Quasilinear symmetric hyperbolic systems

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Abstract

These notes develop an existence theory for quasilinear symmetric hyperbolic systems. The well-posedness theory for the corresponding linear problem is developed using the method originally employed by Friedrichs [4] in his pioneering work on symmetric hyperbolic systems. The nonlinear problem is then solved using a contraction-mapping argument developed by Kato [5] and Fischer-Marsden [3]. We then present several applications: quasilinear wave equations, the wave map problem, the compressible Euler system, the compressible magnetohydrodynamics system, and the shallow water equations.

Contents

1 Linear problem redux

Our goal here is to return to the linear theory and prove some higher-order spatial estimates that are optimized for the quasilinear analysis. First we need some technical results.

1.1 Some technical results

We begin with a product estimate.

Proposition 1.1.1. *Let $n/2 < k \in \mathbb{N}$ and $1 \le i \in \mathbb{N}$. Assume that $f_1, \ldots, f_i \in H^k(\mathbb{R}^n; \mathbb{R})$. Let $\beta_1, \ldots, \beta_i \in \mathbb{N}^n$ be such that $|\beta_1| + \cdots + |\beta_i| = \ell \in \{0, \ldots, k\}$. Then $\partial^{\beta_1} f_1 \cdots \partial^{\beta_i} f_i \in L^2$ and*

$$\left\| \prod_{j=1}^{i} \partial^{\beta_j} f_j \right\|_{L^2} \lesssim \prod_{j=1}^{i} \|f_j\|_{H^k}. \tag{1.1.1}$$

Proof. The result is trivial if $i = 1$, so we may assume that $i \ge 2$. Define

$$A = \{j \in \{1, \ldots, i\} \mid |\beta_j| < k - n/2\} \text{ and } B = \{j \in \{1, \ldots, i\} \mid |\beta_j| \ge k - n/2\}. \tag{1.1.2}$$

If $j \in A$ then $\partial^{\beta_j} f_j \in H^{k-|\beta_j|} \hookrightarrow L^{q_j}$ for all $2 \le q_j \le \infty$. On the other hand, if $j \in B$ then $\partial^{\beta_j} f_j \in H^{k-|\beta_j|} \hookrightarrow L^{p_j}$ for all $2 \le p_j < p_j^*$, where

$$\frac{1}{p_j^*} = \frac{1}{2} - \frac{(k - |\beta_j|)}{n} \in [0, 1/2]. \tag{1.1.3}$$

Note here that we restrict to $p_j < p_j^*$ to easily handle the case of criticality, when $|\beta_j| = k - n/2$.

We now break to cases. First we consider the case in which $|B| \le 1$ and thus $|A| \in \{i-1, i\}$. If $|B| = 1$ we may assume without loss of generality that $\beta_1 \in B$. We may then estimate

$$\left\| \prod_{j=1}^{i} \partial^{\beta_j} f_j \right\|_{L^2} \le \left\| \partial^{\beta_1} f_1 \right\|_{L^2} \left\| \prod_{j=2}^{i} \partial^{\beta_j} f_j \right\|_{L^\infty} \lesssim \prod_{j=1}^{i} \|f_j\|_{H^k}. \tag{1.1.4}$$

This proves the result when $|B| \le 1$.

Now assume that $A = \varnothing$ and thus $B = \{1, \ldots, i\}$. Suppose that for $j = 1, \ldots, i$ we choose $2 \le p_j < p_j^*$ and set

$$\frac{1}{r} = \sum_{j=1}^{i} \frac{1}{p_j}. \tag{1.1.5}$$

By adjusting the p_j we may force $1/r$ to obtain any value $1/r_* < 1/r \le i/2$, where

$$\frac{1}{r_*} = \left(\frac{1}{2} - \frac{k}{n}\right) i + \frac{\ell}{n} = \sum_{j=1}^{i} \frac{1}{p_j^*}. \tag{1.1.6}$$

Now, since $i \ge 2$, $k - n/2 > 0$, and $0 \le \ell \le k$, we have that

$$\ell - \frac{n}{2} < \left(k - \frac{n}{2}\right) i \Rightarrow \left(\frac{n}{2} - k\right) i + \ell < \frac{n}{2} \Rightarrow \left(\frac{1}{2} - \frac{k}{n}\right) i + \frac{\ell}{n} < \frac{1}{2}. \tag{1.1.7}$$

Thus $r_* > 2$, and so we may choose the p_j such that $1/r = 2$. Hence Hölder's inequality tells us that $\partial^{\beta_1} f_1 \cdots \partial^{\beta_i} f_i \in L^2$ and

$$\left\| \prod_{j=1}^{i} \partial^{\beta_j} f_j \right\|_{L^2} \le \prod_{j=1}^{i} \left\| \partial^{\beta_j} f_j \right\|_{L^{p_j}} \lesssim \prod_{j=1}^{i} \| f_j \|_{H^k}. \tag{1.1.8}$$

This proves the result when $A = \varnothing$.

Now suppose that $A \neq \varnothing$ and $|B| \ge 2$. Suppose that for $j \in A$ we choose $2 \le q_j \le \infty$ and for $j \in B$ we choose $2 \le p_j < p_j^*$ and set

$$\frac{1}{s} = \sum_{j \in A} \frac{1}{q_j} \text{ and } \frac{1}{r} = \sum_{j \in B} \frac{1}{p_j}. \tag{1.1.9}$$

We can adjust the p_j to achieve any value $1/r_* < 1/r \le |B|/2$, where now

$$\frac{1}{r_*} = \left(\frac{1}{2} - \frac{k}{n} \right) |B| + \frac{1}{n} \sum_{j \in B} |\beta_j| = \sum_{j \in B} \frac{1}{p_j^*}. \tag{1.1.10}$$

Similarly, we can adjust the q_j to achieve any value $0 \le 1/s \le |A|/2$. Consequently we can choose the q_j and p_j such that

$$\frac{1}{r_*} < \frac{1}{r} + \frac{1}{s} \le \frac{|A| + |B|}{2} = \frac{i}{2} \tag{1.1.11}$$

and $1/r + 1/s$ can be forced to take on any value in the range $(1/r_*, 1]$. Now, $|B| \ge 2$ and $\sum_{j \in B} |\beta_j| \le \ell$ implies that

$$\sum_{j \in B} |\beta_j| - \frac{n}{2} < \left(k - \frac{n}{2} \right) |B| \Rightarrow \left(\frac{n}{2} - k \right) |B| + \sum_{j \in B} |\beta_j| < \frac{n}{2} \Rightarrow \left(\frac{1}{2} - \frac{k}{n} \right) |B| + \frac{1}{n} \sum_{j \in B} |\beta_j| < \frac{1}{2} \tag{1.1.12}$$

and thus $r_* > 2$. We may thus choose the q_j and p_j such that $1/r + 1/s = 2$ and hence Hölder implies that

$$\left\| \prod_{j=1}^{i} \partial^{\beta_j} f_j \right\|_{L^2} = \prod_{j \in A} \left\| \partial^{\beta_j} f_j \right\|_{L^{q_j}} \prod_{j \in B} \left\| \partial^{\beta_j} f_j \right\|_{L^{p_j}} \lesssim \prod_{j=1}^{i} \| f_j \|_{H^k}. \tag{1.1.13}$$

This proves the result when $A \neq \varnothing$ and $B \neq \varnothing$, which was the last remaining case.

□

As a consequence we have the following important result.

Proposition 1.1.2. *Assume that $k > n/2$. Then the following hold.*

1. *$H^k(\mathbb{R}^n; \mathbb{R}^{m \times m})$ is an algebra for each $1 \le m$ and*

$$\|AB\|_{H^k} \lesssim \|A\|_{H^k} \|B\|_{H^k}. \tag{1.1.14}$$

2. *If $A \in H^k(\mathbb{R}^n; \mathbb{R}^{m \times m})$ and $f \in H^k(\mathbb{R}^n; \mathbb{R}^m)$, then $Af \in H^k(\mathbb{R}^n; \mathbb{R}^m)$ and*

$$\|Af\|_{H^k} \lesssim \|A\|_{H^k} \|f\|_{H^k}. \tag{1.1.15}$$

In particular, $A \in \mathcal{L}(H^k(\mathbb{R}^n; \mathbb{R}^m))$ and

$$\|A\|_{\mathcal{L}(H^k)} \lesssim \|A\|_{H^k}. \tag{1.1.16}$$

Proof. These all follow directly from Proposition 1.1.1. □

We will also need the following variant of Proposition 1.1.1.

Proposition 1.1.3. *Let $k > 1 + n/2$ and $0 \le \ell \le k-1$. Suppose that $\nabla f \in H^{k-1}$ and $g \in H^{\ell}(\mathbb{R}^n)$. Let $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha| + |\beta| \le \ell + 1$ and $|\alpha| \ge 1$. Then*

$$\left\|\partial^\alpha f \partial^\beta g\right\|_{L^2} \lesssim \|\nabla f\|_{H^{k-1}} \|g\|_{H^\ell}. \tag{1.1.17}$$

Proof. First note that $|\alpha| \ge 1$ implies that $|\beta| \le \ell$. Then

$$\partial^\alpha f \in H^{k-|\alpha|} \text{ and } \partial^\beta g \in H^{\ell - |\beta|}. \tag{1.1.18}$$

If $k - |\alpha| > n/2$ then we may use the Sobolev L^∞ embedding to estimate

$$\left\|\partial^\alpha f \partial^\beta g\right\|_{L^2} \le \|\partial^\alpha f\|_{L^\infty} \left\|\partial^\beta g\right\|_{L^2} \lesssim \|\nabla f\|_{H^{k-1}} \|g\|_{H^\ell}. \tag{1.1.19}$$

Similarly, if $\ell - |\beta| > n/2$ then we may bound

$$\left\|\partial^\alpha f \partial^\beta g\right\|_{L^2} \le \|\partial^\alpha f\|_{L^2} \left\|\partial^\beta g\right\|_{L^\infty} \lesssim \|\nabla f\|_{H^{k-1}} \|g\|_{H^\ell}. \tag{1.1.20}$$

We thus reduce to proving the result under the assumption that

$$k - |\alpha| \le \frac{n}{2} \text{ and } \ell - |\beta| \le \frac{n}{2}. \tag{1.1.21}$$

Assume this.

We use the Sobolev embeddings to guarantee that

$$\partial^\alpha f \in L^q \text{ for } 2 \le q \le q_0 \text{ and } \partial^\beta g \in L^p \text{ for } 2 \le p \le p_0 \tag{1.1.22}$$

where

$$\frac{1}{q_0} = \frac{n - 2(k - |\alpha|)}{2n} \text{ and } \frac{1}{p_0} = \frac{n - 2(\ell - |\beta|)}{2n}. \tag{1.1.23}$$

The Hölder inequality then guarantees that $\partial^\alpha f \partial^\beta g \in L^r$ for

$$\frac{n - (k + \ell - |\alpha| - |\beta|)}{n} = \frac{1}{q_0} + \frac{1}{p_0} \le \frac{1}{r} \le 1, \tag{1.1.24}$$

i.e. for

$$1 \le r \le \frac{n}{n - (k + \ell - |\alpha| - |\beta|)}. \tag{1.1.25}$$

Since $k > 1 + n/2$ and $|\alpha| + |\beta| \le \ell + 1$ we have that

$$\frac{n}{n - (k + \ell - |\alpha| - |\beta|)} \ge \frac{n}{n - (k-1)} > 2, \tag{1.1.26}$$

and so we find that $\partial^\alpha f \partial^\beta g \in L^2$ with the estimate

$$\left\|\partial^\alpha f \partial^\beta g\right\|_{L^2} \lesssim \|\nabla f\|_{H^{k-1}} \|g\|_{H^\ell}. \tag{1.1.27}$$

□

Next we deal with a higher-order chain rule. We first need a definition.

Definition 1.1.4. *Let X, Y, Z be normed vector spaces. Assume that $1 \leq i \in \mathbb{N}$ and $j_1, \ldots, j_i \in \mathbb{N}\backslash\{0\}$ satisfy $j_1 + \cdots + j_i = k$. We define*

$$\Lambda^i_{j_1,\ldots,j_i} : \mathcal{L}^i(Y;Z) \times \mathcal{L}^{j_1}(X;Y) \times \cdots \times \mathcal{L}^{j_i}(X;Y) \to \mathcal{L}^k(X;Z) \tag{1.1.28}$$

via

$$\begin{aligned}\Lambda^i_{j_1,\ldots,j_i}&(A, B_1, \ldots, B_i)(v_1, \ldots, v_k) \\ &= A[B_1(v_1, \ldots, v_{j_1}), B_2(v_{j_1+1}, \ldots, v_{j_2}), \ldots, B_i(v_{j_1+\cdots+j_{i-1}+1}, \ldots, v_k)].\end{aligned} \tag{1.1.29}$$

We also define that map $\mathcal{S}^k : \mathcal{L}^k(X;Y) \to \mathcal{L}^k_{sym}(X;Y)$ to be the symmetrization operator, given by

$$\mathcal{S}^k T(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{P \in S_k} T(v_{P(1)}, \ldots, v_{P(k)}) \tag{1.1.30}$$

where S_k is the collection of permutations of $\{1, \ldots, k\}$.

