack{0<\beta\le\alpha \\ 1\le|\beta|\le 2N}} \left\| C_{\alpha,\beta} \partial^\beta \tilde{u} \cdot D\partial^{\alpha-\beta}\eta \right\|_{1/2} &\lesssim \left\| D_1^{2N} u \right\|_3 \left\| D_{2N+1}^{4N-1} \eta \right\|_2 + \left\| D\tilde{u} \right\|_{H^2(\Sigma)} \left\| D^{4N} \eta \right\|_{1/2} \\ &\lesssim \sqrt{\mathcal{E}_{2N} \mathcal{D}_{2N}} + \sqrt{\mathcal{K} \mathcal{F}_{2N}}. \end{aligned} \tag{9.9}$$

The only remaining term in G^α is $\partial^\alpha u_3$, which we estimate with trace theory:

$$\left\| \partial^\alpha u_3 \right\|_{H^{1/2}(\Sigma)} \lesssim \left\| D^{4N} u_3 \right\|_1 \lesssim \sqrt{\mathcal{D}_{2N}}. \tag{9.10}$$

We may then combine (9.7), (9.9), and (9.10) for

$$\left\| G^\alpha \right\|_{1/2} \lesssim (1 + \sqrt{\mathcal{E}_{2N}}) \sqrt{\mathcal{D}_{2N}} + \sqrt{\mathcal{K} \mathcal{F}_{2N}}. \tag{9.11}$$

Returning now to (9.5), we square both sides and employ (9.11) and our previous estimate of the term in the exponential to find that

$$\begin{aligned} \sup_{0\le r\le t} \left\| \partial^\alpha \eta(r) \right\|_{1/2}^2 &\le \exp\left(2C \int_0^t \sqrt{\mathcal{K}(r)} dr \right) \\ &\times \left[\left\| \partial^\alpha \eta_0 \right\|_{1/2}^2 + t \int_0^t (1 + \mathcal{E}_{2N}(r)) \mathcal{D}_{2N}(r) dr + \left(\int_0^t \sqrt{\mathcal{K}(r) \mathcal{F}_{2N}(r)} dr \right)^2 \right]. \end{aligned} \tag{9.12}$$

Then the estimate (9.1) follows by summing (9.12) over all $|\alpha| = 4N$, adding the resulting inequality to (9.3), and using the fact that $\left\| \eta \right\|_{4N+1/2}^2 \lesssim \left\| \eta \right\|_{1/2}^2 + \left\| D^{4N} \eta \right\|_{1/2}^2$. □

Now we use this result and the $\mathcal{K}$ estimate of Lemma 3.17 to derive a stronger result.

Proposition 9.2. *There exists a universal constant $0 < \delta < 1$ so that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \lesssim \mathcal{F}_{2N}(0) + t \int_0^t \mathcal{D}_{2N} \tag{9.13}$$

for all $0 \le t \le T$.

Proof. Suppose $\mathcal{G}_{2N}(T) \le \delta \le 1$, for δ to be chosen later. Fix $0 \le t \le T$. Then according to Lemma 3.17, we have that $\mathcal{K} \lesssim \mathcal{E}_{N+2,2}^{(8+2\lambda)/(8+4\lambda)}$, which means that

$$\begin{aligned} \int_0^t \sqrt{\mathcal{K}(r)} dr &\lesssim \int_0^t (\mathcal{E}_{N+2,2}(r))^{(8+2\lambda)/(16+8\lambda)} dr \le \delta^{(8+2\lambda)/(16+8\lambda)} \int_0^t \frac{1}{(1+r)^{1+\lambda/4}} dr \\ &\le \delta^{(8+2\lambda)/(16+8\lambda)} \int_0^\infty \frac{1}{(1+r)^{1+\lambda/4}} dr = \frac{4}{\lambda} \delta^{(8+2\lambda)/(16+8\lambda)}. \end{aligned} \tag{9.14}$$

Since $\delta \le 1$, this implies that for any constant $C > 0$,

$$\exp\left(C \int_0^t \sqrt{\mathcal{K}(r)} dr \right) \lesssim 1. \tag{9.15}$$

Similarly,

$$\begin{aligned} \left(\int_0^t \sqrt{\mathcal{K}(r) \mathcal{F}_{2N}(r)} dr \right)^2 &\lesssim \left(\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \right) \left(\int_0^t \sqrt{\mathcal{K}(r)} dr \right)^2 \\ &\lesssim \left(\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \right) \delta^{(8+2\lambda)/(8+4\lambda)} \end{aligned} \tag{9.16}$$

Then (9.14)–(9.16) and Lemma 9.1 imply that

$$\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \le C\left(\mathcal{F}_{2N}(0) + t\int_0^t \mathcal{D}_{2N}\right) + C\delta^{(8+2\lambda)/(8+4\lambda)}\left(\sup_{0\le r\le t}\mathcal{F}_{2N}(r)\right), \tag{9.17}$$

for some $C > 0$. Then if δ is small enough so that $C\delta^{(8+2\lambda)/(8+4\lambda)} \le 1/2$, we may absorb the right-hand $\mathcal{F}_{2N}$ term onto the left and deduce (9.13). □

This bound on $\mathcal{F}_{2N}$ allows us to estimate to estimate the integral of $\mathcal{K}\mathcal{F}_{2N}$ and $\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}$.

Corollary 9.3. *There exists a universal constant* $0 < \delta < 1$ *so that if* $\mathcal{G}_{2N}(T) \le \delta$*, then*

$$\int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)dr \lesssim \delta^{(8+2\lambda)/(8+4\lambda)}\mathcal{F}_{2N}(0) + \delta^{(8+2\lambda)/(8+4\lambda)}\int_0^t \mathcal{D}_{2N}(r)dr \tag{9.18}$$

and

$$\int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}dr \lesssim \mathcal{F}_{2N}(0) + \delta^{(8+2\lambda)/(16+8\lambda)}\int_0^t \mathcal{D}_{2N}(r)dr \tag{9.19}$$

for $0 \le t \le T$.

Proof. Let $\mathcal{G}_{2N}(T) \le \delta$ with δ as small as in Proposition 9.2 so that estimate (9.13) holds. Lemma 3.17 implies that

$$\mathcal{K}(r) \lesssim (\mathcal{E}_{N+2,2}(r))^{(8+2\lambda)/(8+4\lambda)} \lesssim \delta^{(8+2\lambda)/(8+4\lambda)}\frac{1}{(1+r)^{2+\lambda/2}}. \tag{9.20}$$

This and (9.13) then imply that

$$\begin{aligned} \frac{1}{\delta^{(8+2\lambda)/(8+4\lambda)}}\int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)dr &\lesssim \mathcal{F}_{2N}(0)\int_0^t \frac{dr}{(1+r)^{2+\lambda/2}} \\ &+ \int_0^t \frac{r}{(1+r)^{2+\lambda/2}}\left(\int_0^r \mathcal{D}_{2N}(s)ds\right)dr \lesssim \mathcal{F}_{2N}(0)\int_0^\infty \frac{dr}{(1+r)^{2+\lambda/2}} \\ &+ \left(\int_0^t \mathcal{D}_{2N}(r)dr\right)\left(\int_0^\infty \frac{dr}{(1+r)^{1+\lambda/2}}\right) \lesssim \mathcal{F}_{2N}(0) + \int_0^t \mathcal{D}_{2N}(r)dr, \end{aligned} \tag{9.21}$$

which is estimate (9.18). The estimate (9.19) follows from (9.18), Cauchy-Schwarz, and the fact that $\delta \le 1$:

$$\begin{aligned} \int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}dr &\le \left(\int_0^t \mathcal{D}_{2N}(r)dr\right)^{1/2}\left(\int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)dr\right)^{1/2} \\ &\lesssim \left(\int_0^t \mathcal{D}_{2N}(r)dr\right)^{1/2}\left(\delta^{(8+2\lambda)/(8+4\lambda)}\mathcal{F}_{2N}(0)\right)^{1/2} + \delta^{(8+2\lambda)/(16+8\lambda)}\int_0^t \mathcal{D}_{2N}(r)dr \\ &\lesssim \mathcal{F}_{2N}(0) + \left(\delta^{(8+2\lambda)/(16+8\lambda)} + \delta^{(8+2\lambda)/(8+4\lambda)}\right)\int_0^t \mathcal{D}_{2N}(r)dr \\ &\lesssim \mathcal{F}_{2N}(0) + \delta^{(8+2\lambda)/(16+8\lambda)}\int_0^t \mathcal{D}_{2N}(r)dr. \end{aligned} \tag{9.22}$$

□

9.2. **Boundedness at the** $2N$ **level.** We now show bounds at the $2N$ level in terms of the initial data.

Theorem 9.4. *There exists a universal constant* $0 < \delta < 1$ *so that if* $\mathcal{G}_{2N}(T) \le \delta$*, then*

$$\sup_{0\le r\le t}\mathcal{E}_{2N}(r) + \int_0^t \mathcal{D}_{2N} + \sup_{0\le r\le t}\frac{\mathcal{F}_{2N}(r)}{(1+r)} \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \tag{9.23}$$

for all $0 \le t \le T$.

Proof. Combining the evolution equation estimate of Theorem 7.1 with the comparison estimates of Theorems 8.5 and 8.6, we find that

$$\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(r)dr \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{1+\theta} + \int_0^t (\mathcal{E}_{2N}(r))^{\theta}\mathcal{D}_{2N}(r)dr + \int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}dr + \int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)dr \tag{9.24}$$

for some $\theta > 0$. Let us assume initially that $\delta \le 1$ is as small as in Proposition 9.2 and Corollary 9.3 so that their conclusions hold. We may estimate the last two integrals in (9.24) with Corollary 9.3, using the fact that $\delta \le 1$:

$$\int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}dr + \int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)dr \lesssim \mathcal{F}_{2N}(0) + \delta^{(8+2\lambda)/(16+8\lambda)} \int_0^t \mathcal{D}_{2N}(r)dr. \tag{9.25}$$

On the other hand, $\sup_{0\le r\le t} \mathcal{E}_{2N}(r) \le \mathcal{G}_{2N}(T) \le \delta$, so

$$(\mathcal{E}_{2N}(t))^{1+\theta} + \int_0^t (\mathcal{E}_{2N}(r))^{\theta}\mathcal{D}_{2N}(r)dr \le \delta^{\theta}\mathcal{E}_{2N}(t) + \delta^{\theta}\int_0^t \mathcal{D}_{2N}(r)dr. \tag{9.26}$$

We may then combine (9.24)–(9.26) and write $\psi = \min\{\theta, (8+2\lambda)/(16+8\lambda)\} > 0$ to deduce the bound

$$\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(r)dr \le C\left(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)\right) + C\delta^{\theta}\mathcal{E}_{2N}(t) + C\delta^{\psi}\int_0^t \mathcal{D}_{2N}(r)dr \tag{9.27}$$

for a constant $C > 0$. Then if δ is sufficiently small so that $C\delta^{\theta} \le 1/2$ and $C\delta^{\psi} \le 1/2$, we may absorb the last two terms on the right side of (9.27) into the left, which then yields the estimate

$$\sup_{0\le r\le t} \mathcal{E}_{2N}(r) + \int_0^t \mathcal{D}_{2N}(r)dr \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0). \tag{9.28}$$

We then use this and Proposition 9.2 to estimate

$$\sup_{0\le r\le t} \frac{\mathcal{F}_{2N}(r)}{(1+r)} \lesssim \sup_{0\le r\le t} \frac{\mathcal{F}_{2N}(0)}{(1+r)} + \sup_{0\le r\le t} \frac{r}{(1+r)}\int_0^r \mathcal{D}_{2N}(s)ds \lesssim \mathcal{F}_{2N}(0) + \int_0^t \mathcal{D}_{2N}(r)dr \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0). \tag{9.29}$$

Then (9.23) follows by summing (9.28) and (9.29). □

9.3. **Decay at the $N+2$ level.** Before showing the decay estimates, we first need an interpolation result.