With this notation established we can now state the higher-order chain rule, also known as the Faà di Bruno formula.

Proposition 1.1.5 (Faà di Bruno formula)**.** *Let X, Y, Z be normed vector spaces with $U \subseteq X$ open and $V \subseteq Y$ open. Suppose that $f : U \to V$, $g : V \to Z$ are both $k-$times differentiable for some $1 \leq k \in \mathbb{N}$. Then*

$$D^k(g \circ f) = \mathcal{S}^k \sum_{i=1}^{k} \sum_{j_1 + \cdots j_i = k} \frac{k!}{j_1! \cdots j_i!} \Lambda^i_{j_1,\ldots,j_i}(D^i g \circ f, D^{j_1} f, \ldots, D^{j_i} f), \tag{1.1.31}$$

where $j_d > 0$ for $d = 1, \ldots, i$.

Proof. This can be proved by induction and a tedious exercise in combinatorics. See Chapter 2.4 of the book by Abraham, Marsden, and Ratiu [1].

□

We will now employ this horrible formula to study the composition properties of the Sobolev space H^k.

Theorem 1.1.6. *Let V, W be nontrivial finite dimensional vector spaces over $\mathbb{R}$. Suppose that $u \in H^k(\mathbb{R}^n; V)$ for $k > p + n/2$ for some $p \in \mathbb{N}$. Further suppose that $\Omega \subseteq V$ is an open set such that $u(\mathbb{R}^n) \subseteq \Omega$ and that $f \in C^k(\Omega; W)$ and $Df \in C^{k-1}_b(\Omega; \mathcal{L}(V;W))$. Then the following hold.*

1. *$f \circ u \in C^p(\mathbb{R}^n; W)$, and if $f \in C^k_b(\Omega; W)$ then*

$$\|f \circ u\|_{C^p_b} \leq \|f\|_{C^k_b} P(\|u\|_{H^k}) \tag{1.1.32}$$

   for a polynomial $P : \mathbb{R} \to \mathbb{R}$ with positive coefficients that only depend on n, k, p, $\dim(V)$, and $\dim(W)$.

2. *$D(f \circ u) \in H^{k-1}(\mathbb{R}^n; W)$, and*

$$\|D(f \circ u)\|_{H^{k-1}} \leq \|Df\|_{C^{k-1}_b} Q(\|u\|_{H^k}) \tag{1.1.33}$$

   for a polynomial $Q : \mathbb{R} \to \mathbb{R}$ with positive coefficients that only depend on n, k, p, $\dim(V)$, and $\dim(W)$ and such that $Q(0) = 0$.

Proof. Note first that it suffices to prove the result under the assumption that $V = \mathbb{R}^{d_1}$ and $W = \mathbb{R}^{d_2}$ for some $1 \le d_1, d_2 \in \mathbb{N}$. Assume this.

The inclusion $f \circ u \in C_b^p$ is trivial since $u \in H^k \hookrightarrow C_b^p$ and $f \in C_b^p$. We may then use the Faà di Bruno formula to estimate

$$\|f \circ u\|_{C_b^p} \lesssim \|f\|_{C_b^k} P(\|u\|_{H^k}), \tag{1.1.34}$$

which proves (1.1.32).

In order to prove that $D(f \circ u) \in H^{k-1}$ it thus suffices to shows that $D^\ell(f \circ u) \in L^2$ for $1 \le \ell \le k$. When $\ell = 1$ we have that $D(f \circ u) = Df \circ u Du$, so

$$\|D(f \circ u)\|_{L^2} \le \|Df \circ u\|_{L^\infty} \|Du\|_{L^2} < \infty. \tag{1.1.35}$$

Thus $D(f \circ u) \in L^2$ and

$$\|D(f \circ u)\|_{L^2} \lesssim \|Df\|_{C_b^{k-1}} Q(\|u\|_{H^k}) \tag{1.1.36}$$

for some polynomial with $Q(0) = 0$.

Assume now that $2 \le \ell \le k$. The Faà di Bruno formula tells us that

$$\begin{aligned} \left\|D^\ell(f \circ u)\right\|_{L^2} &\lesssim \sum_{i=1}^{\ell} \sum_{j_1+\cdots j_i=\ell} \left\|\Lambda^i_{j_1,\ldots,j_i}(D^i f \circ u, D^{j_1}u, \ldots, D^{j_i}u)\right\|_{L^2} \\ &\lesssim \|Df\|_{C_b^{k-1}} \sum_{i=1}^{\ell} \sum_{j_1+\cdots j_i=\ell} \left\|\left|D^{j_1}u\right| \cdots \left|D^{j_i}u\right|\right\|_{L^2}. \end{aligned} \tag{1.1.37}$$

Consequently, it suffices to prove that if $1 \le i \le \ell$, $\beta_1, \ldots, \beta_i \in \mathbb{N}^n$ are such that $|\beta_i| = j_i \ge 1$ and $j_1 + \cdots + j_i = \ell$, and $m_1, \ldots, m_i \in \{1, \ldots, d_1\}$, then $\partial^{\beta_1} u_{m_1} \cdots \partial^{\beta_i} u_{m_i} \in L^2$. This follows directly from Proposition 1.1.1, which also provides the estimate

$$\left\|D^\ell(f \circ u)\right\|_{L^2} \lesssim \|Df\|_{C_b^{k-1}} Q(\|u\|_{H^k}) \tag{1.1.38}$$

for a polynomial Q such that $Q(0) = 0$. We obtain (1.1.33) by combining (1.1.36) and (1.1.38). □

1.2 New spatial regularity estimates

We now seek to record estimates for solutions to

$$\begin{cases} A^0 \partial_t u + A^j \partial_j u + Bu = f \\ u(\cdot, 0) = g \end{cases} \tag{1.2.1}$$

with different assumptions on the coefficient matrices. Our approach is a variant of the approaches used by Kato [5] and Fischer and Marsden [3].

For $1 \le k \in \mathbb{N}$ we introduce the function space

$$\mathcal{X}^k(T, m) = \{A \in C_b^{0,1}(\mathbb{R}^n \times [0, T]) \mid \nabla A \in L^\infty([0, T]; H^{k-1}(\mathbb{R}^n; \mathbb{R}^{m\times m}))\}, \tag{1.2.2}$$

which we endow with the norm

$$\|A\|_{\mathcal{X}^k} = \|A\|_{C_b^{0,1}} + \|\nabla A\|_{L^\infty H^{k-1}}. \tag{1.2.3}$$

It's clear that $\mathcal{X}^k(T,m)$ is a Banach space when endowed with this norm. We will also need to make use of the following variant: for $1 \le k \in \mathbb{N}$ we write

$$\mathcal{Y}^k(T,m) = \{A \in L^\infty([0,T]; L^\infty(\mathbb{R}^n;\mathbb{R}^{m\times m})) \mid \nabla A \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^n;\mathbb{R}^{m\times m}))\}, \tag{1.2.4}$$

which we endow with the norm

$$\|A\|_{\mathcal{Y}^k} = \|A\|_{L^\infty L^\infty} + \|\nabla A\|_{L^\infty H^{k-1}}. \tag{1.2.5}$$

Clearly $\mathcal{X}^k(T,m) \hookrightarrow \mathcal{Y}^k(T,m)$ for every $T > 0$, $1 \le m \in \mathbb{N}$.

We now prove a version of the higher-order spatial regularity result under the assumption that the coefficients belong to $\mathcal{X}^k(T,m)$.

Theorem 1.2.1 (Higher-order spatial regularity). *Let $1 + n/2 < k \in \mathbb{N}$ and assume that*

$$A^0, A^1, \ldots, A^n, B \in \mathcal{X}^k(T,m), \tag{1.2.6}$$

that $A^0(x,t), \ldots, A^n(x,t)$ are symmetric for each $x \in \mathbb{R}^n$ and $t \in [0,T]$, and that there exists $\theta > 0$ such that $A^0(x,t) \ge \theta I$ for all $x \in \mathbb{R}^n$ and $t \in [0,T]$. Further suppose that $g \in H^k(\mathbb{R}^n;\mathbb{R}^m)$ and

$$f \in L^2([0,T]; H^k(\mathbb{R}^n;\mathbb{R}^m)) \cap L^\infty([0,T]; H^{k-1}(\mathbb{R}^n;\mathbb{R}^m)). \tag{1.2.7}$$

Let u be the weak solution to

$$\begin{cases} A^0\partial_t u + A^j \partial_j u + Bu = f \\ u(\cdot,0) = g. \end{cases} \tag{1.2.8}$$

Then the following hold.

1. *$u \in L^\infty([0,T]; H^k(\mathbb{R}^n;\mathbb{R}^m))$ and $\partial_t u \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^n;\mathbb{R}^m))$. Moreover, we have the estimates*
$$\|u\|^2_{L^\infty H^k} \le Q \cdot e^{P\cdot T}\left(\|g\|^2_{H^k} + \|f\|^2_{L^2 H^k}\right). \tag{1.2.9}$$
and
$$\|\partial_t u\|^2_{L^\infty H^{k-1}} \le Q \cdot e^{P\cdot T}\left(\|g\|^2_{H^k} + \|f\|^2_{L^2 H^k}\right) + Q \cdot \|f\|^2_{L^\infty H^{k-1}} \tag{1.2.10}$$
for polynomials
$$\begin{aligned} P &= P(\|A^0\|_{\mathcal{X}^k}, \ldots, \|A^n\|_{\mathcal{X}^k}, \|B\|_{\mathcal{X}^k}, 1/\theta) \\ Q &= Q(\|A^0\|_{\mathcal{Y}^k}, \ldots, \|A^n\|_{\mathcal{Y}^k}, \|B\|_{\mathcal{Y}^k}, 1/\theta) \end{aligned} \tag{1.2.11}$$
with positive coefficients.

2. *For each $0 \le s < k$ we have that $u \in C^0_b([0,T]; H^s(\mathbb{R}^n))$ and*
$$\|u\|^2_{C^0_b H^s} \le P \cdot e^{P\cdot T}\left(\|g\|^2_{H^k} + \|f\|^2_{L^2 H^k} + \|f\|^2_{L^\infty H^{k-1}}\right) \tag{1.2.12}$$
for a polynomial $P = P(\|A^0\|_{\mathcal{X}^k}, \ldots, \|A^n\|_{\mathcal{X}^k}, \|B\|_{\mathcal{X}^k}, 1/\theta)$ with positive coefficients.

3. *If $f \in C^0_b(\mathbb{R}^n \times [0,T])$, then $u \in C^1_b(\mathbb{R}^n \times [0,T])$ and is thus a classical solution, i.e. $A^0\partial_t u + A^j\partial_j u + Bu = f$ everywhere in $\mathbb{R}^n \times [0,T]$.*

4. If $h \in H^{k+1}(\mathbb{R}^n;\mathbb{R}^m)$ then

$$\|u-h\|^2_{L^\infty H^k} \leq Q \cdot e^{P \cdot T}\left(\|g-h\|^2_{H^k} + \|f\|^2_{L^2 H^k} + T\,\|h\|^2_{H^{k+1}}\right) \tag{1.2.13}$$

and

$$\|\partial_t u\|^2_{L^\infty H^{k-1}} \leq Q \cdot e^{P \cdot T}\left(\|g-h\|^2_{H^k} + \|f\|^2_{L^2 H^k} + T\,\|h\|^2_{H^{k+1}}\right) + Q \cdot \left(\|f\|^2_{L^\infty H^{k-1}} + \|h\|^2_{H^k}\right) \tag{1.2.14}$$

for polynomials P, Q of the form (1.2.11) with positive coefficients.

Proof. We divide the proof into several steps.

Step 1 – A new approximate problem

The regularity assumptions on the coefficient matrices are weaker than the assumptions we previously used. This causes some technical problems when we attempt to estimate solutions in $L^\infty H^k$. To get around this issue we will introduce a new approximate problem that will give rise to a sequence of approximate solutions that also converge to u. It should be noted that the approximation problem is still of the type first employed by Friedrichs [4].

For $\varepsilon > 0$ consider the approximate problem

$$\begin{cases} (K_\varepsilon A^0)\partial_t u_\varepsilon + K_\varepsilon(A^j \partial_j K_\varepsilon u_\varepsilon) + (K_\varepsilon B)u_\varepsilon = K_\varepsilon f \\ u_\varepsilon(\cdot, 0) = g. \end{cases} \tag{1.2.15}$$

Note first that K_ε is a bounded linear operator on $\mathcal{X}^k(T,m)$, and

$$\|K_\varepsilon\|_{\mathcal{L}(\mathcal{X}^k)} \leq 1. \tag{1.2.16}$$

Also, $K_\varepsilon A^0$ is symmetric and satisfies

$$K_\varepsilon A^0(x,t)\xi \cdot \xi \geq \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) A^0(y,t)\xi \cdot \xi dy \geq \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\theta\,|\xi|^2\,dy = \theta\,|\xi|^2\,, \tag{1.2.17}$$

which means that $K_\varepsilon A^0(x,t) \geq \theta I$ and that $K_\varepsilon A^0(x,t)$ is invertible for each $x \in \mathbb{R}^n$, $t \in [0,T]$.

The properties of $K_\varepsilon A^0$ show that (1.2.15) is equivalent to

$$\begin{cases} \partial_t u_\varepsilon + \mathcal{M}_\varepsilon u_\varepsilon = (K_\varepsilon A^0)^{-1} K_\varepsilon f \\ u_\varepsilon(\cdot, 0) = g, \end{cases} \tag{1.2.18}$$

where

$$\mathcal{M}_\varepsilon v = (K_\varepsilon A^0)^{-1}\left[K_\varepsilon(A^j \partial_j K_\varepsilon v) + (K_\varepsilon B)v\right]. \tag{1.2.19}$$

It's easy to see that

$$K_\varepsilon A^0, (K_\varepsilon A^0)^{-1}, K_\varepsilon B \in L^\infty([0,T]; C^k_b(\mathbb{R}^n;\mathbb{R}^{m\times m})), \tag{1.2.20}$$

from which we deduce that

$$\mathcal{M}_\varepsilon \in L^\infty([0,T];\mathcal{L}(H^k(\mathbb{R}^n;\mathbb{R}^m))) \text{ and } (K_\varepsilon A^0)^{-1} K_\varepsilon f \in L^\infty([0,T]; H^k(\mathbb{R}^n;\mathbb{R}^m)). \tag{1.2.21}$$

The equation (1.2.18) is thus an ODE in the Banach space $H^k(\mathbb{R}^n;\mathbb{R}^m)$, which admits a unique solution

$$u_\varepsilon \in C^{0,1}_b([0,T]; H^k(\mathbb{R}^n;\mathbb{R}^m)) \cap W^{1,\infty}((0,T); H^k(\mathbb{R}^n;\mathbb{R}^m)). \tag{1.2.22}$$

Step 2 – L^2 estimate

We now derive an $\varepsilon-$independent estimate for u_ε in $L^\infty L^2$. To do this we take the dot product of (1.2.15) with u_ε and integrate over $\mathbb{R}^n$ to find that

$$\frac{d}{dt}\int_{\mathbb{R}^n}\frac{1}{2}(K_\varepsilon A^0)u_\varepsilon\cdot u_\varepsilon = \int_{\mathbb{R}^n} K_\varepsilon f\cdot u_\varepsilon - (K_\varepsilon B)u_\varepsilon\cdot u_\varepsilon + \frac{1}{2}(K_\varepsilon\partial_t A^0)u_\varepsilon\cdot u_\varepsilon - K_\varepsilon(A^j\partial_j K_\varepsilon u_\varepsilon)\cdot u_\varepsilon. \quad (1.2.23)$$

Then

$$\begin{aligned}\int_{\mathbb{R}^n} -K_\varepsilon(A^j\partial_j K_\varepsilon u_\varepsilon)\cdot u_\varepsilon &= -\int_{\mathbb{R}^n} A^j\partial_j K_\varepsilon u_\varepsilon\cdot K_\varepsilon u_\varepsilon \\ &= \int_{\mathbb{R}^n} -\partial_j\left(\frac{A^j}{2}K_\varepsilon u_\varepsilon\cdot K_\varepsilon u_\varepsilon\right) + \frac{\partial_j A^j}{2}K_\varepsilon u_\varepsilon\cdot K_\varepsilon u_\varepsilon = \int_{\mathbb{R}^n}\frac{\partial_j A^j}{2}K_\varepsilon u_\varepsilon\cdot K_\varepsilon u_\varepsilon, \quad (1.2.24)\end{aligned}$$

and so we may estimate

$$\begin{aligned}\int_{\mathbb{R}^n} -K_\varepsilon(A^j\partial_j K_\varepsilon u_\varepsilon)\cdot u_\varepsilon &\le \frac{1}{2}\left\|\partial_j A^j\right\|_{L^\infty}\int_{\mathbb{R}^n}\left|K_\varepsilon u_\varepsilon\right|^2 \le \frac{1}{2}\left\|\partial_j A^j\right\|_{L^\infty}\int_{\mathbb{R}^n}\left|u_\varepsilon\right|^2 \\ &\le \frac{1}{\theta}\left\|\partial_j A^j\right\|_{L^\infty}\int_{\mathbb{R}^n}\frac{1}{2}(K_\varepsilon A^0)u_\varepsilon\cdot u_\varepsilon. \quad (1.2.25)\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^n}(K_\varepsilon B)u_\varepsilon\cdot u_\varepsilon \le \left\|K_\varepsilon B\right\|_{L^\infty}\int_{\mathbb{R}^n}\left|u_\varepsilon\right|^2 \le \frac{2}{\theta}\left\|B\right\|_{L^\infty}\int_{\mathbb{R}^n}\frac{1}{2}(K_\varepsilon A^0)u_\varepsilon\cdot u_\varepsilon, \quad (1.2.26)$$

$$\int_{\mathbb{R}^n}\frac{1}{2}(K_\varepsilon\partial_t A^0)u_\varepsilon\cdot u_\varepsilon \le \frac{1}{\theta}\left\|\partial_t A^0\right\|_{L^\infty}\int_{\mathbb{R}^n}\frac{1}{2}(K_\varepsilon A^0)u_\varepsilon\cdot u_\varepsilon, \quad (1.2.27)$$

and

$$\int_{\mathbb{R}^n} K_\varepsilon f\cdot u_\varepsilon \le \int_{\mathbb{R}^n}\frac{1}{2}\left|f\right|^2 + \frac{1}{\theta}\int_{\mathbb{R}^n}\frac{1}{2}(K_\varepsilon A^0)u_\varepsilon\cdot u_\varepsilon. \quad (1.2.28)$$

Thus

$$\frac{d}{dt}\int_{\mathbb{R}^n}(K_\varepsilon A^0)u_\varepsilon\cdot u_\varepsilon \le \frac{1}{\theta}\left(1+\left\|\partial_t A^0\right\|_{L^\infty}+\left\|\partial_j A^j\right\|_{L^\infty}+2\left\|B\right\|_{L^\infty}\right)\int_{\mathbb{R}^n}(K_\varepsilon A^0)u_\varepsilon\cdot u_\varepsilon + \int_{\mathbb{R}^n}\left|f\right|, \quad (1.2.29)$$

Step 3 – Temporal derivative estimates

We now turn to the proof of estimates for the temporal derivatives of u_ε. These will play an essential role in allowing us to derive control of the spatial derivatives in the next step. We begin by using the first equation in (1.2.15) to estimate

$$\begin{aligned}\theta\left\|\partial_t u_\varepsilon(\cdot,t)\right\|_{L^2}^2 &\le \int_{\mathbb{R}^n}(K_\varepsilon A^0)\partial_t u_\varepsilon\cdot\partial_t u_\varepsilon(t) \\ &\le \left\|K_\varepsilon f(\cdot,t)-(K_\varepsilon B)u_\varepsilon(\cdot,t)-K_\varepsilon(A^j\partial_j K_\varepsilon u_\varepsilon)(\cdot,t)\right\|_{L^2}\left\|\partial_t u_\varepsilon(\cdot,t)\right\|_{L^2} \quad (1.2.30)\end{aligned}$$

and hence

$$\begin{aligned}\left\|\partial_t u_\varepsilon\right\|_{L^2} &\le \frac{1}{\theta}\left(\left\|K_\varepsilon f\right\|_{L^2}+\left\|(K_\varepsilon B)u_\varepsilon\right\|_{L^2}+\left\|K_\varepsilon(A^j\partial_j K_\varepsilon u_\varepsilon\right\|_{L^2}\right) \\ &\le \frac{1}{\theta}\left(1+\left\|B\right\|_{L^\infty}+\max_j\left\|A^j\right\|_{L^\infty}\right)\left(\left\|f\right\|_{L^2}+\left\|u\right\|_{H^1}\right). \quad (1.2.31)\end{aligned}$$

Thus

$$\|\partial_t u_\varepsilon\|_{L^2}^2 \leq Q \cdot \left(\|f\|_{L^2}^2 + \|u\|_{H^1}^2\right) \tag{1.2.32}$$

for a polynomial $Q = Q(\|A^0\|_{\mathcal{Y}^k}, \ldots, \|A^n\|_{\mathcal{Y}^k}, \|B\|_{\mathcal{Y}^k}, 1/\theta)$.

Now we bootstrap. Suppose that for $0 \leq \ell < k-1$

$$\|\partial_t u_\varepsilon\|_{H^\ell}^2 \leq Q \cdot \left(\|f\|_{H^\ell}^2 + \|u\|_{H^{\ell+1}}^2\right) \tag{1.2.33}$$

for a polynomial $Q = Q(\|A^0\|_{\mathcal{Y}^k}, \ldots, \|A^n\|_{\mathcal{Y}^k}, \|B\|_{\mathcal{Y}^k}, 1/\theta)$. We claim that

$$\|\partial_t u_\varepsilon\|_{H^{\ell+1}}^2 \leq P \cdot \left(\|f\|_{H^{\ell+1}}^2 + \|u\|_{H^{\ell+2}}^2\right) \tag{1.2.34}$$

for a polynomial $Q = Q(\|A^0\|_{\mathcal{Y}^k}, \ldots, \|A^n\|_{\mathcal{Y}^k}, \|B\|_{\mathcal{Y}^k}, 1/\theta)$.

Let $\alpha \in \mathbb{N}^n$ with $|\alpha| = \ell + 1 \geq 1$. Then we may apply ∂^α to (1.2.15) to find that

$$(K_\varepsilon A^0)\partial_t \partial^\alpha u_\varepsilon = \partial^\alpha K_\varepsilon f - \partial^\alpha[(K_\varepsilon B)u_\varepsilon] - K_\varepsilon[\partial^\alpha(A^j\partial_j K_\varepsilon u_\varepsilon)] - \sum_{\beta<\alpha} \frac{\alpha!}{\beta!(\alpha-\beta!)}\partial^{\alpha-\beta}K_\varepsilon A^0 \partial^\beta \partial_t u_\varepsilon. \tag{1.2.35}$$

We may then argue as above to estimate

$$\|\partial^\alpha \partial_t u_\varepsilon\|_{L^2}^2 \leq C\left(\|\partial^\alpha K_\varepsilon f\|_{L^2}^2 + \|\partial^\alpha[(K_\varepsilon B)u_\varepsilon]\|_{L^2}^2 + \left\|K_\varepsilon[\partial^\alpha(A^j\partial_j K_\varepsilon u_\varepsilon)]\right\|_{L^2}^2 \right.$$
$$\left. + \sum_{\beta<\alpha}\left\|\partial^{\alpha-\beta}K_\varepsilon A^0 \partial^\beta \partial_t u_\varepsilon\right\|_{L^2}^2\right). \tag{1.2.36}$$

We will estimate each of the terms on the right. For the first we estimate

$$\|\partial^\alpha K_\varepsilon f\|_{L^2}^2 \leq \|f\|_{H^{\ell+1}}^2. \tag{1.2.37}$$