Proposition 9.5. *There exists a universal $0 < \delta < 1$ so that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\mathcal{D}_{N+2,m}(t) \lesssim \bar{\mathcal{D}}_{N+2,m}(t), \; \mathcal{E}_{N+2,m}(t) \lesssim \bar{\mathcal{E}}_{N+2,m}(t), \tag{9.30}$$

and

$$\bar{\mathcal{E}}_{N+2,m}(t) \lesssim (\mathcal{E}_{2N}(t))^{1/(m+\lambda+1)}(\bar{\mathcal{D}}_{N+2,m}(t))^{(m+\lambda)/(m+\lambda+1)} \tag{9.31}$$

for $m = 1, 2$ and $0 \le t \le T$.

Proof. The bound $\mathcal{G}_{2N}(T) \le \delta$ and Theorems 8.159 and 8.161 imply that

$$\mathcal{D}_{N+2,m} \le C\bar{\mathcal{D}}_{N+2,m} + C\mathcal{E}_{2N}^{\theta}\mathcal{D}_{N+2,m} \le C\bar{\mathcal{D}}_{N+2,m} + C\delta^{\theta}\mathcal{D}_{N+2,m} \tag{9.32}$$

and

$$\mathcal{E}_{N+2,m} \le C\bar{\mathcal{E}}_{N+2,m} + C\mathcal{E}_{2N}^{\theta}\mathcal{E}_{N+2,m} \le C\bar{\mathcal{E}}_{N+2,m} + C\delta^{\theta}\mathcal{E}_{N+2,m} \tag{9.33}$$

for constants $C > 0$ and $\theta > 0$. Then if δ is small enough so that $C\delta^{\theta} \le 1/2$, we may absorb the second term on the right side of (9.32) and (9.33) into the left to deduce the bounds in (9.30).

We now turn to the proof of (9.31). According to Remark 2.6, we have that

$$\bar{\mathcal{E}}_{N+2,m} \lesssim \left\|\bar{D}_m^{2N+4}u\right\|_0^2 + \left\|\bar{D}_m^{2N+4}\eta\right\|_0^2, \tag{9.34}$$

and by Lemma A.12, we also know that

$$\left\|\bar{D}_m^{2N+4}u\right\|_0^2 \lesssim \left\|\bar{D}_m^{2N+4}\mathbb{D}u\right\|_0^2 = \bar{\mathcal{D}}_{N+2,m}. \tag{9.35}$$

On the other hand, the definition of $\mathcal{D}_{N+2,m}$, given by (2.54) when $m=1$ and (2.55) when $m=2$, together with (9.30) implies that

$$\left\|\bar{D}_{m+1}^{2N+4}\eta\right\|_0^2 \le \mathcal{D}_{N+2,m} \lesssim \bar{\mathcal{D}}_{N+2,m}. \tag{9.36}$$

We may then combine (9.34)–(9.36) to see that

$$\bar{\mathcal{E}}_{N+2,m} \lesssim \bar{\mathcal{D}}_{N+2,m} + \left\|\bar{D}^m\eta\right\|_0^2. \tag{9.37}$$

In the case $m=1$ we use the H^0 interpolation estimates of Lemma 3.1 to bound

$$\left\|\bar{D}^m\eta\right\|_0^2 = \left\|D\eta\right\|_0^2 \lesssim (\mathcal{E}_{2N})^{1/(2+\lambda)}(\mathcal{D}_{N+2,1})^{(1+\lambda)/(2+\lambda)}. \tag{9.38}$$

In the case $m=2$ we use the H^0 interpolation estimates of $D^2\eta$ from Lemma 3.1 and the H^0 estimate of $\partial_t\eta$ from Proposition 3.16 to bound

$$\left\|\bar{D}^m\eta\right\|_0^2 = \left\|D^2\eta\right\|_0^2 + \left\|\partial_t\eta\right\|_0^2 \lesssim (\mathcal{E}_{2N})^{1/(3+\lambda)}(\mathcal{D}_{N+2,1})^{(2+\lambda)/(3+\lambda)}. \tag{9.39}$$

Together, (9.38) and (9.39) may be written as

$$\left\|\bar{D}^m\eta\right\|_0^2 \lesssim (\mathcal{E}_{2N})^{1/(m+\lambda+1)}(\mathcal{D}_{N+2,1})^{(m+\lambda)/(m+\lambda+1)}. \tag{9.40}$$

Now, according to Lemma 2.7, we can bound

$$\bar{\mathcal{D}}_{N+2,m} \le \mathcal{D}_{N+2,m} \lesssim (\mathcal{E}_{2N})^{1/(m+\lambda+1)}(\mathcal{D}_{N+2,m})^{(m+\lambda)/(m+\lambda+1)}. \tag{9.41}$$

Then we use the estimates (9.40) and (9.41) to bound the right side of (9.37); the bound (9.31) follows from the resulting inequality and (9.30). □

Now we show that the extra integral term appearing in Theorem 7.2 can essentially be absorbed into $\bar{\mathcal{E}}_{N+2,m}$.

Lemma 9.6. *Let F^2 be defined by (2.19) with $\partial^\alpha = \partial_t^{N+2}$. There exists a universal $0<\delta<1$ so that if $\mathcal{G}_{2N}(T)\le\delta$, then*

$$\frac{2}{3}\bar{\mathcal{E}}_{N+2,m}(t) \le \bar{\mathcal{E}}_{N+2,m}(t) - 2\int_\Omega J(t)\partial_t^{N+1}p(t)F^2(t) \le \frac{4}{3}\bar{\mathcal{E}}_{N+2,m}(t) \tag{9.42}$$

for all $0\le t\le T$.

Proof. Suppose that δ is as small as in Proposition 9.5. Then we combine estimate (5.4) of Theorem 5.2, Lemma 2.4, and estimate (9.30) of Proposition 9.5 to see that

$$\begin{aligned} \|J\|_{L^\infty}\left\|\partial_t^{N+1}p\right\|_0\left\|F^2\right\|_0 &\lesssim \sqrt{\mathcal{E}_{N+2,m}}\sqrt{\mathcal{E}_{2N}^\theta\mathcal{E}_{N+2,m}} \\ &= \mathcal{E}_{2N}^{\theta/2}\mathcal{E}_{N+2,m} \lesssim \mathcal{E}_{2N}^{\theta/2}\bar{\mathcal{E}}_{N+2,m} \lesssim \delta^{\theta/2}\bar{\mathcal{E}}_{N+2,m} \end{aligned} \tag{9.43}$$

for some $\theta>0$. This estimate and Cauchy-Schwarz then imply that

$$\left|2\int_\Omega J\partial_t^{N+1}pF^2\right| \le 2\|J\|_{L^\infty}\left\|\partial_t^{N+1}p\right\|_0\left\|F^2\right\|_0 \le C\delta^{\theta/2}\bar{\mathcal{E}}_{N+2,m} \le \frac{1}{3}\bar{\mathcal{E}}_{N+2,m} \tag{9.44}$$

if δ is small enough. The bound (9.42) then follows easily from (9.44). □

Now we prove decay at the $N+2$ level.

Theorem 9.7. *There exists a universal constant $0<\delta<1$ so that if $\mathcal{G}_{2N}(T)\le\delta$, then*

$$\sup_{0\le r\le t}(1+r)^{m+\lambda}\mathcal{E}_{N+2,m}(r) \lesssim \mathcal{E}_{2N}(0)+\mathcal{F}_{2N}(0) \tag{9.45}$$

for all $0\le t\le T$ and for $m\in\{1,2\}$.

Proof. Let δ be as small as in Theorem 9.4, Proposition 9.5, and Lemma 9.6. Theorem 7.2 and the estimate (9.30) of Proposition 9.5 imply that

$$\partial_t \left(\bar{\mathcal{E}}_{N+2,m} - 2\int_\Omega J \partial_t^{N+1} p F^2 \right) + \bar{\mathcal{D}}_{N+2,m} \le C \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,m} \le C\delta^\theta \bar{\mathcal{D}}_{N+2,m} \le \frac{1}{2}\bar{\mathcal{D}}_{N+2,m} \tag{9.46}$$

if δ is small enough (here $\theta > 0$). On the other hand, Theorem 9.4, (9.31) of Proposition 9.5, and (9.42) of Lemma 9.6 imply that

$$\begin{aligned} 0 &\le \frac{2}{3}\bar{\mathcal{E}}_{N+2,m} \le \bar{\mathcal{E}}_{N+2,m} - 2\int_\Omega J\partial_t^{N+1} p F^2 \le \frac{4}{3}\bar{\mathcal{E}}_{N+2,m} \\ &\le C(\mathcal{E}_{2N})^{1/(m+\lambda+1)} (\bar{\mathcal{D}}_{N+2,m})^{(m+\lambda)/(m+\lambda+1)} \le C_0 \mathcal{Z}_0^{1/(m+\lambda+1)} (\bar{\mathcal{D}}_{N+2,m})^{(m+\lambda)/(m+\lambda+1)} \end{aligned} \tag{9.47}$$

for all $0 \le t \le T$, where we have written $\mathcal{Z}_0 := \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)$, and C_0 is a universal constant which we may assume satisfies $C_0 \ge 1$. Let us write

$$h(t) = \bar{\mathcal{E}}_{N+2,m}(t) - 2\int_\Omega J(t)\partial_t^{N+1} p(t) F^2(t) \ge 0, \tag{9.48}$$

as well as

$$s = \frac{1}{m+\lambda} \text{ and } C_1 = \frac{1}{2C_0^{1+s}\mathcal{Z}_0^s}. \tag{9.49}$$

In these three terms we should distinguish between the cases $m = 1$ and $m = 2$, but to avoid notational clutter we will abuse notation and only write $h(t)$, s, and C_1. We may then combine (9.46) with (9.47) and use our new notation to derive the differential inequality

$$\partial_t h(t) + C_1 (h(t))^{1+s} \le 0 \tag{9.50}$$

for $0 \le t \le T$.

Since $h(t) \ge 0$, we may integrate (9.50) to find that for any $0 \le r \le T$,

$$h(r) \le \frac{h(0)}{[1 + sC_1(h(0))^s r]^{1/s}}. \tag{9.51}$$

Notice that Remark 2.6 implies that $\bar{\mathcal{E}}_{N+2,m} \le (3/2)\mathcal{E}_{2N}$. Then (9.47) implies that $h(0) \le (4/3)\bar{\mathcal{E}}_{N+2,m}(0) \le 2\mathcal{E}_{2N}(0) \le 2\mathcal{Z}_0$, which in turn implies that

$$sC_1(h(0))^s = \frac{s}{2C_0^{1+s}} \left(\frac{h(0)}{\mathcal{Z}_0} \right)^s \le \frac{s}{2C_0^{1+s}} 2^s = \frac{s}{C_0^{1+s}} 2^{s-1} \le 1 \tag{9.52}$$

since $0 < s < 1$ and $C_0 \ge 1$. A simple computation shows that

$$\sup_{r\ge 0} \frac{(1+r)^{1/s}}{(1+Mr)^{1/s}} = \frac{1}{M^{1/s}} \tag{9.53}$$

when $0 \le M \le 1$ and $s > 0$. This, (9.51), and (9.52) then imply that

$$(1+r)^{1/s} h(r) \le h(0) \frac{(1+r)^{1/s}}{[1 + sC_1(h(0))^s r]^{1/s}} \le h(0) \left(\frac{2C_0^{1+s}}{s} \right)^{1/s} \frac{\mathcal{Z}_0}{h(0)} = \left(\frac{2C_0^{1+s}}{s} \right)^{1/s} \mathcal{Z}_0. \tag{9.54}$$

Now we use (9.30) of Proposition 9.5 together with (9.47) to bound

$$\mathcal{E}_{N+2,m}(r) \lesssim \bar{\mathcal{E}}_{N+2,m}(r) \lesssim h(r) \text{ for } 0 \le r \le T. \tag{9.55}$$

The estimate (9.45) then follows from (9.54), (9.55), and the fact that $s = 1/(m+\lambda)$ and $\mathcal{Z}_0 = \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)$. □

9.4. **A priori estimates for $\mathcal{G}_{2N}$.** We now collect the results of Theorems 9.4 and 9.7 into a single bound on $\mathcal{G}_{2N}$, as defined by (2.58). The estimate recorded specifically names the constant in the inequality with $C_1 > 0$ so that it can be referenced later.

Theorem 9.8. *There exists a universal $0 < \delta < 1$ so that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\mathcal{G}_{2N}(t) \le C_1(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) \tag{9.56}$$

for all $0 \le t \le T$, where $C_1 > 0$ is a universal constant.

Proof. Let δ be as small as in Theorems 9.4 and 9.7. Then the conclusions of the theorems hold, and we may sum them to deduce (9.56). □

10. Specialized local well-posedness

10.1. **Propagation of $\mathcal{I}_\lambda$ bounds.** To prove Theorem 1.3, we will combine our a priori estimates, Theorem 9.8, with a local well-posedness result. Theorem 1.1 is not quite enough since it does not address the boundedness of $\|\mathcal{I}_\lambda u(t)\|_0^2$, $\|\mathcal{I}_\lambda \eta(t)\|_0^2$, and $\|\mathcal{I}_\lambda p(t)\|_0^2$ for $t > 0$. In order to prove these bounds, we will first study the cutoff operators $\mathcal{I}_\lambda^m$, which we define now. Let $m \ge 1$ be an integer. For a function f defined on Ω, we define the cutoff Riesz potential $\mathcal{I}_\lambda^m f$ by

$$\mathcal{I}_\lambda^m f(x', x_3) = \int_{-b}^0 \int_{\{|\xi| \ge 1/m\}} \hat{f}(\xi, x_3) \, |\xi|^{-\lambda} \, e^{2\pi i x' \cdot \xi} d\xi dx_3. \tag{10.1}$$

Similarly, for f defined on Σ, we set

$$\mathcal{I}_\lambda^m f(x') = \int_{\{|\xi| \ge 1/m\}} \hat{f}(\xi) \, |\xi|^{-\lambda} \, e^{2\pi i x' \cdot \xi} d\xi. \tag{10.2}$$

The operator $\mathcal{I}_\lambda^m$ is clearly bounded on $H^0(\Omega)$ and $H^0(\Sigma)$, which allows us to apply it to our solutions and then study the evolution of $\mathcal{I}_\lambda^m u$ and $\mathcal{I}_\lambda^m \eta$.

Before doing so, we will record some estimates for terms involving $\mathcal{I}_\lambda^m$ that are analogous to the $\mathcal{I}_\lambda$ estimates in Sections 4.3 and 6.2 and Appendix A.2. We begin with the analog of Lemmas A.3 and A.4, which were the starting point for our $\mathcal{I}_\lambda$ estimates.

Lemma 10.1. *If $\mathcal{I}_\lambda h \in H^0(\Omega)$, then $\|\mathcal{I}_\lambda^m h\|_0^2 \le \|\mathcal{I}_\lambda h\|_0^2$. A similar estimate holds if $\mathcal{I}_\lambda h \in H^0(\Sigma)$. As a consequence, the results of Lemmas A.3 and A.4 hold with $\mathcal{I}_\lambda$ replaced by $\mathcal{I}_\lambda^m$ and with the constants in the inequalities independent of m.*

Proof. Suppose that $\mathcal{I}_\lambda h \in H^0(\Omega)$ for some h. Then, writing $\hat{\ }$ for the horizontal Fourier transform, we easily see that

$$\|\mathcal{I}_\lambda^m h\|_0^2 = \int_{-b}^0 \int_{\{|\xi| \ge 1/m\}} \left| \hat{h}(\xi, x_3) \right|^2 |\xi|^{-2\lambda} \, d\xi dx_3 \le \|\mathcal{I}_\lambda h\|_0^2 . \tag{10.3}$$

The corresponding estimate in case $\mathcal{I}_\lambda h \in H^0(\Sigma)$ follows similarly. Then the estimates of Lemmas A.3 and A.4 may be combined with these inequalities to replace $\mathcal{I}_\lambda$ with $\mathcal{I}_\lambda^m$. □

We do not want our estimates for $\mathcal{I}_\lambda^m$ to be given in terms of $\mathcal{E}_{2N}$ since this energy contains $\mathcal{I}_\lambda$ terms. Instead, we desire estimates in terms of a modified energy, which we write as

$$\mathfrak{E}_{2N} = \mathcal{E}_{2N} - \|\mathcal{I}_\lambda u\|_0^2 - \|\mathcal{I}_\lambda \eta\|_0^2 . \tag{10.4}$$

Lemma 10.1 allows us prove the following modification of Proposition 4.3. The proof is a simple adaptation of the one for Proposition 4.3, and is thus omitted.

Proposition 10.2. *We have that*

$$\left\|\mathcal{I}_\lambda^m G^1\right\|_1^2 + \left\|\mathcal{I}_\lambda^m G^2\right\|_2^2 + \left\|\mathcal{I}_\lambda^m \partial_t G^2\right\|_0^2 + \left\|\mathcal{I}_\lambda^m G^3\right\|_1^2 + \left\|\mathcal{I}_\lambda^m G^4\right\|_1^2 \lesssim \mathfrak{E}_{2N}^2. \tag{10.5}$$

Here the constant in the inequality does not depend on m.

We may similarly modify the proof of Lemma 4.4.

Lemma 10.3. *We have that*

$$\|\mathcal{I}_\lambda^m[(AK)\partial_3 u_1 + (BK)\partial_3 u_2]\|_0^2 + \sum_{i=1}^2 \|\mathcal{I}_\lambda^m[u\partial_i K]\|_0^2 \lesssim \mathfrak{E}_{2N}^2 \tag{10.6}$$

and

$$\|\mathcal{I}_\lambda^m[(1-K)u]\|_0^2 + \left\|\mathcal{I}_\lambda^m[(1-K)G^2]\right\|_0^2 \lesssim \mathfrak{E}_{2N}^2. \tag{10.7}$$

Here the constants in the inequalities do not depend on m.

Then Lemma 10.3 leads to a modification of Lemma 6.5.

Lemma 10.4. *It holds that*

$$\|\mathcal{I}_\lambda^m p\|_0^2 \lesssim \|\mathcal{I}_\lambda^m \eta\|_0^2 + \mathfrak{E}_{2N} \text{ and } \|\mathcal{I}_\lambda^m Dp\|_0^2 \lesssim \mathfrak{E}_{2N}. \tag{10.8}$$

Here the constants in the inequalities do not depend on m.

In turn, Lemma 10.4 gives a variant of Lemma 6.6.

Lemma 10.5. *It holds that*

$$\left|\int_\Omega \mathcal{I}_\lambda^m p \mathcal{I}_\lambda^m G^2\right| \lesssim \mathfrak{E}_{2N} \|\mathcal{I}_\lambda^m \eta\|_0 + \mathfrak{E}_{2N}. \tag{10.9}$$

Here the constant in the inequality does not depend on m.

These results now allow us to study the boundedness of $\mathcal{I}_\lambda u$, etc. We first apply the operator $\mathcal{I}_\lambda^m$ to the equations (2.23), which is possible since $\mathcal{I}_\lambda^m$ is bounded on $H^0(\Omega)$ and $H^0(\Sigma)$. Then the energy evolution for $\mathcal{I}_\lambda^m u$ and $\mathcal{I}_\lambda^m \eta$ allows us to derive bounds for these quantities, which yield bounds for $\mathcal{I}_\lambda u$ and $\mathcal{I}_\lambda \eta$ after passing to the limit $m \to \infty$.