For the second we expand with Leibniz and then estimate

$$\begin{aligned}\|\partial^\alpha[(K_\varepsilon B)u_\varepsilon]\|_{L^2}^2 &\lesssim \|(K_\varepsilon B)\partial^\alpha u_\varepsilon\|_{L^2}^2 + \sum_{\beta<\alpha}\left\|\partial^{\alpha-\beta}(K_\varepsilon B)\partial^\beta u_\varepsilon\right\|_{L^2}^2 \\ &\lesssim \|K_\varepsilon B\|_{L^\infty}^2 \|u_\varepsilon\|_{H^{\ell+1}}^2 + \|\nabla(K_\varepsilon B)\|_{H^{k-1}}^2 \|u_\varepsilon\|_{H^{\ell+1}}^2 \\ &\lesssim (\|B\|_{L^\infty L^\infty}^2 + \|\nabla B\|_{L^\infty H^{k-1}}^2)\|u_\varepsilon\|_{H^{\ell+1}}^2 \lesssim \|B\|_{\mathcal{Y}^k}^2 \|u_\varepsilon\|_{H^{\ell+1}}^2,\end{aligned} \tag{1.2.38}$$

where in the second inequality we have employed Proposition 1.1.3. For the third term we argue similarly with Leibniz and Proposition 1.1.3:

$$\begin{aligned}\left\|K_\varepsilon[\partial^\alpha(A^j\partial_j K_\varepsilon u_\varepsilon)]\right\|_{L^2}^2 &\lesssim \left\|A^j \partial^\alpha \partial_j K_\varepsilon u_\varepsilon\right\|_{L^2}^2 + \sum_{\beta<\alpha}\left\|\partial^{\alpha-\beta}A^j\partial^\beta\partial_j K_\varepsilon u_\varepsilon\right\|_{L^2}^2 \\ &\lesssim \left\|A^j\right\|_{L^\infty}^2 \|K_\varepsilon \partial^\alpha \partial_j u_\varepsilon\|_{L^2}^2 + \left\|\nabla A^j\right\|_{L^\infty H^{k-1}}^2 \|\partial_j K_\varepsilon u_\varepsilon\|_{H^{\ell+1}}^2 \\ &\lesssim \max_{1\leq j\leq n}\left(\left\|A^j\right\|_{L^\infty L^\infty}^2 + \left\|\nabla A^j\right\|_{L^\infty H^{k-1}}^2\right)\|u_\varepsilon\|_{H^{\ell+2}}^2 \lesssim \max_{1\leq j\leq n}\left\|A^j\right\|_{\mathcal{Y}^k}\|u_\varepsilon\|_{H^{\ell+2}}^2.\end{aligned} \tag{1.2.39}$$

Finally, for the fourth term we again use Proposition 1.1.3, this time in conjunction with the hypothesis (1.2.33), to bound

$$\sum_{\beta<\alpha}\left\|\partial^{\alpha-\beta}K_\varepsilon A^0\partial^\beta\partial_t u_\varepsilon\right\|_{L^2}^2 \lesssim \left\|A^0\right\|_{\mathcal{Y}^k}^2 \|\partial_t u_\varepsilon\|_{H^\ell}^2 \lesssim \left\|A^0\right\|_{\mathcal{Y}^k}^2 Q \cdot \left(\|f\|_{H^\ell}^2 + \|u\|_{H^{\ell+1}}^2\right). \tag{1.2.40}$$

Combining all of these estimates and summing over α with $|\alpha| = \ell + 1$ then shows that (1.2.34) holds, as claimed.

Now that the claim is proved, a finite induction then shows that

$$\|\partial_t u_\varepsilon\|_{H^{k-1}}^2 \leq Q \cdot \left(\|f\|_{H^{k-1}}^2 + \|u\|_{H^k}^2\right) \tag{1.2.41}$$

for a polynomial $Q = Q(\|A^0\|_{\mathcal{Y}^k}, \ldots, \|A^n\|_{\mathcal{Y}^k}, \|B\|_{\mathcal{Y}^k}, 1/\theta)$.

Step 4 – H^k estimate

Now we adapt the energy estimates to control spatial derivatives. Let $\alpha \in \mathbb{N}^n$ with $1 \leq |\alpha| \leq k$. Applying ∂^α to (1.2.15) shows that

$$\begin{aligned}(K_\varepsilon A^0)\partial_t \partial^\alpha u_\varepsilon + K_\varepsilon[A^j \partial_j K_\varepsilon \partial^\alpha u_\varepsilon] + (K_\varepsilon B)\partial^\alpha u_\varepsilon = \partial^\alpha K_\varepsilon f \\ - \sum_{\beta<\alpha} \frac{\alpha!}{\beta!(\alpha-\beta!)} \left[K_\varepsilon \partial^{\alpha-\beta} A^0 \partial^\beta \partial_t u_\varepsilon + K_\varepsilon \partial^{\alpha-\beta} B \partial^\beta u_\varepsilon + K_\varepsilon[\partial^{\alpha-\beta} A^j \partial_j K_\varepsilon \partial^\beta u_\varepsilon]\right].\end{aligned} \tag{1.2.42}$$

We may then argue as in Step 1 to find that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n} (K_\varepsilon A^0)\partial^\alpha u_\varepsilon \cdot \partial^\alpha u_\varepsilon \\ &\quad \leq \frac{1}{\theta}\left(4 + \left\|\partial_t A^0\right\|_{L^\infty} + \left\|\partial_j A^j\right\|_{L^\infty} + 2\|B\|_{L^\infty}\right) \int_{\mathbb{R}^n} (K_\varepsilon A^0)\partial^\alpha u_\varepsilon \cdot \partial^\alpha u_\varepsilon + \int_{\mathbb{R}^n} |\partial^\alpha f|^2 \\ &+ \sum_{\beta<\alpha} \frac{\alpha!}{\beta!(\alpha-\beta!)} \int_{\mathbb{R}^n} \left|K_\varepsilon \partial^{\alpha-\beta} A^0 \partial^\beta \partial_t u_\varepsilon\right|^2 + \left|K_\varepsilon \partial^{\alpha-\beta} B \partial^\beta u_\varepsilon\right|^2 + \left|K_\varepsilon[\partial^{\alpha-\beta} A^j \partial_j K_\varepsilon \partial^\beta u_\varepsilon]\right|^2.\end{aligned} \tag{1.2.43}$$

We then employ Proposition 1.1.3 and the estimate (1.2.41) to bound

$$\begin{aligned}&\sum_{\beta<\alpha} \frac{\alpha!}{\beta!(\alpha-\beta!)} \int_{\mathbb{R}^n} \left|K_\varepsilon \partial^{\alpha-\beta} A^0 \partial^\beta \partial_t u_\varepsilon\right|^2 + \left|K_\varepsilon \partial^{\alpha-\beta} B \partial^\beta u_\varepsilon\right|^2 + \left|K_\varepsilon[\partial^{\alpha-\beta} A^j \partial_j K_\varepsilon \partial^\beta u_\varepsilon]\right|^2 \\ &\qquad \leq P(\left\|A^0\right\|_{\mathcal{X}^k}, \ldots, \|A^n\|_{\mathcal{X}^k}, \|B\|_{\mathcal{X}^k}, 1/\theta) \|u_\varepsilon\|_{H^k}^2\end{aligned} \tag{1.2.44}$$

for a polynomial P. We may further estimate

$$\|u_\varepsilon\|_{H^k}^2 \leq \frac{1}{\theta} \sum_{|\beta|\leq k} \int_{\mathbb{R}^n} (K_\varepsilon A_0)\partial^\beta u_\varepsilon \cdot \partial^\beta u_\varepsilon. \tag{1.2.45}$$

We then plug these into (1.2.43), sum over $1 \leq |\alpha| \leq k$, and add the resulting inequality to (1.2.29) to deduce that

$$\begin{aligned}&\frac{d}{dt} \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} (K_\varepsilon A^0)\partial^\alpha u_\varepsilon \cdot \partial^\alpha u_\varepsilon \leq \|f\|_{H^k}^2 \\ &\qquad + P(\left\|A^0\right\|_{\mathcal{X}^k}, \ldots, \|A^n\|_{\mathcal{X}^k}, \|B\|_{\mathcal{X}^k}, 1/\theta) \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} (K_\varepsilon A^0)\partial^\alpha u_\varepsilon \cdot \partial^\alpha u_\varepsilon.\end{aligned} \tag{1.2.46}$$

Gronwall's lemma tells us that

$$\begin{aligned}\theta \|u_\varepsilon(\cdot,t)\|_{H^k}^2 &= \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\partial^\alpha u_\varepsilon(\cdot,t)|^2 \leq \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} (K_\varepsilon A^0)\partial^\alpha u_\varepsilon \cdot \partial^\alpha u_\varepsilon(t) \\ &\leq e^{tP} \left(\sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} (K_\varepsilon A^0(0))\partial^\alpha g \cdot \partial^\alpha g + \int_0^t \|f(\cdot,s)\|_{H^k}^2 \, ds \right)\end{aligned} \tag{1.2.47}$$

for each $t \in [0,T]$, where $P = P(\|A^0\|_{\mathcal{X}^k}, \ldots, \|A^n\|_{\mathcal{X}^k}, \|B\|_{\mathcal{X}^k}, 1/\theta)$ is a polynomial with positive coefficients. Since A^0 is continuous on $\mathbb{R}^n \times [0,T]$ we may estimate

$$\left\|A^0(\cdot,0)\right\|_{L^\infty} \le \left\|A^0\right\|_{L^\infty L^\infty} \le \left\|A^0\right\|_{\mathcal{Y}^k}. \tag{1.2.48}$$

Thus

$$\|u_\varepsilon\|^2_{L^\infty H^k} \le Q \cdot e^{PT}\left(\|g\|^2_{H^k} + \|f\|^2_{L^2 H^k}\right) \tag{1.2.49}$$

where P and Q are polynomials of the form listed in (1.2.11) with positive coefficients. We combine (1.2.49) with (1.2.41) to further deduce that

$$\|\partial_t u_\varepsilon\|^2_{L^\infty H^{k-1}} \le Q \cdot e^{PT}\left(\|g\|^2_{H^k} + \|f\|^2_{L^2 H^k}\right) + Q\,\|f\|^2_{L^\infty H^{k-1}} \tag{1.2.50}$$

where P, Q are are polynomials of the form (1.2.11) positive coefficients.

Step 5 – Passing to the limit

The estimates (1.2.49) and (1.2.50) allow us to extract weak$-*$ limits

$$\begin{aligned} u_\varepsilon &\overset{*}{\rightharpoonup} v \text{ weakly-}* \text{ in } L^\infty H^k \\ \partial_t u_\varepsilon &\overset{*}{\rightharpoonup} \partial_t v \text{ weakly-}* \text{ in } L^\infty H^{k-1} \end{aligned} \tag{1.2.51}$$

and Simon's theorem further implies that $u_\varepsilon \to v$ strongly in $C^0_b H^s$ for all $0 \le s < k$. The argument from class works here to show that $v(\cdot,0) = g$.

For $\varphi \in C^\infty_c((0,T))$ and $\psi \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^m)$ we deduce from (1.2.15) that

$$\int_0^T \varphi\left((K_\varepsilon A^0)\partial_t u_\varepsilon, \psi\right)_{L^2} + \int_0^T \varphi\left(K_\varepsilon(A^j\partial_j K_\varepsilon u_\varepsilon), \psi\right)_{L^2} + \int_0^T \varphi\left((K_\varepsilon B)u_\varepsilon, \psi\right)_{L^2} = \int_0^T \varphi\left(K_\varepsilon f, \psi\right)_{L^2}. \tag{1.2.52}$$

Upon passing to the limit along our extracted subsequence, we find that

$$\int_0^T \varphi\left(A^0\partial_t v, \psi\right)_{L^2} + \int_0^T \varphi\left(A^j\partial_j v, \psi\right)_{L^2} + \int_0^T \varphi\left(Vv, \psi\right)_{L^2} = \int_0^T \varphi\left(f, \psi\right)_{L^2} \tag{1.2.53}$$

for all such φ and ψ. Thus $v \in L^\infty H^k$, $\partial_t v \in L^\infty H^{k-1}$, and

$$\begin{cases} A^0\partial_t v + A^j\partial_j v + Bv = f & \text{in } \mathbb{R}^n \times [0,T] \\ v(\cdot,0) = g & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2.54}$$

The uniqueness of weak solutions then shows that $v = u$. The estimates (1.2.9) and (1.2.10) follow from (1.2.49), (1.2.50), and weak$-*$ lower semi-continuity. The estimate (1.2.12) follows directly from (1.2.9), (1.2.10), and an interpolation argument. Finally, the inclusion $u \in C^1_b(\mathbb{R}^n \times [0,T])$ follows from the same argument used in class.