Proposition 10.6. *Suppose (u,p,η) are solutions on the time interval $[0,T]$ and that $\|\mathcal{I}_\lambda u_0\|_0^2 + \|\mathcal{I}_\lambda \eta_0\|_0^2 < \infty$ and $\sup_{0\le t\le T} \mathfrak{E}_{2N}(t) \le 1$. Then*

$$\begin{aligned} \sup_{0\le t\le T} \left(\|\mathcal{I}_\lambda u(t)\|_0^2 + \|\mathcal{I}_\lambda p(t)\|_0^2 + \|\mathcal{I}_\lambda \eta(t)\|_0^2 \right) + \int_0^T \|\mathcal{I}_\lambda u(t)\|_1^2 \, dt \\ \lesssim e^T \left(\|\mathcal{I}_\lambda u_0\|_0^2 + \|\mathcal{I}_\lambda \eta_0\|_0^2 \right) + e^T \sup_{0\le t\le T} \mathfrak{E}_{2N}(t). \end{aligned} \tag{10.10}$$

Proof. Since $\mathcal{I}_\lambda^m$ is a bounded operator on $H^0(\Omega)$ and $H^0(\Sigma)$, we are free to apply it to the equations (2.23). After doing so we then use Lemma 2.3 to see that

$$\begin{aligned} \partial_t \left(\frac{1}{2}\int_\Omega |\mathcal{I}_\lambda^m u|^2 + \frac{1}{2}\int_\Sigma |\mathcal{I}_\lambda^m \eta|^2 \right) + \frac{1}{2}\int_\Omega |\mathbb{D}\mathcal{I}_\lambda^m u|^2 = \int_\Omega \mathcal{I}_\lambda^m u \cdot \mathcal{I}_\lambda^m G^1 + \mathcal{I}_\lambda^m p \mathcal{I}_\lambda^m G^2 \\ + \int_\Sigma -\mathcal{I}_\lambda^m u \cdot \mathcal{I}_\lambda^m G^3 + \mathcal{I}_\lambda^m \eta \mathcal{I}_\lambda^m G^4. \end{aligned} \tag{10.11}$$

We will estimate each term on the right side of this equation. First, we use Cauchy-Schwarz and Lemma 10.2 to estimate the first and fourth terms:

$$\begin{aligned} \left|\int_\Omega \mathcal{I}_\lambda^m u \cdot \mathcal{I}_\lambda^m G^1\right| + \left|\int_\Sigma \mathcal{I}_\lambda^m \eta \mathcal{I}_\lambda^m G^4\right| \le \|\mathcal{I}_\lambda^m u\|_0 \left\|\mathcal{I}_\lambda^m G^1\right\|_0 + \|\mathcal{I}_\lambda^m \eta\|_0 \left\|\mathcal{I}_\lambda^m G^4\right\|_0 \\ \le \frac{1}{2}\|\mathcal{I}_\lambda^m u\|_0^2 + \frac{1}{4}\|\mathcal{I}_\lambda^m \eta\|_0^2 + \frac{1}{2}\left\|\mathcal{I}_\lambda^m G^1\right\|_0^2 + \left\|\mathcal{I}_\lambda^m G^4\right\|_0^2 \le \frac{1}{2}\|\mathcal{I}_\lambda^m u\|_0^2 + \frac{1}{4}\|\mathcal{I}_\lambda^m \eta\|_0^2 + C\mathfrak{E}_{2N}^2 \end{aligned} \tag{10.12}$$

for $C > 0$ independent of m. For the second term we use Lemma 10.5 and Cauchy's inequality for

$$\left|\int_\Omega \mathcal{I}_\lambda^m p \mathcal{I}_\lambda^m G^2\right| \le C \|\mathcal{I}_\lambda^m \eta\|_0 \mathfrak{E}_{2N} + C\mathfrak{E}_{2N} \le \frac{1}{4}\|\mathcal{I}_\lambda^m \eta\|_0^2 + C(\mathfrak{E}_{2N} + \mathfrak{E}_{2N}^2), \tag{10.13}$$

where again $C > 0$ is independent of m. Finally, for the third term we use trace theory, Lemma 10.2, and Lemma A.12 to bound

$$\left| \int_\Sigma \mathcal{I}_\lambda^m u \cdot \mathcal{I}_\lambda^m G^3 \right| \le \|\mathcal{I}_\lambda^m u\|_{H^0(\Sigma)} \left\|\mathcal{I}_\lambda^m G^3\right\|_0 \le C \|\mathcal{I}_\lambda^m u\|_1 \left\|\mathcal{I}_\lambda^m G^3\right\|_0 \le C \|\mathbb{D}\mathcal{I}_\lambda^m u\|_0 \mathfrak{E}_{2N} \le \frac{1}{4} \|\mathbb{D}\mathcal{I}_\lambda^m u\|_0^2 + C\mathfrak{E}_{2N}^2, \tag{10.14}$$

with $C > 0$ independent of m. Now we use (10.12)–(10.14) to estimate the right side of (10.11); after rearranging the resulting bound, we find that

$$\partial_t \left(\|\mathcal{I}_\lambda^m u\|_0^2 + \|\mathcal{I}_\lambda^m \eta\|_0^2 \right) + \frac{1}{2} \|\mathbb{D}\mathcal{I}_\lambda^m u\|_0^2 \le \|\mathcal{I}_\lambda^m u\|_0^2 + \|\mathcal{I}_\lambda^m \eta\|_0^2 + C(\mathfrak{E}_{2N} + \mathfrak{E}_{2N}^2) \tag{10.15}$$

for a constant $C > 0$ that does not depend on m.

The inequality (10.15) may be viewed as the differential inequality

$$\partial_t \mathcal{E}_{\lambda,m} + \frac{1}{2}\mathcal{D}_{\lambda,m} \le \mathcal{E}_{\lambda,m} + C(\mathfrak{E}_{2N} + \mathfrak{E}_{2N}^2), \tag{10.16}$$

where we have written $\mathcal{E}_{\lambda,m} = \|\mathcal{I}_\lambda^m u\|_0^2 + \|\mathcal{I}_\lambda^m \eta\|_0^2$ and $\mathcal{D}_{\lambda,m} = \|\mathbb{D}\mathcal{I}_\lambda^m u\|_0^2$. Applying Gronwall's lemma to (10.16) and using the fact that $\mathfrak{E}_{2N}(t) \le 1$ then shows that

$$\mathcal{E}_{\lambda,m}(t) + \frac{1}{2}\int_0^t \mathcal{D}_{\lambda,m}(s) ds \le \mathcal{E}_{\lambda,m}(0)e^t + C\int_0^t e^{t-s}\mathfrak{E}_{2N}(s) ds \le \mathcal{E}_{\lambda,m}(0)e^t + C(e^t - 1)\sup_{0\le s\le t} \mathfrak{E}_{2N}(s) \tag{10.17}$$

where again $C > 0$ is independent of m. It is a simple matter to verify, using the definitions of $\mathcal{I}_\lambda^m$ and $\mathcal{I}_\lambda$, the Fourier transform in (x_1, x_2), and the monotone convergence theorem, that as $m \to \infty$,

$$\mathcal{E}_{\lambda,m}(s) = \|\mathcal{I}_\lambda^m u(s)\|_0^2 + \|\mathcal{I}_\lambda^m \eta(s)\|_0^2 \to \|\mathcal{I}_\lambda u(s)\|_0^2 + \|\mathcal{I}_\lambda \eta(s)\|_0^2 \tag{10.18}$$

for both $s = 0$ and $s = t$, and

$$\int_0^t \mathcal{D}_{\lambda,m}(s) ds \to \int_0^t \|\mathbb{D}\mathcal{I}_\lambda u(s)\|_0^2 \, ds. \tag{10.19}$$

Now, according to these two convergence results, we may pass to the limit $m \to \infty$ in (10.17); the resulting estimate and Lemma A.12 then imply that

$$\sup_{0\le t\le T} \left(\|\mathcal{I}_\lambda u(t)\|_0^2 + \|\mathcal{I}_\lambda \eta(t)\|_0^2 \right) + \int_0^T \|\mathcal{I}_\lambda u(t)\|_1^2 \, dt \lesssim \left(\|\mathcal{I}_\lambda u_0\|_0^2 + \|\mathcal{I}_\lambda \eta_0\|_0^2 \right) e^T + (e^T - 1) \sup_{0\le t\le T} \mathfrak{E}_{2N}(t). \tag{10.20}$$

On the other hand, from Lemma 10.4, we know that

$$\|\mathcal{I}_\lambda^m p(t)\|_0^2 \lesssim \|\mathcal{I}_\lambda^m \eta(t)\|_0^2 + \mathfrak{E}_{2N}(t). \tag{10.21}$$

We may then argue as above, employing the monotone convergence theorem, to pass to the limit $m \to \infty$ in this estimate. We then find that

$$\sup_{0\le t\le T} \|\mathcal{I}_\lambda p(t)\|_0^2 \lesssim \sup_{0\le t\le T} \|\mathcal{I}_\lambda \eta(t)\|_0^2 + \sup_{0\le t\le T} \mathfrak{E}_{2N}(t). \tag{10.22}$$

The estimate (10.10) then follows by combining (10.20) and (10.22). □

10.2. **Local well-posedness.** We now record the specialized version of the local well-posedness theorem. We include estimates for $\mathcal{I}_\lambda u$, $\mathcal{I}_\lambda \eta$, and $\mathcal{I}_\lambda p$. We also separate estimates for $\mathcal{E}_{2N}$ and $\mathcal{D}_{2N}$ from estimates for $\mathcal{F}_{2N}$ and $\mathfrak{E}_{2N}$, the latter of which is defined by (10.4).

Theorem 10.7. *Suppose that initial data are given satisfying the compatibility conditions of Theorem 1.1 and* $\|u(0)\|_{4N}^2 + \|\eta(0)\|_{4N+1/2}^2 + \|\mathcal{I}_\lambda u(0)\|_0^2 + \|\mathcal{I}_\lambda \eta(0)\|_0^2 < \infty$*. Let* $\varepsilon > 0$*. There exists a* $\delta_0 = \delta_0(\varepsilon) > 0$ *and a*

$$T_0 = C(\varepsilon) \min \left\{ 1, \frac{1}{\|\eta(0)\|_{4N+1/2}^2} \right\} > 0, \tag{10.23}$$

where $C(\varepsilon) > 0$ *is a constant depending on* ε*, so that if* $0 < T \le T_0$ *and* $\|u(0)\|_{4N}^2 + \|\eta(0)\|_{4N}^2 \le \delta_0$*, then there exists a unique solution* (u, p, η) *to* (1.9) *on the interval* $[0, T]$ *that achieves the initial data. The solution obeys the estimates*

$$\begin{aligned} &\sup_{0\le t\le T} \mathcal{E}_{2N}(t) + \sup_{0\le t\le T} \|\mathcal{I}_\lambda p(t)\|_0^2 + \int_0^T \mathcal{D}_{2N}(t)dt \\ &+ \int_0^T \left(\left\| \partial_t^{2N+1} u(t) \right\|_{({}_0H^1)^*}^2 + \left\| \partial_t^{2N} p(t) \right\|_0^2 \right) dt \le C_2 \left(\varepsilon + \|\mathcal{I}_\lambda u(0)\|_0^2 + \|\mathcal{I}_\lambda \eta(0)\|_0^2 \right), \end{aligned} \tag{10.24}$$

$$\sup_{0\le t\le T} \mathfrak{E}_{2N}(t) \le \varepsilon, \text{ and } \sup_{0\le t\le T} \mathcal{F}_{2N}(t) \le C_2 \mathcal{F}_{2N}(0) + \varepsilon \tag{10.25}$$

for $C_2 > 0$ *a universal constant. Here* $\mathfrak{E}_{2N}$ *is as defined by* (10.4)*.*

Proof. The result follows directly from Proposition 10.6 and Theorem 1.1. □

Remark 10.8. *The finiteness of the terms in* (10.24)–(10.25) *justifies all of the computations leading to Theorem 9.8. In particular, it shows that* $\partial_t^{2N+1} u$ *and* $\partial_t^{2N} p$ *are well-defined.*

11. Global well-posedness and decay: proof of Theorem 1.3

In order to combine the local existence result, Theorem 10.7, with the a priori estimates of Theorem 9.8, we must be able to estimate $\mathcal{G}_{2N}$ in terms of the estimates given in (10.24)–(10.25). We record this estimate now.

Proposition 11.1. *Let* $\mathfrak{E}_{2N}$ *be as defined by* (10.4)*. There exists a universal constant* $C_3 > 0$ *with the following properties. If* $0 \le T$*, then we have the estimate*

$$\mathcal{G}_{2N}(T) \le \sup_{0\le t\le T} \mathcal{E}_{2N}(t) + \int_0^{T_2} \mathcal{D}_{2N}(t)dt + \sup_{0\le t\le T} \mathcal{F}_{2N}(t) + C_3(1+T)^{2+\lambda} \sup_{0\le t\le T} \mathfrak{E}_{2N}(t). \tag{11.1}$$

If $0 < T_1 \le T_2$*, then we have the estimate*

$$\begin{aligned} \mathcal{G}_{2N}(T_2) \le C_3 \mathcal{G}_{2N}(T_1) &+ \sup_{T_1\le t\le T_2} \mathcal{E}_{2N}(t) + \int_{T_1}^{T_2} \mathcal{D}_{2N}(t)dt \\ &+ \frac{1}{(1+T_1)} \sup_{T_1\le t\le T_2} \mathcal{F}_{2N}(t) + C_3(T_2-T_1)^2(1+T_2)^{2+\lambda} \sup_{T_1\le t\le T_2} \mathfrak{E}_{2N}(t). \end{aligned} \tag{11.2}$$

Proof. We will only prove the estimate (11.2); the bound (11.1) follows from a similar, but easier argument. The definition of $\mathcal{G}_{2N}(T_2)$ allows us to estimate

$$\begin{aligned} \mathcal{G}_{2N}(T_2) \le \mathcal{G}_{2N}(T_1) &+ \sup_{T_1\le t\le T_2} \mathcal{E}_{2N}(t) + \int_{T_1}^{T_2} \mathcal{D}_{2N}(t)dt \\ &+ \sup_{T_1\le t\le T_2} \frac{\mathcal{F}_{2N}(t)}{(1+t)} + \sum_{m=1}^{2} \sup_{T_1\le t\le T_2} \left((1+t)^{m+\lambda} \mathcal{E}_{N+2,m}(t) \right). \end{aligned} \tag{11.3}$$

Since $N \geq 3$ it is easy to verify that

$$\sum_{j=0}^{N+2} \left\| \partial_t^{j+1} u \right\|_{2(N+2)-2j}^2 + \left\| \partial_t^j u \right\|_{2(N+2)-2j}^2 + \left\| \partial_t^{j+1} \eta \right\|_{2(N+2)-2j}^2 + \left\| \partial_t^j \eta \right\|_{2(N+2)-2j}^2 \lesssim \mathfrak{E}_{2N} \tag{11.4}$$

and

$$\sum_{j=0}^{N+1} \left\| \partial_t^{j+1} p \right\|_{2(N+2)-2j-1}^2 + \left\| \partial_t^j p \right\|_{2(N+2)-2j-1}^2 \lesssim \mathfrak{E}_{2N}. \tag{11.5}$$

For $j = 0, \dots, 2N$, we may then integrate $\partial_t \left[(1+t)^{(m+\lambda)/2} \partial_t^j u(t) \right]$ in time from T_1 to $T_1 \leq t \leq T_2$ to deduce the bound

$$\begin{aligned} \left\| (1+t)^{(m+\lambda)/2} \partial_t^j u(t) \right\|_{2N+4-2j} &\leq \left\| (1+T_1)^{(m+\lambda)/2} \partial_t^j u(T_1) \right\|_{2N+4-2j} \\ &+ \int_{T_1}^{T_2} (1+s)^{(m+\lambda)/2} \left\| \partial_t^{j+1} u(s) \right\|_{2N+4-2j} + \frac{(m+\lambda)}{2} (1+s)^{(m+\lambda-2)/2} \left\| \partial_t^j u(s) \right\|_{2N+4-2j} \\ &\lesssim \sqrt{\mathcal{G}_{2N}(T_1)} + (T_2 - T_1)(1+T_2)^{1+\lambda/2} \sqrt{\sup_{T_1 \leq t \leq T_2} \mathfrak{E}_{2N}(t)}. \end{aligned} \tag{11.6}$$

Squaring both sides of this then yields, for $j = 0, \dots, N+2$,

$$\sup_{T_1 \leq t \leq T_2} \left((1+t)^{m+\lambda} \left\| \partial_t^j u(t) \right\|_{2(N+2)-2j}^2 \right) \lesssim \mathcal{G}_{2N}(T_1) + (T_2 - T_1)^2 (1+T_2)^{2+\lambda} \sup_{T_1 \leq t \leq T_2} \mathfrak{E}_{2N}(t). \tag{11.7}$$

Similar estimates hold for $j = 0, \dots, N+2$ with $\partial_t^j u$ replaced by $\partial_t^j \eta$ and for $j = 0, \dots, N+1$ with $\left\| \partial_t^j u(t) \right\|_{2(N+2)-2j}^2$ replaced by $\left\| \partial_t^j p(t) \right\|_{2(N+2)-2j-1}^2$. From these we may then estimate

$$\sum_{m=1}^{2} \sup_{T_1 \leq t \leq T_2} \left((1+t)^{m+\lambda} \mathcal{E}_{N+2,m}(t) \right) \lesssim \mathcal{G}_{2N}(T_1) + (T_2 - T_1)^2 (1+T_2)^{2+\lambda} \sup_{T_1 \leq t \leq T_2} \mathfrak{E}_{2N}(t). \tag{11.8}$$

Then (11.2) follows from (11.3), (11.8), and the trivial bound

$$\sup_{T_1 \leq t \leq T_2} \frac{\mathcal{F}_{2N}(t)}{(1+t)} \leq \frac{1}{(1+T_1)} \sup_{T_1 \leq t \leq T_2} \mathcal{F}_{2N}(t). \tag{11.9}$$

□

We now turn to our main result.

Theorem 11.2. *Suppose the initial data (u_0, η_0) satisfy the compatibility conditions of Theorem 1.1. There exists a $\kappa > 0$ so that if $\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa$, then there exists a unique solution (u, p, η) on the interval $[0, \infty)$ that achieves the initial data. The solution obeys the estimate*

$$\mathcal{G}_{2N}(\infty) \leq C_1 \left(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \right) < C_1 \kappa, \tag{11.10}$$

where $C_1 > 0$ is given by Theorem 9.8.

Proof. Let $0 < \delta < 1$ and $C_1 > 0$ be the constants from Theorem 9.8, $C_2 > 0$ be the constant from Theorem 10.7, and $C_3 > 0$ be the constant from Proposition 11.1. According to (11.1) of Proposition 11.1, if a solution exists on the interval $[0, T]$ with $T < 1$ and obeys the estimates (10.24)–(10.25), then

$$\mathcal{G}_{2N}(T) \leq C_2 \kappa + \varepsilon \left[C_2 + 1 + C_3 (2)^{2+\lambda} \right]. \tag{11.11}$$

If ε is chosen so that the latter term in (11.11) equals $\delta/2$, then we may choose κ sufficiently small so that $C_2 \kappa < \delta/2$ and $\kappa < \delta_0(\varepsilon)$ (with $\delta_0(\varepsilon)$ given by Theorem 10.7); then Theorem 10.7 provides a unique solution on $[0, T]$ obeying the estimates (10.24)–(10.25), and hence $\mathcal{G}_{2N}(T) \leq \delta$. According to Remark 10.8, all of the computations leading to Theorem 9.8 are justified by the estimates (10.24)–(10.25).

Let us now define

$$T_*(\kappa) = \sup\{T > 0 \mid \text{for every choice of initial data satisfying the compatibility conditions and } \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa \text{ there exists a unique solution on } [0,T] \text{ that achieves the data and satisfies } \mathcal{G}_{2N}(T) \le \delta\}. \tag{11.12}$$

By the above analysis, $T_*(\kappa)$ is well-defined and satisfies $T_*(\kappa) > 0$ if κ is small enough, i.e. there is a $\kappa_1 > 0$ so that $T_* : (0, \kappa_1] \to (0, \infty]$. It is easily verified that T_* is non-increasing on $(0, \kappa_1]$. Let us now set

$$\varepsilon = \frac{\delta}{3} \min\left\{ \frac{1}{1 + C_2}, \frac{1}{C_3} \right\} \tag{11.13}$$

and then define $\kappa_0 \in (0, \kappa_1]$ by

$$\kappa_0 = \min\left\{ \frac{\delta}{3C_1(C_3 + 2C_2)}, \frac{\delta_0(\varepsilon)}{C_1}, \kappa_1 \right\}, \tag{11.14}$$

where $\delta_0(\varepsilon)$ is given by Theorem 10.7 with ε given by (11.13). We claim that $T_*(\kappa_0) = \infty$. Once the claim is established, the proof of the theorem is complete since then $T_*(\kappa) = \infty$ for all $0 < \kappa \le \kappa_0$.

Suppose, by way of contradiction, that $T_*(\kappa_0) < \infty$. We will show that solutions can actually be extended past $T_*(\kappa_0)$ and that these solutions satisfy $\mathcal{G}_{2N}(T_2) \le \delta$ for $T_2 > T_*(\kappa_0)$, contradicting the definition of $T_*(\kappa_0)$. We begin by extending the solutions. By the definition of $T_*(\kappa_0)$, we know that for every $0 < T_1 < T_*(\kappa_0)$ and for any choice of data satisfying the compatibility conditions and the bound $\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa_0$, there exists a unique solution on $[0, T_1]$ that achieves the initial data and satisfies $\mathcal{G}_{2N}(T_1) \le \delta$. Then by Theorem 9.8, we know that actually

$$\mathcal{G}_{2N}(T_1) \le C_1(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) < C_1\kappa_0. \tag{11.15}$$

In particular, this and (11.14) imply that

$$\mathcal{E}_{2N}(T_1) + \frac{\mathcal{F}_{2N}(T_1)}{(1 + T_1)} < C_1\kappa_0 \le \delta_0(\varepsilon) \text{ for all } 0 < T_1 < T_*(\kappa_0), \tag{11.16}$$

where ε is given by (11.13). We view $(u(T_1), p(T_1), \eta(T_1))$ as initial data for a new problem; since (u, p, η) are already solutions, they satisfy the compatibility conditions needed to use them as data. Then since $\mathcal{E}_{2N}(T_1) < \delta_0(\varepsilon)$, we can use Theorem 10.7 with ε given by (11.13) to extend solutions to $[T_1, T_2]$ for any T_2 satisfying

$$0 < T_2 - T_1 \le T_0 = C(\varepsilon) \min\{1, \mathcal{F}_{2N}(T_1)^{-1}\}. \tag{11.17}$$

In light of (11.16), we may bound

$$\bar{T} := C(\varepsilon) \min\left\{ 1, \frac{1}{\delta_0(\varepsilon)(1 + T_*(\kappa_0))} \right\} \le T_0. \tag{11.18}$$