Step 6 – Proof of the fourth item

Let $h \in H^{k+1}$. Then $u - h \in L^\infty H^k$, $\partial_t(u-h) \in L^\infty H^{k-1}$, and

$$\begin{cases} A^0\partial_t(u-h) + A^j\partial_j(u-h) + B(u-h) = f - Bh - A^j\partial_j h & \text{in } \mathbb{R}^n \times [0,T] \\ v(\cdot,0) = g - h & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2.55}$$

Proposition 1.1.3 tells us that

$$\left\|Bh + A^j\partial_j h\right\|_{L^2 H^k} \le T\left(\|B\|_{\mathcal{Y}^k} + \max_j \left\|A^j\right\|_{\mathcal{Y}^k}\right)\|h\|_{H^{k+1}} \tag{1.2.56}$$

and

$$\left\|Bh + A^j \partial_j h\right\|_{L^\infty H^{k-1}} \le \left(\|B\|_{\mathcal{Y}^k} + \max_j \left\|A^j\right\|_{\mathcal{Y}^k} \right) \|h\|_{H^{k+1}}. \tag{1.2.57}$$

To deduce (1.2.12) and (1.2.13) we simply combine these with the estimates (1.2.9) and (1.2.10) applied to $u - h$ with f replaced by $f - Bh - A^j \partial_j h \in L^2 H^k \cap L^\infty H^{k-1}$ and g replaced by $g - h \in H^k$. □

2 The quasilinear problem

Our goal now is to solve the quasilinear problem

$$\begin{cases} A^0(u)\partial_t u + A^j(u)\partial_j u + B(u)u = f + F(u) & \text{in } \mathbb{R}^n \times [0,T] \\ u(\cdot, 0) = g & \text{in } \mathbb{R}^n. \end{cases} \tag{2.0.1}$$

This is clearly not the most general form of the problem. We would ideally replace $A^0(u)$ with $A^0(x,t,u)$ and do the same for the other coefficient matrices and for F to achieve full generality. This can be done using the scheme we will develop, but it requires more work than we have time for. We will thus content ourselves with studying this problem. As we will see in the next section, this is already good enough to solve several important problems in physics.

In order to solve (2.0.1) we will employ the Banach fixed point theorem. Our strategy is a variant of that employed by Kato [5] and Fischer and Marsden [3] and proceeds as follows.

1. We will find a metric space compatible with the estimates for the linear problem given in Theorem 1.2.1.

2. We will show that with a weak (and someone strange) choice of a metric, this metric space is complete. The choice of the funny metric will be important in the contraction argument.

3. We will record some technical results needed to show that the nonlinear coefficient terms and the nonlinear forcing term belong to the spaces needed to apply Theorem 1.2.1.

4. We will define on the metric space whose fixed point gives a solution to the quasilinear problem (2.0.1), and we will show that for small T this map is contractive.

Remark 2.0.1. *The result proved by Kato [5] is far more general and powerful than what we aim for here.*

2.1 More technical warm-up

We now prove a couple more technical results that will be needed to solve the quasilinear problem. First we present a standard approximation argument.

Lemma 2.1.1. *Suppose that $g \in H^k(\mathbb{R}^n; \mathbb{R}^m)$ for some $k \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists $h \in H^{k+1}(\mathbb{R}^n; \mathbb{R}^m)$ such that*

$$\|g - h\|_{H^k} < \varepsilon. \tag{2.1.1}$$

Proof. Choose h to be a mollified version of g with appropriate mollification parameter. □

Next we present a Lipschitz estimates.

Proposition 2.1.2. *Suppose that $u \in L^\infty([0,T];H^k(\mathbb{R}^n))$ and $\partial_t u \in L^\infty([0,T];H^{k-1}(\mathbb{R}^n))$ for some $k > 1 + n/2$. Then $u \in C_b^{0,1}(\mathbb{R}^n \times [0,T])$ and we have the estimates*

$$\|u\|_{C_b^0} \lesssim \|u\|_{L^\infty H^k} \tag{2.1.2}$$

and

$$\|u\|_{C_b^{0,1}} \lesssim \|u\|_{L^\infty H^k} + \|\partial_t u\|_{L^\infty H^{k-1}}. \tag{2.1.3}$$

Proof. We know that $u \in W^{1,\infty}H^{k-1} \hookrightarrow C_b^0 H^{k-1} \hookrightarrow C_b^0 C_b^0 \hookrightarrow C_b^0$. Thus u is continuous on $\mathbb{R}^n \times [0,T]$ and so we may estimate

$$\|u\|_{C_b^0} = \sup_{(x,t)\in\mathbb{R}^n\times[0,T]} \leq \|u\|_{L^\infty L^\infty} \lesssim \|u\|_{L^\infty H^k}. \tag{2.1.4}$$

Let $x, y \in \mathbb{R}^n$ and $t, s \in [0,T]$. Then

$$|u(x,t) - u(y,s)| \leq |u(x,t) - u(y,t)| + |u(y,t) - u(y,s)|. \tag{2.1.5}$$

We estimate

$$|u(x,t) - u(y,t)| \leq \|\nabla u(\cdot,t)\|_{L^\infty} |x-y| \lesssim \|u\|_{L^\infty H^k} |x-y| \tag{2.1.6}$$

and

$$|u(y,t) - u(y,s)| \leq \int_{s\wedge t}^{s\vee t} |\partial_t u(y,r)|\, dr \lesssim \int_{s\wedge t}^{s\vee t} \|\partial_t u(\cdot,r)\|_{L^\infty} dr \lesssim \|\partial_t u\|_{L^\infty H^{k-1}} |t-s|. \tag{2.1.7}$$

Thus

$$[u]_{C^{0,1}} \lesssim \|u\|_{L^\infty H^k} + \|\partial_t u\|_{L^\infty H^{k-1}}. \tag{2.1.8}$$

We now combine the above two estimates to deduce that

$$\|u\|_{C_b^{0,1}} = \|u\|_{C_b^0} + [u]_{C^{0,1}} \lesssim \|u\|_{L^\infty H^k} + \|\partial_t u\|_{L^\infty H^{k-1}}. \tag{2.1.9}$$

□

The next result guarantees that we can compose certain matrix-valued functions with u. We will use this to handle the coefficient matrices in the nonlinear problem.

Theorem 2.1.3. *Suppose that $u \in L^\infty([0,T];H^k(\mathbb{R}^n;\mathbb{R}^m))$ and $\partial_t u \in L^\infty([0,T];H^{k-1}(\mathbb{R}^n;\mathbb{R}^m))$ for some $k > 1 + n/2$. Further suppose that $\Omega \subseteq \mathbb{R}^m$ is an open set such that $u(\mathbb{R}^n,t) \subseteq \Omega$ for all $t \in [0,T]$. Let $M \in C_b^k(\Omega;\mathbb{R}^{m\times m}) \cap C_b^{0,1}(\Omega;\mathbb{R}^{m\times m})$. Then $M \circ u \in \mathcal{X}^k(T,m) \hookrightarrow \mathcal{Y}^k(T,m)$. Moreover, we have the estimates*

$$\|M \circ u\|_{\mathcal{X}^k} \leq \left(\|M\|_{C_b^k} + \|M\|_{C_b^{0,1}}\right) P\left(\|u\|_{L^\infty H^k} + \|\partial_t u\|_{L^\infty H^{k-1}}\right) \tag{2.1.10}$$

and

$$\|M \circ u\|_{\mathcal{Y}^k} \leq \|M\|_{C_b^k} P(\|u\|_{L^\infty H^k}) \tag{2.1.11}$$

for some polynomial $P : \mathbb{R} \to \mathbb{R}$ with positive universal coefficients.

Proof. The inclusion and the estimates follow immediately from Theorem 1.1.6, Proposition 2.1.2, and the usual L^∞ Sobolev embedding. □

We will also need the following variant for the nonlinear forcing term.

Theorem 2.1.4. *Suppose that* $u \in L^\infty([0,T]; H^k(\mathbb{R}^n;\mathbb{R}^m))$ *and* $\partial_t u \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^n;\mathbb{R}^m))$ *for some* $k > 1 + n/2$. *Further suppose that* $\Omega \subseteq \mathbb{R}^m$ *is an open set such that* $u(\mathbb{R}^n, t) \subseteq \Omega$ *for all* $t \in [0,T]$. *Let* $F \in C^k(\Omega;\mathbb{R}^m)$ *be such that* $DF \in C_b^{k-1}(\Omega; \mathcal{L}(\mathbb{R}^m))$ *and*

$$|F(z)| \le a\,|z|^p + b\,|z|^q \text{ for } a, b \in [0,\infty) \text{ and } p, q \in [1,\infty). \tag{2.1.12}$$

Then $F \circ u \in L^\infty([0,T]; H^k(\mathbb{R}^n;\mathbb{R}^m))$ *and* $F \circ u \in C_b^0(\mathbb{R}^n \times [0,T])$. *Moreover, we have the estimate*

$$\|F \circ u\|_{L^\infty H^k} \le \left(a + b + \|DF\|_{C_b^{k-1}}\right) P(\|u\|_{L^\infty H^k}). \tag{2.1.13}$$

for some polynomial $P : \mathbb{R} \to \mathbb{R}$ *with positive universal coefficients.*

Proof. First note that

$$\begin{aligned} \int_{\mathbb{R}^n} |F \circ u(\cdot,t)|^2 \le 2 \int_{\mathbb{R}^n} a^2\,|u(\cdot,t)|^{2p} + b^2\,|u(\cdot,t)|^{2q} &\le 2(a^2+b^2)\left(\|u(\cdot,t)\|_{L^{2p}}^{2p} + \|u(\cdot,t)\|_{L^{2q}}^{2q}\right) \\ &\lesssim 2(a^2+b^2)\left(\|u(\cdot,t)\|_{H^k}^{2p} + \|u(\cdot,t)\|_{H^k}^{2q}\right) \end{aligned} \tag{2.1.14}$$

and hence

$$\|F \circ u\|_{L^\infty L^2} \le (a+b) P(\|u\|_{L^\infty H^k}) \tag{2.1.15}$$

for some polynomial $P : \mathbb{R} \to \mathbb{R}$ with positive coefficients (the degree of the polynomial is determined by $\max\{p,q\}$). On the other hand, Theorem 1.1.6 provides us with the bound

$$\|D(F \circ u)\|_{L^\infty H^{k-1}} \le \|DF\|_{C_b^{k-1}} P(\|u\|_{L^\infty H^k}). \tag{2.1.16}$$

These two estimates then imply (2.1.13). The inclusion $F \circ u \in C_b^0$ follows since F is continuous and u is bounded and continuous.

□

2.2 The metric space

Definition 2.2.1. *Suppose that* $T > 0$, $1 + n/2 < k \in \mathbb{N}$, $h \in H^{k+1}(\mathbb{R}^n;\mathbb{R}^m)$. *For* $\sigma_1, \sigma_2 \in (0,\infty)$ *we define*

$$\begin{aligned} S(k,T,h,\sigma_1,\sigma_2) = \{u \in L^\infty([0,T]; H^k(\mathbb{R}^n;\mathbb{R}^m)) \mid \partial_t u \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^n;\mathbb{R}^m)), \\ \text{with the estimates } \|u - h\|_{L^\infty H^k} \le \sigma_1 \text{ and } \|\partial_t u\|_{L^\infty H^{k-1}} \le \sigma_2\}. \end{aligned} \tag{2.2.1}$$

We endow this space with the metric

$$d(u,v) = \|u - v\|_{L^\infty L^2}. \tag{2.2.2}$$

At first glance this seems like a very strange choice for a metric to place on $S(k,T,h,\sigma_1,\sigma_2)$. If we were to use the "natural" metric built into the definition we would be able to trivially show that the space is complete. However, it would fail to serve a useful purpose in our existence theory. Consequently we must show that the space is still complete when endowed with this weak metric.

Theorem 2.2.2. $S(k,T,h,\sigma_1,\sigma_2)$ *is a complete metric space.*

Proof. Assume that $\{v_j\}_{j=0}^\infty \subseteq S(k,T,h,\sigma_1,\sigma_2)$ is Cauchy. Since $L^\infty L^2$ is a Banach space, we have that there exists $v \in L^\infty L^2$ such that $v_j \to v$ in $L^\infty L^2$ as $j \to \infty$, i.e.

$$\|v - v_j\|_{L^\infty L^2} \to 0 \text{ as } j \to \infty. \tag{2.2.3}$$

To conclude we must only prove that $v \in S(k,T,h,\sigma_1,\sigma_2)$.