Notice that $\bar{T}$ depends on ε (given by (11.13)) and $T_*(\kappa_0)$, but is independent of T_1. Let

$$\gamma = \min\left\{ \bar{T}, T_*(\kappa_0), \frac{1}{(1 + 2T_*(\kappa_0))^{1+\lambda/2}} \right\}, \tag{11.19}$$

and then let us choose $T_1 = T_*(\kappa_0) - \gamma/2$ and $T_2 = T_*(\kappa_0) + \gamma/2$. The choice of γ implies that

$$0 < T_1 < T_*(\kappa_0) < T_2 < 2T_*(\kappa_0) \text{ and } 0 < \gamma = T_2 - T_1 \le \bar{T} \le T_0. \tag{11.20}$$

Then Theorem 10.7 allows us to extend solutions to the interval $[0, T_2]$, and it provides estimates on the extended interval $[T_1, T_2]$:

$$\begin{aligned} &\sup_{T_1 \le t \le T_2} \mathcal{E}_{2N}(t) + \sup_{T_1 \le t \le T_2} \|\mathcal{I}_\lambda p(t)\|_0^2 + \int_{T_1}^{T_2} \mathcal{D}_{2N}(t) dt \\ &+ \int_{T_1}^{T_2} \left(\left\| \partial_t^{2N+1} u(t) \right\|_{({}_0H^1)^*}^2 + \left\| \partial_t^{2N} p(t) \right\|_0^2 \right) dt \le C_2 \left(\varepsilon + \|\mathcal{I}_\lambda u(T_1)\|_0^2 + \|\mathcal{I}_\lambda \eta(T_1)\|_0^2 \right), \end{aligned} \tag{11.21}$$

$$\sup_{T_1 \le t \le T_2} \mathfrak{E}_{2N}(t) \le \varepsilon, \text{ and } \sup_{T_1 \le t \le T_2} \mathcal{F}_{2N}(t) \le C_2 \mathcal{F}_{2N}(T_1) + \varepsilon. \tag{11.22}$$

Having extended the existence interval, we will now show that $\mathcal{G}_{2N}(T_2) \le \delta$. We combine the estimates (11.21)–(11.22) with (11.15)–(11.16) and the bound (11.2) of Proposition 11.1 to see that

(11.23)
$$\mathcal{G}_{2N}(T_2) < C_1 C_3 \kappa_0 + C_2(\varepsilon + C_1\kappa_0) + \frac{C_1 C_2 \kappa_0 (1+T_1) + \varepsilon}{(1+T_1)} + \varepsilon C_3 (T_2 - T_1)^2 (1+T_2)^{2+\lambda}$$
$$\le \kappa_0 C_1 (C_3 + 2C_2) + \varepsilon(1+C_2) + \varepsilon C_3 \gamma^2 (1 + 2T_*(\kappa_0))^{2+\lambda} \le \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta,$$

where the second inequality follows from (11.20) and the third follows from the choice of ε, κ_0, and γ given in (11.13), (11.14), and (11.19), respectively. Hence $\mathcal{G}_{2N}(T_2) \le \delta$, contradicting the definition of $T_*(\kappa_0)$. We deduce then that $T_*(\kappa_0) = \infty$, which completes the proof of the claim and the theorem. □

With this result in hand, it is a simple matter to prove Theorem 1.3.

Proof of Theorem 1.3. We set $N = 5$ in Theorem 11.2 to deduce all of the conclusions of Theorem 1.3 except the estimates (1.19)–(1.20). Proposition 3.9 implies that

$$\|u\|^2_{C^2(\Omega)} \le C(r) (\mathcal{E}_8)^{r/(2+r)} (\mathcal{E}_{6,2})^{2/(2+r)} \tag{11.24}$$

for any $r \in (0,1)$, where $C(r) > 0$ is a constant depending on r. Let $0 \le \rho < \lambda$ and then choose $r \in (0,1)$ so that

$$0 < r \le 2\left(\frac{2+\lambda}{2+\rho}\right) - 2 \Rightarrow (2+\rho) \le (2+\lambda)\left(\frac{2}{2+r}\right). \tag{11.25}$$

Then $C(r) = C(\rho)$ and the bound $\mathcal{G}_8(\infty) \le C_1 \kappa$ implies that

$$\sup_{t\ge 0} (1+t)^{2+\rho} \|u(t)\|^2_{C^2(\Omega)} \le C(\rho) C_1 \kappa \sup_{t \ge 0} (1+t)^{2+\rho} \left(\frac{1}{(1+t)^{2+\lambda}}\right)^{2/(2+r)} \le C(\rho) C_1 \kappa, \tag{11.26}$$

which is (1.19). The estimate (1.20) follows similarly by using the interpolation estimates of Lemma 3.1 for the η terms and the interpolation estimates of Theorem 3.14 for $\|u\|_2^2$. In this case, though, no use of $r \in (0,1)$ is necessary because it does not appear in the interpolations. □

Appendix A. Analytic tools

A.1. **Products in Sobolev spaces.** We will need some estimates of the product of functions in Sobolev spaces.

Lemma A.1. *The following hold for sufficiently smooth subsets of* $\mathbb{R}^n$.

(1) *Let* $0 \le r \le s_1 \le s_2$ *be such that* $s_1 > n/2$. *Let* $f \in H^{s_1}$, $g \in H^{s_2}$. *Then* $fg \in H^r$ *and*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{A.1}$$

(2) *Let* $0 \le r \le s_1 \le s_2$ *be such that* $s_2 > r + n/2$. *Let* $f \in H^{s_1}$, $g \in H^{s_2}$. *Then* $fg \in H^r$ *and*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{A.2}$$

(3) *Let* $0 \le r \le s_1 \le s_2$ *be such that* $s_2 > r + n/2$. *Let* $f \in H^{-r}(\Sigma)$, $g \in H^{s_2}(\Sigma)$. *Then* $fg \in H^{-s_1}(\Sigma)$ *and*

$$\|fg\|_{-s_1} \lesssim \|f\|_{-r} \|g\|_{s_2}. \tag{A.3}$$

Proof. The proofs of (A.1) and (A.2) are standard; the bounds are first proved in $\mathbb{R}^n$ with the Fourier transform, and then the bounds in sufficiently nice subsets of $\mathbb{R}^n$ are deduced by use of an extension operator. To prove (A.3) we argue by duality. For $\varphi \in H^{s_1}$ we use (A.2)bound

$$\int_\Sigma \varphi f g \lesssim \|\varphi g\|_r \|f\|_{-r} \lesssim \|\varphi\|_{s_1} \|g\|_{s_2} \|f\|_{-r}, \tag{A.4}$$

so that taking the supremum over φ with $\|\varphi\|_{s_1} \le 1$ we get (A.3). □

We will also need the following variant.

Lemma A.2. *Suppose that $f \in C^1(\Sigma)$ and $g \in H^{1/2}(\Sigma)$. Then*

$$\|fg\|_{1/2} \lesssim \|f\|_{C^1} \|g\|_{1/2}. \tag{A.5}$$

Proof. Consider the operator $F : H^k \to H^k$ given by $F(g) = fg$ for $k = 0, 1$. It is a bounded operator for $k = 0, 1$ since

$$\|fg\|_0 \le \|f\|_{C^1} \|g\|_0 \text{ and } \|fg\|_1 \lesssim \|f\|_{C^1} \|g\|_1. \tag{A.6}$$

Then the theory of interpolation of operators implies that F is bounded from $H^{1/2}$ to itself, with operator norm less than a constant times $\sqrt{\|f\|_{C^1}}\sqrt{\|f\|_{C^1}} = \|f\|_{C^1}$, which is the desired result. □

A.2. **Estimates of the Riesz potential $\mathcal{I}_\lambda$.** Consider $\Omega = \mathbb{R}^2 \times (-b, 0)$ for $b > 0$. For a function f, defined on Ω, we define the Riesz potential $\mathcal{I}_\lambda f$ by

$$\mathcal{I}_\lambda f(x', x_3) = \int_{-b}^0 \int_{\mathbb{R}^2} \hat{f}(\xi, x_3) |\xi|^{-\lambda} e^{2\pi i x' \cdot \xi} d\xi dx_3. \tag{A.7}$$

Similarly, for f defined on Σ, we set

$$\mathcal{I}_\lambda f(x') = \int_{\mathbb{R}^2} \hat{f}(\xi) |\xi|^{-\lambda} e^{2\pi i x' \cdot \xi} d\xi. \tag{A.8}$$

We have a product estimate that is a fractional analog of the Leibniz rule.

Lemma A.3. *Let $\lambda \in (0, 1)$. If $f \in H^0(\Omega)$ and $g, Dg \in H^1(\Omega)$, then*

$$\|\mathcal{I}_\lambda(fg)\|_0 \lesssim \|f\|_0 \|g\|_1^\lambda \|Dg\|_1^{1-\lambda}. \tag{A.9}$$

If $f \in H^0(\Sigma)$ and $g \in H^1(\Sigma)$, then

$$\|\mathcal{I}_\lambda(fg)\|_{H^0(\Sigma)} \lesssim \|f\|_{H^0(\Sigma)} \|g\|_{H^0(\Sigma)}^\lambda \|Dg\|_{H^0(\Sigma)}^{1-\lambda}. \tag{A.10}$$

Proof. The Hardy-Littlewood-Sobolev inequality (see, for example, Theorem 4.3 of [19]) implies that $\mathcal{I}_\lambda : L^{2/(1+\lambda)}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a bounded linear operator for $\lambda \in (0, 1)$. We may then employ Fubini and apply this result on each slice $\{x_3 = z\}$ for $z \in (-b, 0)$ to estimate

$$\begin{aligned} \int_\Omega |\mathcal{I}_\lambda(fg)|^2 &= \int_{-b}^0 \int_{\mathbb{R}^2} |\mathcal{I}_\lambda(fg)|^2 dx' dx_3 \lesssim \int_{-b}^0 \left(\int_{\mathbb{R}^2} |fg|^{2/(1+\lambda)} dx' \right)^{1+\lambda} dx_3 \\ &\le \int_{-b}^0 \left(\int_{\mathbb{R}^2} |f|^2 dx' \right) \left(\int_{\mathbb{R}^2} |g|^{2/\lambda} dx' \right)^\lambda dx_3 \le \sup_{-b \le x_3 \le 0} \|g(\cdot, x_3)\|_{L^{2/\lambda}(\mathbb{R}^2)}^2 \int_\Omega |f|^2, \end{aligned} \tag{A.11}$$

where in the second inequality we have applied Hölder's inequality. By the Gagliardo-Nirenberg interpolation inequality on $\mathbb{R}^2$ we may bound

$$\|g(\cdot, x_3)\|_{L^{2/\lambda}(\mathbb{R}^2)} \lesssim \|g(\cdot, x_3)\|_{L^2(\mathbb{R}^2)}^\lambda \|Dg(\cdot, x_3)\|_{L^2(\mathbb{R}^2)}^{1-\lambda}, \tag{A.12}$$

but by trace theory we also have

$$\|g(\cdot, x_3)\|_{L^2(\mathbb{R}^2)} \lesssim \|g\|_1 \text{ and } \|Dg(\cdot, x_3)\|_{L^2(\mathbb{R}^2)} \lesssim \|Dg\|_1, \tag{A.13}$$

so that

$$\sup_{-b \le x_3 \le 0} \|g(\cdot, x_3)\|_{L^{2/\lambda}(\mathbb{R}^2)}^2 \lesssim \|g\|_1^\lambda \|Dg\|_1^{1-\lambda}. \tag{A.14}$$

Chaining together (A.11) and (A.14) then yields the estimate (A.9). A similar argument, not employing Fubini or trace theory, provides the estimate (A.10). □

Our next result shows how $\mathcal{I}_\lambda$ interacts with horizontal derivatives in Ω.