Since $h \in H^k$, for each j we have that $\|v_j\|_{L^\infty H^k} \le \sigma_1 + \|h\|_{H^k}$ and $\|\partial_t v_j\|_{L^\infty H^{k-1}} \le \sigma_2$. Up to the extraction of a subsequence we have that

$$\begin{aligned} v_j &\overset{*}{\rightharpoonup} v \text{ weakly-}* \text{ in } L^\infty H^k \\ v_j - h &\overset{*}{\rightharpoonup} v \text{ weakly-}* \text{ in } L^\infty H^k \\ \partial_t v_j &\overset{*}{\rightharpoonup} \partial_t v \text{ weakly-}* \text{ in } L^\infty H^{k-1}, \end{aligned} \tag{2.2.4}$$

which in particular means that $v \in L^\infty H^k$ and $\partial_t v \in L^\infty H^k$. To complete the proof we must only show that the required estimates hold. However, these follow from the corresponding bounds on $v_j - h$ and $\partial_t v_j$ and weak$-*$ lower semicontinuity.

□

Proposition 2.2.3. *Suppose that $v \in S(k,T,h,\sigma_1,\sigma_2)$ and $g \in H^k(\mathbb{R}^n;\mathbb{R}^m)$. Further suppose that $\Omega \subseteq \mathbb{R}^m$ is an open set such that*

$$N(g(\mathbb{R}^n), r) = \{z \in \mathbb{R}^m \mid \operatorname{dist}(z, g(\mathbb{R}^n)) < r\} \subseteq \Omega \tag{2.2.5}$$

for some $r > 0$. Then the following hold.

1. *There exists a universal constant $\gamma_0 > 0$ such that*

$$\sup_{t\in[0,T]} \operatorname{dist}(v(\mathbb{R}^n,t), g(\mathbb{R}^n)) \le \gamma_0(\|g-h\|_{H^k} + \sigma_1). \tag{2.2.6}$$

2. *There exists a constant $\delta = \delta(r) > 0$ such that if $\|g-h\|_{H^k} \le \delta$ and $\sigma_1 \le \delta$, then*

$$v(\mathbb{R}^n, t) \subseteq \Omega \textit{ for all } t \in [0,T]. \tag{2.2.7}$$

Proof. We know that

$$\|v-g\|_{L^\infty H^k} \le \|v-h\|_{L^\infty H^k} + \|g-h\|_{H^k} \le \sigma_1 + \|g-h\|_{H^k} \tag{2.2.8}$$

Consequently, Proposition 2.1.2 tells us that there exists a universal constant $\gamma_0 > 0$ such that

$$\|v-g\|_{C_b^0} \le \gamma_0(\sigma_1 + \|g-h\|_{H^k}), \tag{2.2.9}$$

from which the first item easily follows. To prove the second we simply set $\delta = r/(3\gamma_0)$ and use Proposition 2.2.3. □

2.3 The mapping

We now aim to define a solution mapping. First we must clearly specify the assumptions on the nonlinearities.

Assumption 2.3.1. *Assume the following.*

1. $1 \leq n, m \in \mathbb{N}$ *and* $1 + n/2 < k \in \mathbb{N}$.

2. $g \in H^k(\mathbb{R}^n; \mathbb{R}^m)$, *and* $\Omega \subseteq \mathbb{R}^m$ *is an open set such that*

$$N(g(\mathbb{R}^n), r) = \{z \in \mathbb{R}^m \mid \operatorname{dist}(z, g(\mathbb{R}^n)) < r\} \subseteq \Omega \tag{2.3.1}$$

   for some $r > 0$.

3. $A^0, A^1, \dots, A^n, B \in C_b^k(\Omega; \mathbb{R}^{m\times m}) \cap C_b^{0,1}(\Omega; \mathbb{R}^{m\times m})$. *Also,* $A^0(z), A^1(z), \dots, A^n(z)$ *are symmetric matrices for all* $z \in \Omega$, *and there exists* $\theta > 0$ *such that* $A^0(z) \geq \theta I$ *for all* $z \in \Omega$.

4. $F \in C^k(\Omega; \mathbb{R}^m)$ *is such that* $DF \in C_b^{k-1}(\Omega; \mathcal{L}(\mathbb{R}^m))$ *and*

$$|F(z)| \leq a\,|z|^p + b\,|z|^q \ \text{ for } a, b \in [0, \infty) \text{ and } p, q \in [1, \infty). \tag{2.3.2}$$

   Also,

$$\sup_{\substack{z,w\in\Omega \\ z\neq w}} \frac{|F(z) - F(w)|}{|z - w|} = [F]_{C^{0,1}} < \infty. \tag{2.3.3}$$

5. *The constants* $\lambda_1, \lambda_2, \lambda_3 \in [0, \infty)$ *are given by*

$$\begin{aligned}
\lambda_1 &= \max\{\|A^0\|_{C_b^k}, \|A^1\|_{C_b^k}, \dots, \|A^n\|_{C_b^k}, \|B\|_{C_b^k}\}, \\
\lambda_2 &= \max\{\|A^0\|_{C_b^{0,1}}, \|A^1\|_{C_b^{0,1}}, \dots, \|A^n\|_{C_b^{0,1}}, \|B\|_{C_b^{0,1}}\}, \\
\lambda_3 &= a + b + \|DF\|_{C_b^{k-1}}.
\end{aligned} \tag{2.3.4}$$

Now we construct the mapping.

Theorem 2.3.2. *Assume Assumption 2.3.1. Let* $T_* \in (0, \infty)$ *and suppose that*

$$f \in L^\infty([0, T_*]; H^k(\mathbb{R}^n; \mathbb{R}^m)) \cap C_b^0(\mathbb{R}^n \times [0, T_*]). \tag{2.3.5}$$

Let $h \in H^{k+1}(\mathbb{R}^n; \mathbb{R}^m)$, $0 < T \leq T_*$, $\sigma_1, \sigma_2 \in (0, \infty)$, *and assume that*

$$0 < \sigma_1 \leq \delta \text{ and } \|g - h\|_{H^k} \leq \delta, \tag{2.3.6}$$

where $\delta = \delta(r) > 0$ *is as in Proposition 2.2.3. Then the following hold.*

1. *For every* $v \in S(k, T, h, \sigma_1, \sigma_2)$ *we have that*

$$A^0 \circ v, A^1 \circ v, \dots A^n \circ v, B \circ v \in \mathcal{X}^k(T, m) \tag{2.3.7}$$

   and $F \circ v \in L^\infty([0, T]; H^k(\mathbb{R}^n; \mathbb{R}^m)) \cap C_b^0(\mathbb{R}^n \times [0, T]; \mathbb{R}^m)$.

2. *For every* $v \in S(k,T,h,\sigma_1,\sigma_2)$ *there exists a unique* $u \in C^1_b(\mathbb{R}^n \times [0,T];\mathbb{R}^m)$ *such that* $u \in L^\infty([0,T];H^k(\mathbb{R}^n;\mathbb{R}^m))$*,* $\partial_t u \in L^\infty([0,T];H^{k-1}(\mathbb{R}^n;\mathbb{R}^m))$*, and* u *is the unique solution to*

$$\begin{cases} A^0(v)\partial_t u + A^j(v)\partial_j u + B(v)u = f + F(v) & \text{in } \mathbb{R}^n \times [0,T] \\ u(\cdot,0) = g & \text{in } \mathbb{R}^n. \end{cases} \tag{2.3.8}$$

3. *There exist continuous functions*

$$\bar{T} : (0,\infty)^2 \times [0,\infty)^3 \to (0,T_*], \bar{\varepsilon} : (0,\infty)\times[0,\infty) \to (0,\delta], \text{ and } \bar{\zeta} : (0,\infty)\times[0,\infty)^2 \to (0,\infty) \tag{2.3.9}$$

such that if

$$\begin{aligned} &\|g-h\|_{H^k} \le \bar{\varepsilon}(\sigma_1, \|g\|_{H^k}), \\ &\bar{\zeta}(\sigma_1, \|f\|_{L^\infty_{T_*}H^{k-1}}, \|g\|_{H^k}) \le \sigma_2, \text{ and} \\ &0 < T \le \bar{T}(\sigma_1,\sigma_2,\|f\|_{L^\infty_{T_*}H^k}, \|g\|_{H^k}, \|h\|_{H^{k+1}}), \end{aligned} \tag{2.3.10}$$

then $u \in S(k,T,h,\sigma_1,\sigma_2)$*. Here* $\|f\|_{L^\infty_{T_*}H^\ell}$ *denotes the norm with the temporal supremum evaluated over* $[0,T_*]$*.*

Proof. To begin we note that the assumption (2.3.6) and Proposition 2.2.3 imply that $v(\mathbb{R}^n,t) \subseteq \Omega$ for every $t \in [0,T]$. Consequently, $A^0 \circ v, A^1 \circ v, \dots, A^n \circ v$, $B \circ v$, and $F \circ v$ are all well-defined, and we may apply Theorems 2.1.3 and 2.1.4 to deduce the first item. The second item then follows directly from Theorem 1.2.1 applied with forcing $f + F(v)$, coefficient matrices $A^0 \circ v, \dots, A^n \circ v, B \circ v$, and initial data g.

We now turn to the proof of the third item. Note that

$$\|f\|_{L^\infty H^\ell} = \|f\|_{L^\infty_T H^\ell} \le \|f\|_{L^\infty_{T_*} H^\ell} \tag{2.3.11}$$

for every $0 < T \le T_*$ and $0 \le \ell \le k$. The fourth item of Theorem 1.2.1 then tells us that

$$\|u-h\|^2_{L^\infty H^k} \le Q \cdot e^{P\cdot T}\left(\|g-h\|^2_{H^k} + T\|f\|^2_{L^\infty_{T_*}H^k} + T\|F(v)\|^2_{L^\infty H^k} + T\|h\|^2_{H^{k+1}}\right) \tag{2.3.12}$$

and

$$\begin{aligned} \|\partial_t u\|^2_{L^\infty H^{k-1}} \le Q \cdot e^{P\cdot T}\left(\|g-h\|^2_{H^k} + T\|f\|^2_{L^\infty H^k} + T\|F(v)\|^2_{L^\infty_{T_*}H^k} + T\|h\|^2_{H^{k+1}}\right) \\ + Q\cdot\left(\|f\|^2_{L^\infty_{T_*}H^{k-1}} + \|F(v)\|^2_{L^\infty H^{k-1}} + \|h\|^2_{H^k}\right) \end{aligned} \tag{2.3.13}$$

for polynomials P, Q of the form (1.2.11) with positive coefficients. Here we have used the fact that $f, F(v) \in L^\infty H^k$ rather than $L^2 H^k$ in order to introduce the factor of T in various places.

Note that

$$\|u\|_{L^\infty H^k} \le \|u-h\|_{L^\infty H^k} + \|h-g\|_{H^k} + \|g\|_{H^k} \le \sigma_1 + \delta + \|g\|_{H^k}. \tag{2.3.14}$$

Theorem 2.1.3 then implies that

$$P \le P_0(\lambda_1+\lambda_2, \sigma_1+\sigma_2+\delta+\|g\|_{H^k}, 1/\theta) \text{ and } Q \le Q_0(\lambda_1, \sigma_1+\delta+\|g\|_{H^k}, 1/\theta) \tag{2.3.15}$$

for polynomials $P_0, Q_0 : \mathbb{R}^3 \to \mathbb{R}$ with positive coefficients. Similarly, Theorem 2.1.4 implies that

$$\|F \circ v\|_{L^\infty H^k}^2 \le \lambda_3^2 R_0(\sigma_1 + \delta + \|g\|_{H^k}) \tag{2.3.16}$$

for a polynomial $R_0 : \mathbb{R} \to \mathbb{R}$ with positive coefficients.