Lemma A.4. *Let $\lambda \in (0,1)$. If $f \in H^k(\Omega)$ for $k \geq 1$ an integer, then*

$$\left\| \mathcal{I}_\lambda D^k f \right\|_0 \lesssim \left\| D^{k-1} f \right\|_0^\lambda \left\| D^k f \right\|_0^{1-\lambda}. \tag{A.15}$$

Proof. On a fixed horizontal slice $\{x_3 = z\}$ for $z \in (-b, 0)$, Parseval's theorem implies that

$$\begin{aligned} \int_{\mathbb{R}^2} \left| \mathcal{I}_\lambda D^k f(x', x_3) \right|^2 dx' &\lesssim \int_{\mathbb{R}^2} |\xi|^{2(k-\lambda)} \left| \hat{f}(\xi, x_3) \right|^2 d\xi \\ &= \int_{\mathbb{R}^2} \left(|\xi|^{2(k-1)} \left| \hat{f}(\xi, x_3) \right|^2 \right)^\lambda \left(|\xi|^{2k} \left| \hat{f}(\xi, x_3) \right|^2 \right)^{1-\lambda} d\xi \\ &\lesssim \left(\int_{\mathbb{R}^2} \left| D^{k-1} f(x', x_3) \right|^2 dx' \right)^\lambda \left(\int_{\mathbb{R}^2} \left| D^k f(x', x_3) \right|^2 dx' \right)^{1-\lambda}. \end{aligned} \tag{A.16}$$

Here in the second inequality we have used Hölder and Parseval. Integrating both sides of this inequality with respect to $x_3 \in (-b, 0)$ and again applying Hölder's inequality yields the estimate (A.15). □

A.3. **Poisson integral.** For a function f defined on $\Sigma = \mathbb{R}^2$, the Poisson integral in $\mathbb{R}^2 \times (-\infty, 0)$ is defined by

$$\mathcal{P} f(x', x_3) = \int_{\mathbb{R}^2} \hat{f}(\xi) e^{2\pi |\xi| x_3} e^{2\pi i x' \cdot \xi} d\xi. \tag{A.17}$$

Although $\mathcal{P} f$ is defined in all of $\mathbb{R}^2 \times (-\infty, 0)$, we will only need bounds on its norm in the restricted domain $\Omega = \mathbb{R}^2 \times (-b, 0)$. This yields a couple improvements of the usual estimates of $\mathcal{P} f$ on the set $\mathbb{R}^2 \times (-\infty, 0)$.

Lemma A.5. *Let $\mathcal{P} f$ be the Poisson integral of a function f that is either in $\dot{H}^q(\Sigma)$ or $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$ (here $\dot{H}^s$ is the usual homogeneous Sobolev space of order s). Then*

$$\left\| \nabla^q \mathcal{P} f \right\|_0^2 \lesssim \int_{\mathbb{R}^2} |\xi|^{2q} \left| \hat{f}(\xi) \right|^2 \left(\frac{1 - e^{-4\pi b |\xi|}}{|\xi|} \right) d\xi, \tag{A.18}$$

and in particular

$$\left\| \nabla^q \mathcal{P} f \right\|_0^2 \lesssim \|f\|_{\dot{H}^{q-1/2}(\Sigma)}^2 \text{ and } \left\| \nabla^q \mathcal{P} f \right\|_0^2 \lesssim \|f\|_{\dot{H}^q(\Sigma)}^2. \tag{A.19}$$

Proof. Employing Fubini, the horizontal Fourier transform, and Parseval, we may bound

$$\begin{aligned} \left\| \nabla^q \mathcal{P} f \right\|_0^2 \lesssim \int_{\mathbb{R}^2} \int_{-b}^0 |\xi|^{2q} \left| \hat{f}(\xi) \right|^2 e^{4\pi |\xi| x_3} dx_3 d\xi &\leq \int_{\mathbb{R}^2} |\xi|^{2q} \left| \hat{f}(\xi) \right|^2 \left(\int_{-b}^0 e^{4\pi |\xi| x_3} dx_3 \right) d\xi \\ &\lesssim \int_{\mathbb{R}^2} |\xi|^{2q} \left| \hat{f}(\xi) \right|^2 \left(\frac{1 - e^{-4\pi b |\xi|}}{|\xi|} \right) d\xi. \end{aligned} \tag{A.20}$$

This is (A.18). To deduce (A.19) from (A.18), we simply note that

$$\frac{1 - e^{-4\pi b |\xi|}}{|\xi|} \leq \min \left\{ 4\pi b, \frac{1}{|\xi|} \right\}, \tag{A.21}$$

which means we are free to bound the right hand side of (A.20) by either $\|f\|_{\dot{H}^{q-1/2}(\Sigma)}^2$ or $\|f\|_{\dot{H}^q(\Sigma)}^2$. □

A.4. **Interpolation estimates.** Assume that $\Sigma = \mathbb{R}^2$ and $\Omega = \Sigma \times (-b, 0)$. We begin with an interpolation result for Poisson integrals, as defined by A.17.

Lemma A.6. *Let $\mathcal{P}f$ be the Poisson integral of f, defined on Σ. Let $\lambda \geq 0$, $q, s \in \mathbb{N}$, and $r \geq 0$. Then the following estimates hold.*

(1) *Let*

$$\theta = \frac{s}{q+s+\lambda} \text{ and } 1-\theta = \frac{q+\lambda}{q+s+\lambda}. \tag{A.22}$$

Then

$$\|\nabla^q \mathcal{P}f\|_0^2 \lesssim \left(\|\mathcal{I}_\lambda f\|_0^2\right)^\theta \left(\left\|D^{q+s} f\right\|_0^2\right)^{1-\theta}. \tag{A.23}$$

(2) *Let $r+s>1$,*

$$\theta = \frac{r+s-1}{q+s+r+\lambda}, \text{ and } 1-\theta = \frac{q+\lambda+1}{q+s+r+\lambda}. \tag{A.24}$$

Then

$$\|\nabla^q \mathcal{P}f\|_{L^\infty}^2 \lesssim \left(\|\mathcal{I}_\lambda f\|_0^2\right)^\theta \left(\left\|D^{q+s} f\right\|_r^2\right)^{1-\theta}. \tag{A.25}$$

(3) *Let $s>1$. Then*

$$\|\nabla^q \mathcal{P}f\|_{L^\infty}^2 \lesssim \|D^q f\|_s^2. \tag{A.26}$$

Proof. Employing Fubini, the horizontal Fourier transform, and Parseval, we may bound

$$\begin{aligned} \|\nabla^q \mathcal{P}f\|_0^2 &\lesssim \int_{\mathbb{R}^2}\int_{-b}^0 |\xi|^{2q}\left|\hat{f}(\xi)\right|^2 e^{4\pi|\xi|x_3} dx_3 d\xi \lesssim \int_{\mathbb{R}^2} |\xi|^{2q}\left|\hat{f}(\xi)\right|^2 d\xi. \\ &= \int_{\mathbb{R}^2} \left(|\xi|^{2(q+s)}\left|\hat{f}(\xi)\right|^2\right)^\theta \left(|\xi|^{-2\lambda}\left|\hat{f}(\xi)\right|^2\right)^{1-\theta} d\xi \end{aligned} \tag{A.27}$$

for θ and $1-\theta$ defined by (A.22). An application of Hölder's inequality and a second application of Parseval's theorem then provides the estimate (A.23).

For the L^∞ estimate (A.25), we use the definition of $\mathcal{P}f$ in conjunction with the trivial estimate $\exp(2\pi|\xi|x_3) \leq 1$ in Ω to bound

$$\|\nabla^q \mathcal{P}f\|_{L^\infty} \lesssim \int_{\mathbb{R}^2} |\xi|^q \left|\hat{f}(\xi)\right| d\xi. \tag{A.28}$$

For $R>0$ we split into high and low frequencies to see that

$$\begin{aligned} \int_{\mathbb{R}^2} |\xi|^q\left|\hat{f}(\xi)\right| d\xi &= \int_{B_R} |\xi|^{q+\lambda}|\xi|^{-\lambda}\left|\hat{f}(\xi)\right| d\xi + \int_{B_R^c} |\xi|^{q+s}\langle\xi\rangle^r\langle\xi\rangle^{-r}|\xi|^{-s}\left|\hat{f}(\xi)\right| d\xi \\ &\leq \left(\int_{B_R}|\xi|^{2(q+\lambda)} d\xi\right)^{1/2} \|\mathcal{I}_\lambda f\|_0 + \left(\int_{B_R^c}|\xi|^{-2s}\langle\xi\rangle^{-2r} d\xi\right)^{1/2} \left\|D^{q+s}f\right\|_r \\ &\lesssim R^{q+\lambda+1}\|\mathcal{I}_\lambda f\|_0 + R^{-(r+s-1)}\left\|D^{q+s}f\right\|_r. \end{aligned} \tag{A.29}$$

The condition $r+s>1$ guarantees that integral over B_R^c is finite. Minimizing the right side with respect to $R \in (0,\infty)$ then yields (A.25).

The estimate (A.26) follows from the easy bound

$$\int_{\mathbb{R}^2} |\xi|^q\left|\hat{f}(\xi)\right| d\xi \lesssim \|D^q f\|_s \left(\int_{\mathbb{R}^2}\langle\xi\rangle^{-2s} d\xi\right)^{1/2} \lesssim \|D^q f\|_s, \tag{A.30}$$

which holds when $s>1$. □

The next result is a similar interpolation result for functions defined only on Σ.

Lemma A.7. *Let f be defined on Σ. Let $\lambda \geq 0$. Then the following estimates hold.*

(1) *Let* $q, s \in (0, \infty)$ *and*

$$\theta = \frac{s}{q+s+\lambda} \text{ and } 1-\theta = \frac{q+\lambda}{q+s+\lambda}. \tag{A.31}$$

Then

$$\|D^q f\|_0^2 \lesssim \left(\|\mathcal{I}_\lambda f\|_0^2\right)^\theta \left(\left\|D^{q+s} f\right\|_0^2\right)^{1-\theta}. \tag{A.32}$$

(2) *Let* $q, s \in \mathbb{N}$, $r \geq 0$, $r+s > 1$,

$$\theta = \frac{r+s-1}{q+s+r+\lambda}, \text{ and } 1-\theta = \frac{q+\lambda+1}{q+s+r+\lambda}. \tag{A.33}$$

Then

$$\|D^q f\|_{L^\infty}^2 \lesssim \left(\|\mathcal{I}_\lambda f\|_0^2\right)^\theta \left(\left\|D^{q+s} f\right\|_r^2\right)^{1-\theta}. \tag{A.34}$$

Proof. For the H^0 estimate we use

$$\|D^q f\|_0^2 \lesssim \int_{\mathbb{R}^2} |\xi|^{2q} \left|\hat{f}(\xi)\right|^2 d\xi \tag{A.35}$$

and argue as in Lemma A.6. For the L^∞ estimate we bound

$$\|D^q f\|_{L^\infty} \lesssim \int_{\mathbb{R}^2} |\xi|^q \left|\hat{f}(\xi)\right| d\xi. \tag{A.36}$$

and again argue as in Lemma A.6. □

Now we record a similar result for functions defined on Ω that are not Poisson integrals. The result follows from estimates on fixed horizontal slices.

Lemma A.8. *Let f be a function on Ω. Let $\lambda \geq 0$, $q, s \in \mathbb{N}$, and $r \geq 0$. Then the following estimates hold.*

(1) *Let*

$$\theta = \frac{s}{q+s+\lambda} \text{ and } 1-\theta = \frac{q+\lambda}{q+s+\lambda}. \tag{A.37}$$

Then

$$\|D^q f\|_0^2 \lesssim \left(\|\mathcal{I}_\lambda f\|_0^2\right)^\theta \left(\left\|D^{q+s} f\right\|_0^2\right)^{1-\theta}. \tag{A.38}$$

(2) *Let* $r+s > 1$,

$$\theta = \frac{r+s-1}{q+s+r+\lambda}, \text{ and } 1-\theta = \frac{q+\lambda+1}{q+s+r+\lambda}. \tag{A.39}$$

Then

$$\|D^q f\|_{L^\infty}^2 \lesssim \left(\|\mathcal{I}_\lambda f\|_1^2\right)^\theta \left(\left\|D^{q+s} f\right\|_{r+1}^2\right)^{1-\theta} \tag{A.40}$$

and

$$\|D^q f\|_{L^\infty(\Sigma)}^2 \lesssim \left(\|\mathcal{I}_\lambda f\|_1^2\right)^\theta \left(\left\|D^{q+s} f\right\|_{r+1}^2\right)^{1-\theta} \tag{A.41}$$

Proof. We employ the horizontal Fourier transform and Parseval in conjunction with Fubini to bound

$$\|D^q f\|_0^2 \lesssim \int_{-b}^0 \int_{\mathbb{R}^2} |\xi|^{2q} \left|\hat{f}(\xi, x_3)\right|^2 d\xi dx_3. \tag{A.42}$$

For a fixed x_3 we may argue as in Lemma A.6 to show that

$$\int_{\mathbb{R}^2} |\xi|^{2q} \left|\hat{f}(\xi, x_3)\right|^2 d\xi \leq \left(\|\mathcal{I}_\lambda f(\cdot, x_3)\|_0^2\right)^\theta \left(\left\|D^{q+s} f(\cdot, x_3)\right\|_0^2\right)^{1-\theta} \tag{A.43}$$

for θ and $1-\theta$ given by (A.37). Combining these two inequalities with Hölder's inequality then shows that

$$\|D^q f\|_0^2 \lesssim \int_{-b}^0 \left(\|\mathcal{I}_\lambda f(\cdot,x_3)\|_0^2\right)^\theta \left(\left\|D^{q+s} f(\cdot,x_3)\right\|_0^2\right)^{1-\theta} dx_3 \leq \left(\|\mathcal{I}_\lambda f\|_0^2\right)^\theta \left(\left\|D^{q+s} f\right\|_0^2\right)^{1-\theta}, \tag{A.44}$$

which is (A.38).

Now for the L^∞ estimate we first work on a horizontal slice $\{x_3 = z\}$ for some $z \in [-b,0]$. Indeed, using the horizontal Fourier transform on the slice, we have

$$\|D^q f(\cdot,x_3)\|_{L^\infty} \lesssim \int_{\mathbb{R}^2} |\xi|^q \left|\hat{f}(\xi,x_3)\right| d\xi. \tag{A.45}$$

We may then argue as in Lemma A.6 to show that

$$\int_{\mathbb{R}^2} |\xi|^q \left|\hat{f}(\xi,x_3)\right| d\xi \lesssim \left(\|\mathcal{I}_\lambda f(\cdot,x_3)\|_0\right)^\theta \left(\left\|D^{q+s} f(\cdot,x_3)\right\|_r\right)^{1-\theta} \tag{A.46}$$

for θ and $1-\theta$ given by (A.39). By the usual trace theory

$$\|\mathcal{I}_\lambda f(\cdot,x_3)\|_0 \lesssim \|\mathcal{I}_\lambda f\|_1 \text{ and } \left\|D^{q+s} f(\cdot,x_3)\right\|_r \lesssim \left\|D^{q+s} f\right\|_{r+1}. \tag{A.47}$$

Combining (A.45)–(A.47) and taking the supremum over $x_3 \in [-b,0]$ then gives (A.40). A similar argument yields (A.41). □

A.5. **Transport estimate.** Consider the equation

$$\begin{cases} \partial_t \eta + u \cdot D\eta = g & \text{in } \Sigma \times (0,T) \\ \eta(t=0) = \eta_0 \end{cases} \tag{A.48}$$

with $T \in (0,\infty]$. We have the following estimate of the transport of regularity for solutions to (A.48), which is a particular case of a more general result proved in [10].

Lemma A.9 (Proposition 2.1 of [10])**.** *Let η be a solution to* (A.48)*. Then there is a universal constant $C > 0$ so that for any $0 \leq s < 2$*

$$\sup_{0\leq r\leq t} \|\eta(r)\|_{H^s} \leq \exp\left(C\int_0^t \|Du(r)\|_{H^{3/2}} dr\right)\left(\|\eta_0\|_{H^s} + \int_0^t \|g(r)\|_{H^s} dr\right). \tag{A.49}$$

Proof. Use $p = p_2 = 2$, $N = 2$, and $\sigma = s$ in Proposition 2.1 of [10] along with the embedding $H^{3/2} \hookrightarrow B^1_{2,\infty} \cap L^\infty$. □

A.6. **Poincaré-type inequalities.** Let Σ and Ω be as before.

Lemma A.10. *It holds that*

$$\|f\|^2_{L^2(\Omega)} \lesssim \|f\|^2_{L^2(\Sigma)} + \|\partial_3 f\|^2_{L^2(\Omega)} \tag{A.50}$$

for all $f \in H^1(\Omega)$. Also, if $f \in W^{1,\infty}(\Omega)$, then

$$\|f\|^2_{L^\infty(\Omega)} \lesssim \|f\|^2_{L^\infty(\Sigma)} + \|\partial_3 f\|^2_{L^\infty(\Omega)}. \tag{A.51}$$

Proof. By density we may assume that f is smooth. Writing $x = (x', x_3)$ for $x' \in \Sigma$ and $x_3 \in (-b(x'),0)$, we have

$$\left|f(x',x_3)\right|^2 = \left|f(x',0)\right|^2 - 2\int_{x_3}^0 f(x',z)\partial_3 f(x',z) dz \leq \left|f(x',0)\right|^2 + 2\int_{-b(x')}^0 \left|f(x',z)\right| \left|\partial_3 f(x',z)\right| dz. \tag{A.52}$$

We may integrate this with respect to $x_3 \in (-b(x'),0)$ to get

$$\int_{-b(x')}^0 \left|f(x',x_3)\right|^2 dx_3 \lesssim \left|f(x',0)\right|^2 + 2\int_{-b(x')}^0 \left|f(x',z)\right| \left|\partial_3 f(x',z)\right| dz. \tag{A.53}$$

Now we integrate over $x' \in \Sigma$ to find

$$\text{(A.54)} \quad \int_\Omega |f(x)|^2 \, dx \lesssim \|f\|_{L^2(\Sigma)}^2 + 2 \int_\Omega |f(x)| \, |\partial_3 f(x)| \, dx \\ \leq \|f\|_{L^2(\Sigma)}^2 + \varepsilon \|f\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon} \|\partial_3 f\|_{L^2(\Omega)}^2$$

for any $\varepsilon > 0$. Choosing $\varepsilon > 0$ sufficiently small then yields (A.50). The estimate (A.51) follows similarly, taking suprema rather than integrating. □

A simple modification of the proof of Lemma A.10 yields the following estimates.

Lemma A.11. *It holds that $\|f\|_{H^0(\Sigma)} \lesssim \|\partial_3 f\|_{H^0(\Omega)}$ for $f \in H^1(\Omega)$ so that $f = 0$ on Σ_b. It also holds that $\|f\|_{L^\infty(\Sigma)} \lesssim \|\partial_3 f\|_{L^\infty(\Omega)}$ for $f \in W^{1,\infty}(\Omega)$ so that $f = 0$ on Σ_b.*

We will need a version of Korn's inequality, which is proved, for instance, in Lemma 2.7 [4].

Lemma A.12. *It holds that $\|u\|_1 \lesssim \|\mathbb{D}u\|_0$ for all $u \in H^1(\Omega; \mathbb{R}^3)$ so that $u = 0$ on Σ_b.*

We also record the standard Poincaré inequality, which applies for functions taking either vector or scalar values.

Lemma A.13. *It holds that $\|f\|_0 \lesssim \|f\|_1 \lesssim \|\nabla f\|_0$ for all $f \in H^1(\Omega)$ so that $f = 0$ on Σ_b. Also, $\|f\|_{L^\infty(\Omega)} \lesssim \|f\|_{W^{1,\infty}(\Omega)} \lesssim \|\nabla f\|_{L^\infty(\Omega)}$ for all $f \in W^{1,\infty}(\Omega)$ so that $f = 0$ on Σ_b.*

A.7. **An elliptic estimate.** The proof of the following estimate may be found in [4].

Lemma A.14. *Suppose (u, p) solve*

$$\text{(A.55)} \quad \begin{cases} -\Delta u + \nabla p = \phi \in H^{r-2}(\Omega) \\ \operatorname{div} u = \psi \in H^{r-1}(\Omega) \\ (pI - \mathbb{D}(u)) e_3 = \alpha \in H^{r-3/2}(\Sigma) \\ u|_{\Sigma_b} = 0. \end{cases}$$

Then for $r \geq 2$,

$$\text{(A.56)} \quad \|u\|_{H^r}^2 + \|p\|_{H^{r-1}}^2 \lesssim \|\phi\|_{H^{r-2}}^2 + \|\psi\|_{H^{r-1}}^2 + \|\alpha\|_{H^{r-3/2}}^2 .$$

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