Define the continuous functions $\bar{T}_0, \bar{T}_1 : (0,\infty)^2 \times [0,\infty)^3 \to (0,\infty)$ by

$$\bar{T}_0(s_1, s_2, w_1, w_2, w_3) = \frac{\log 2}{1 + P_0(\lambda_1 + \lambda_2, s_1 + s_2 + \delta + w_2, 1/\theta)} \tag{2.3.17}$$

and

$$\bar{T}_1(s_1, s_2, w_1, w_2, w_3) = \frac{1}{4} \frac{s_1^2}{1 + Q_0(\lambda_1, s_1 + \delta + w_2, 1/\theta)[w_3^2 + w_1^2 + \lambda_3^2 R_0(s_1 + \delta + w_2)]}. \tag{2.3.18}$$

We then set $\bar{T} : (0,\infty)^2 \times [0,\infty)^3 \to (0, T_*]$ via

$$\bar{T}(s_1, s_2, w_1, w_2, w_3) = \min\{\bar{T}_0(s_1, s_2, w_1, w_2, w_3), \bar{T}_1(s_1, s_2, w_1, w_2, w_3), T_*\}, \tag{2.3.19}$$

which is clearly continuous. Next let $\bar{\varepsilon} : (0,\infty) \times [0,\infty) \to (0,\delta]$ be the continuous function defined by

$$\bar{\varepsilon}(s_1, w) = \sqrt{\min\left\{\delta^2, \frac{s_1^2}{4[1 + Q_0(\lambda_1, s_1 + \delta + w, 1/\theta)]}\right\}}. \tag{2.3.20}$$

Finally, let $\bar{\zeta} : (0,\infty) \times [0,\infty)^2 \to (0,\infty)$ be the continuous function given by

$$\bar{\zeta}(s_1, w_1, w_2) = \sqrt{s_1^2 + Q_0(\lambda_1, s_1 + \delta + w_2, 1/\theta)[w_1^2 + 2\delta^2 + 2w_2^2 + \lambda_3^2 R_0(s_1 + \delta + w_2)]} \tag{2.3.21}$$

Now assume that (2.3.10) holds. The definition of $\bar{T}$ guarantees that $e^{P \cdot T} \le e^{\log 2} = 2$,

$$Q \cdot e^{P \cdot T} \left(T \|f\|_{L^\infty_{T_*} H^k}^2 + T \|F(v)\|_{L^\infty H^k}^2 + T \|h\|_{H^{k+1}}^2 \right) \le \frac{\sigma_1^2}{2}, \tag{2.3.22}$$

while the definition of $\bar{\varepsilon}$ guarantees that

$$Q \cdot e^{P \cdot T} \|g - h\|_{H^k}^2 \le \frac{\sigma_1^2}{2}. \tag{2.3.23}$$

Consequently, (2.3.12) implies

$$\|u - h\|_{L^\infty H^k}^2 \le \frac{\sigma_1^2}{2} + \frac{\sigma_1^2}{2} = \sigma_1^2. \tag{2.3.24}$$

Finally, the definition of $\bar{\zeta}$ and (2.3.13) guarantee that

$$\begin{aligned} \|\partial_t u\|_{L^\infty H^{k-1}}^2 \le \sigma_1^2 + Q_0(\lambda_1, \sigma_1 + \delta + \|g\|_{H^k}, 1/\theta)[\|f\|_{L^\infty_{T_*} H^{k-1}}^2 + 2\delta^2 + 2\|g\|_{H^k}^2 + \lambda_3^2 R_0(\sigma_1 + \delta + \|g\|_{H^k})] \\ = \bar{\zeta}(\sigma_1) \le \sigma_2^2. \end{aligned} \tag{2.3.25}$$

Thus $u \in S(k, T, h, \sigma_1, \sigma_2)$, which proves the third item.

□

2.4 The fixed point

Theorem 2.4.1. *Assume Assumption 2.3.1. Let $T_* \in (0,\infty)$ and suppose that*

$$f \in L^\infty([0,T_*]; H^k(\mathbb{R}^n;\mathbb{R}^m)) \cap C^0_b(\mathbb{R}^n \times [0,T_*]). \tag{2.4.1}$$

Assume the following.

I. *Let $\sigma_1 = \delta = \delta(r) > 0$ be the constant from Proposition 2.2.3.*

II. *Let $\varepsilon = \bar{\varepsilon}(\sigma_1, \|g\|_{H^k}) \in (0,\delta]$, where $\bar{\varepsilon}$ is the function given in the third item of Theorem 2.3.2.*

III. *Let $\sigma_2 = \bar{\zeta}(\sigma_1, \|f\|_{L^\infty_{T_*}H^{k-1}}, \|g\|_{H^k}) \in (0,\infty)$, where $\bar{\zeta}$ is the function given in the third item of Theorem 2.3.2.*

IV. *Let $h \in H^{k+1}(\mathbb{R}^n;\mathbb{R}^m)$ be such that $\|g-h\|_{H^k} < \varepsilon \le \delta$. The existence of such an h is guaranteed by Proposition 2.1.1.*

V. *Let $0 < T \le \bar{T}(\sigma_1, \sigma_2, \|f\|_{L^\infty_{T_*}H^k}, \|g\|_{H^k}, \|h\|_{H^{k+1}}) \in (0,T_*]$, where $\bar{T}$ is the function given in the third item of Theorem 2.3.2.*

Then the following hold.

1. *For each $v \in S(k,T,h,\sigma_1,\sigma_2)$ there exists a unique $u \in S(k,T,h,\sigma_1,\sigma_2)$ such that u is the unique solution to*

$$\begin{cases} A^0(v)\partial_t u + A^j(v)\partial_j u + B(v)u = f + F(v) & \text{in } \mathbb{R}^n \times [0,T] \\ u(\cdot,0) = g & \text{in } \mathbb{R}^n. \end{cases} \tag{2.4.2}$$

2. *Let $\Phi : S(k,T,h,\sigma_1,\sigma_2) \to S(k,T,h,\sigma_1,\sigma_2)$ be given by $\Phi(v) = u$, where u is as in the previous item. Then Φ is Lipschitz and obeys the estimate*

$$d(\Phi(v_1),\Phi(v_2)) \le \sqrt{\frac{T[\lambda_2^2\gamma((\sigma_1+\delta+\|g\|_{H^k})^2+\sigma_2^2)+[F]^2_{C^{0,1}}]}{\theta}}\, e^{P\cdot T} d(v_1,v_2) \tag{2.4.3}$$

where $P = P(\lambda_1+\lambda_2, \sigma_1+\sigma_2+\delta+\|g\|_{H^k}, 1/\theta)$ is a polynomial with positive coefficients and $\gamma > 0$ is a universal constant.

Proof. The first item follows directly from Theorem 2.3.2 in light of the assumptions I–V. We now turn to the proof of the second item. Assume that $v_1, v_2 \in S(k,T,h,\sigma_1,\sigma_2)$, write $u_1 = \Phi(v_1)$, $u_2 = \Phi(v_2)$, and $u = u_1 - u_2$. Note that $u_1, u_2 \in S(k,T,h,\sigma_1,\sigma_2)$ implies that

$$\|u_i\|_{L^\infty H^k} \le \|u_i - h\|_{L^\infty H^k} + \|g-h\|_{H^k} + \|g\|_{H^k} \le \sigma_1 + \delta + \|g\|_{H^k} \tag{2.4.4}$$

and

$$\|\partial_t u_i\|_{L^\infty H^{k-1}} \tag{2.4.5}$$

for $i = 1,2$.

Next we compute

$$\begin{cases} A^0(v_1)\partial_t u + A^j(v_1)\partial_j u + B(v_1)u = Z & \text{in } \mathbb{R}^n \times [0,T] \\ u(\cdot,0) = 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{2.4.6}$$

where

$$Z = (A^0(v_2) - A^0(v_1))\partial_t u_2 + (A^j(v_2) - A^j(v_1))\partial_j u_2 + (B(v_2) - B(v_1))u_2 + (F(v_1) - F(v_2)). \quad (2.4.7)$$

We may easily estimate

$$\begin{aligned}\|Z\|^2_{L^\infty L^2} &\le \lambda_2^2 \|v_1 - v_2\|^2_{L^\infty L^2} \left(\|\partial_t u_1\|^2_{L^\infty L^\infty} + \|u_2\|^2_{L^\infty W^{1,\infty}}\right) + [F]^2_{C^{0,1}} \|v_1 - v_2\|^2_{L^\infty L^2} \\ &\le \lambda_2^2 \gamma \|v_1 - v_2\|^2_{L^\infty L^2} \left(\|\partial_t u_1\|^2_{L^\infty H^{k-1}} + \|u_2\|^2_{L^\infty H^k}\right) + [F]^2_{C^{0,1}} \|v_1 - v_2\|^2_{L^\infty L^2} \\ &\le \left(\lambda_2^2 \gamma((\sigma_1 + \delta + \|g\|_{H^k})^2 + \sigma_2^2) + [F]^2_{C^{0,1}}\right) \|v_1 - v_2\|^2_{L^\infty L^2} \quad (2.4.8)\end{aligned}$$

for $\gamma > 0$ a universal constant. We may then apply the basic L^2 estimate to u to bound

$$\|u\|^2_{L^\infty L^2} \le \frac{e^{\mu T}}{\theta} \int_0^T \|Z(\cdot, t)\|^2_{L^2}\, dt \le \frac{T e^{\mu T}}{\theta} \|Z\|^2_{L^\infty L^2} \quad (2.4.9)$$

where

$$\mu = \frac{1}{\theta}\left(1 + \left\|\partial_t(A^0 \circ v_1)\right\|_{L^\infty} + \left\|\partial_j(A^j \circ v_1)\right\|_{L^\infty} + 2\|B \circ v_1\|_{L^\infty}\right). \quad (2.4.10)$$

We may then employ Theorem 2.1.3 to deduce that

$$\mu \le P(\lambda_1 + \lambda_2, \sigma_1 + \sigma_2 + \delta + \|g\|_{H^k}, 1/\theta) \quad (2.4.11)$$

for a polynomial $P : \mathbb{R}^3 \to \mathbb{R}$ with positive coefficients. The estimate (2.4.3) then follows from (2.4.8) and (2.4.10).

□

Finally, we have all the tools needed to produce our solution.

Theorem 2.4.2. *Assume Assumption 2.3.1. Let* $T_* \in (0, \infty)$ *and suppose that*

$$f \in L^\infty([0, T_*]; H^k(\mathbb{R}^n; \mathbb{R}^m)) \cap C^0_b(\mathbb{R}^n \times [0, T_*]). \quad (2.4.12)$$

Then there exists a

$$0 < T_0 = T_0(\|f\|_{L^\infty_{T_*} H^k}, \|g\|_{H^k}) \le T_* \quad (2.4.13)$$

such that if $0 < T \le T_0$ *then there exists a unique* $u \in C^1_b(\mathbb{R}^n \times [0, T]; \mathbb{R}^m)$ *such that* $u \in L^\infty([0, T]; H^k(\mathbb{R}^n; \mathbb{R}^m))$, $\partial_t u \in L^\infty([0, T]; H^{k-1}(\mathbb{R}^n; \mathbb{R}^m))$, *and* u *is the solution to*

$$\begin{cases} A^0(u)\partial_t u + A^j(u)\partial_j u + B(u)u = f + F(u) & \text{in } \mathbb{R}^n \times [0, T] \\ u(\cdot, 0) = g & \text{in } \mathbb{R}^n. \end{cases} \quad (2.4.14)$$

Moreover, $u \in S(k, T, h, \sigma_1, \sigma_2)$, *where* σ_1, σ_2 *and* h *are as in Theorem 2.4.1.*

Proof. Let σ_1, σ_2 and h be as in Theorem 2.4.1. Set $T_1 = \bar{T}(\sigma_1, \sigma_2, \|f\|_{L^\infty_{T_*} H^k}, \|g\|_{H^k}, \|h\|_{H^{k+1}}) \in (0, T_*]$. Set

$$0 < T_2 = \frac{\log 2}{P(\lambda_1 + \lambda_2, \sigma_1 + \sigma_2 + \delta + \|g\|_{H^k}, 1/\theta)} \quad (2.4.15)$$

where P is the polynomial from Theorem 2.4.1. Finally,

$$0 < T_3 = \frac{\theta}{16[\lambda_2^2 \gamma((\sigma_1 + \delta + \|g\|_{H^k})^2 + \sigma_2^2) + [F]^2_{C^{0,1}}]} \quad (2.4.16)$$

where $\gamma > 0$ is the universal constant from Theorem 2.4.1. Note that σ_1 and σ_2 are determined by the data g and f, T_1, T_2, T_3 are determined by them as well. Finally, set

$$0 < T_0 = \min\{T_1, T_2, T_3\} \leq T_*. \tag{2.4.17}$$

Theorem 2.4.1 and the bounds on T_2 and T_3 then imply that $\Phi : S(k, T, h, \sigma_1, \sigma_2) \to S(k, T, h, \sigma_1, \sigma_2)$ is such that

$$d(\Phi(v_1), \Phi(v_2)) \leq \frac{1}{2} d(v_1, v_2), \tag{2.4.18}$$

and hence Φ is a contraction. Since $S(k, T, h, \sigma_1, \sigma_2)$ is a complete metric space we may apply the contraction mapping principle to deduce the existence of a unique $u \in S(k, T, h, \sigma_1, \sigma_2)$ such that $\Phi(u) = u$.

□

Remark 2.4.3. *This result is not technically a well-posedness result since it fails to establish that the solution depends continuously (in some topology) on the data. For the sake of time we will ignore this issue here. However, with a little more work we could establish it in the framework we have developed. To see details of the proof we refer to Theorem III in Kato's seminal paper [5].*

3 Examples

3.1 Quasilinear wave equations

We now turn our attention to the quasilinear wave equation

$$\begin{cases} \partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u = f + F(u, \partial_t u, \nabla u) & \text{in } \mathbb{R}^n \times [0, T] \\ u(\cdot, 0) = g \text{ and } \partial_t u(\cdot, 0) = h & \text{in } \mathbb{R}^n. \end{cases} \tag{3.1.1}$$

Here we assume that

$$A \in C^k(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n; \mathbb{R}^{n \times n}) \tag{3.1.2}$$

is such that

$$A(z, w, p) = A^T(z, w, p) \text{ for all } (z, w, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \tag{3.1.3}$$

and there exists $\theta > 0$ such that

$$A(z, w, p) \geq \theta I \text{ for all } (z, w, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n. \tag{3.1.4}$$

We also assume that

$$F \in C^k(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n; \mathbb{R}). \tag{3.1.5}$$

We now produce solutions.

Theorem 3.1.1. *Let $k > 1 + n/2$ and assume* (3.1.2)–(3.1.5)*. Let $g \in H^{k+1}(\mathbb{R}^n; \mathbb{R})$, $h \in H^k(\mathbb{R}^n; \mathbb{R})$, and $f \in L^\infty([0, T]; H^k(\mathbb{R}^n; \mathbb{R}))$. Then there exists $T \in (0, \infty]$ and $u \in C_b^2(\mathbb{R}^n \times [0, T]; \mathbb{R})$ such that the following hold.*

1. *$u \in L^\infty([0, T]; H^{k+1}(\mathbb{R}^n; \mathbb{R}))$, $\partial_t u \in L^\infty([0, T]; H^k(\mathbb{R}^n; \mathbb{R}))$, $\partial_t^2 u \in L^\infty([0, T]; H^{k-1}(\mathbb{R}^n; \mathbb{R}))$.*

2. *u solves*

$$\begin{cases} \partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u = f + F(u, \partial_t u, \nabla u) & \text{in } \mathbb{R}^n \times [0, T] \\ u(\cdot, 0) = g \text{ and } \partial_t u(\cdot, 0) = h & \text{in } \mathbb{R}^n. \end{cases} \tag{3.1.6}$$

3. If $T < \infty$ then

$$\limsup_{t \to T^-} \left(\|u(\cdot,t)\|_{H^{k+1}} + \|\partial_t u(\cdot,t)\|_{H^k} \right) = \infty. \tag{3.1.7}$$

Proof. Write $A^0, A^1, \ldots, A^n, B \in C^k(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n; \mathbb{R}^{(n+2)\times(n+2)})$ via

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n1} & \cdots & a_{nn}, \end{pmatrix} \tag{3.1.8}$$

$$A^j = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -a_{1j} & \cdots & -a_{nj} \\ 0 & -a_{1j} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -a_{nj} & 0 & \cdots & 0 \end{pmatrix} \tag{3.1.9}$$

and

$$B = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.1.10}$$

The assumptions guarantee that $A^0, \ldots, A^n$ are symmetric and $A^0 \geq \min\{\theta, 1\} I$. Set $\mathcal{F} \in C^k(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n; \mathbb{R}^{n+2})$ via

$$\mathcal{F}(z,w,p) = (0, F(z,w,p), 0, \ldots, 0). \tag{3.1.11}$$

The assumptions on the data allow us to find Ω and r such that Assumptions 2.3.1 hold. We may then apply Theorem 2.4.2 to produce solutions on a time interval $[0, T_0]$ to

$$\begin{cases} A^0(U)\partial_t U + A^j(U)\partial_j U + B(U)U = f + \mathcal{F}(U) \\ U(\cdot, 0) = (g, h, \nabla g) \in H^k. \end{cases} \tag{3.1.12}$$

As in the linear analysis we deduce from the structure of the matrices that $U = (u, \partial_t u, \nabla u)$ and that (3.1.6) holds.

We may then iterate as long as condition in the third item holds, which extends the solution to $[T_0, T_1]$. We continue this process until either the solution exists on $[0, \infty]$ or else the condition in the third item holds.

□

Remark 3.1.2. *It's possible to prove variants of this result under the the assumption that $A \in C^k(\Omega; \mathbb{R}^{n\times n})$, where $\Omega \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ is some open set. Then, of course, solutions can cease to exist by exiting Ω.*

3.2 Wave maps

In the theory we've developed in class we only considered equations where the solution belongs pointwise to some linear space, say $\mathbb{R}^m$ for $m \geq 1$. However, it's also perfectly natural to consider functions taking values in more general manifolds $\mathcal{M}$. One particular case of this is to consider maps taking values in $\mathbb{S}^m \subseteq \mathbb{R}^{m+1}$. These arise, for instance, in physical theories known as $\sigma-$models. We will now consider the wave map problem, i.e. we seek $u : \mathbb{R}^n \times [0,T] \to \mathbb{S}^m \subseteq \mathbb{R}^{m+1}$ such that

$$\partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u \perp T_u \mathbb{S}^m. \tag{3.2.1}$$

This condition is essentially the correct geometric analog of $Lu = 0$: it says that the vector $\partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u$ does not have to vanish, but it must be orthogonal to the tangent space of $\mathbb{S}^m$ at each point. This yields $\partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u = 0$ when the manifold is $\mathbb{R}^m$ in place of $\mathbb{S}^m$. Here we're assuming that $A \in C^k(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \times \mathbb{R}^{(m+1)\times n})$ with $A = A^T$ and $A \geq \theta I$.

The problem (3.2.1) is rather implicit as stated and can be reformulated in a more convenient way. The sphere makes this particularly easy, as the condition $z \perp T_x \mathbb{S}^m$ for $x \in \mathbb{S}^m$ and $z \in \mathbb{R}^{m+1}$ is equivalent to $z = \lambda x$ for some $\lambda \in \mathbb{R}$. Then (3.2.1) is equivalent to

$$\partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u = \lambda u \text{ for some function } \lambda : \mathbb{R}^n \times [0,T] \to \mathbb{R}. \tag{3.2.2}$$

We now seek to determine λ.

To this end let's suppose that $v : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{m+1}$ is, say, C^2, and let's define

$$\mu(x,t) = |v(x,t)|^2 . \tag{3.2.3}$$

Then

$$\partial_t \mu = 2v \cdot \partial_t v \text{ and } \partial_t^2 \mu = 2 |\partial_t v|^2 + 2v \cdot \partial_t^2 v \tag{3.2.4}$$

and

$$\partial_i \mu = 2v \cdot \partial_i v \text{ and } \partial_i \partial_j \mu = 2\partial_i v \cdot \partial_j v + 2v \cdot \partial_i \partial_j v, \tag{3.2.5}$$

which means that

$$\partial_t^2 \mu - A : D^2 \mu = 2(|\partial_t v|^2 - A_{ij} \partial_i v \cdot \partial_j v) + 2v \cdot (\partial_t^2 v - A : D^2 v). \tag{3.2.6}$$

Now, using this result with $u = v$ we have that $\mu = 1$, and so

$$\begin{aligned} \lambda = \lambda |u|^2 = \lambda u \cdot u &= (\partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u) \cdot u \\ &= \frac{1}{2}(\partial_t^2 \mu - \Delta \mu) + (A(u, \partial_t u, \nabla u)_{ij} \partial_i u \cdot \partial_j u - |\partial_t u|^2) \\ &= A(u, \partial_t u, \nabla u)_{ij} \partial_i u \cdot \partial_j u - |\partial_t u|^2 . \end{aligned} \tag{3.2.7}$$

Thus

$$\partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u = u(A(u, \partial_t u, \nabla u)_{ij} \partial_i u \cdot \partial_j u - |\partial_t u|^2), \tag{3.2.8}$$

which is a more explicit form of the wave map equations. We also know that

$$u(\cdot, 0) = g \text{ and } \partial_t u(\cdot, 0) = h \tag{3.2.9}$$

satisfy

$$|g(x)| = 1 \text{ and } g(x) \cdot h(x) = 0 \text{ for all } x \in \mathbb{R}^n. \tag{3.2.10}$$

The latter follows from the fact that $\partial_t \mu = 0$ at $t = 0$. From this we see that the wave map problem is a semilinear system of second-order hyperbolic equations with some conditions on the data.

In fact, we can use our analysis of μ to push things a bit farther. Suppose that $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{m+1}$ is C^2 and satisfies (3.2.8) with data satisfying (3.2.10). The point is that we assume we have a solution to the PDE but we don't know that it satisfies the desired geometric property, namely that $u(x,t) \in \mathbb{S}^m$ for all x, t. However, we know that for $\mu = |u|^2$,

$$\begin{aligned}\partial_t^2 \mu - A(u, \partial_t u, \nabla u) : D^2\mu &= 2(|\partial_t u|^2 - A_{ij}(u, \partial_t u, \nabla u)\partial_i u \cdot \partial_j u) + 2u \cdot (\partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u) \\ &= 2(1-\mu)(|\partial_t u|^2 - A_{ij}(u, \partial_t u, \nabla u)\partial_i u \cdot \partial_j u) \quad (3.2.11)\end{aligned}$$

and so

$$\begin{cases} \partial_t^2 \mu - A(u, \partial_t u, \nabla u) : D^2\mu = 2(1-\mu)(|\partial_t u|^2 - A_{ij}(u, \partial_t u, \nabla u)\partial_i u \cdot \partial_j u) \\ \mu(\cdot, 0) = 1 \text{ and } \partial_t \mu(\cdot, 0) = 2g \cdot h = 0. \end{cases} \quad (3.2.12)$$

The uniqueness of solutions to forced wave equations of this form then shows that

$$\mu(x,t) = 1 \text{ for all } x \in \mathbb{R}^n \text{ and } t \in [0,T]. \quad (3.2.13)$$

The upshot of this analysis is that the wave map problem

$$\begin{cases} u : \mathbb{R}^n \times [0,T] \to \mathbb{S}^m \\ \partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u = u(A(u, \partial_t u, \nabla u)_{ij}\partial_i u \cdot \partial_j u - |\partial_t u|^2) \\ u(\cdot, 0) = g \text{ and } \partial_t u(\cdot, 0) = h \\ |g| = 1 \text{ and } g \cdot h = 0 \end{cases} \quad (3.2.14)$$

is equivalent to

$$\begin{cases} u : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{m+1} \\ \partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u = u(A(u, \partial_t u, \nabla u)_{ij}\partial_i u \cdot \partial_j u - |\partial_t u|^2) \\ u(\cdot, 0) = g \text{ and } \partial_t u(\cdot, 0) = h \\ |g| = 1 \text{ and } g \cdot h = 0. \end{cases} \quad (3.2.15)$$

In other words, we can recover the geometric condition $u(x,t) \in \mathbb{S}^m$ from the PDE itself and some conditions on the initial data. This is good news, as it opens the door for us to apply the techniques of symmetric hyperbolic systems.

The bad news is that the condition $|g| = 1$ is not compatible with $g \in H^k(\mathbb{R}^n; \mathbb{R}^{m+1})$ since certainly $g \notin L^2(\mathbb{R}^n; \mathbb{R}^{m+1})$. To get around this we will consider only solutions to the wave map problem that are perturbations of a fixed direction $\xi \in \mathbb{S}^m$. In other words, we posit that

$$u(x,t) = \xi + v(x,t). \quad (3.2.16)$$

Then (3.2.15) is equivalent to

$$\begin{cases} v : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{m+1} \\ \partial_t^2 v - A(\xi + v, \partial_t v, \nabla v) : D^2 v = (\xi + v)(A(\xi + v, \partial_t v, \nabla v)_{ij}\partial_i v \cdot \partial_j v - |\partial_t v|^2) \\ v(\cdot, 0) = \tilde{g} \text{ and } \partial_t v(\cdot, 0) = h \\ |\tilde{g}|^2 + 2\tilde{g} \cdot \xi = 0 \text{ and } (\xi + \tilde{g}) \cdot h = 0. \end{cases} \quad (3.2.17)$$

The key to this formulation is that we can find $\tilde{g} \in H^{k+1}$ and $h \in H^k$ satisfying the last condition.

Now define

$$\Psi(v, \partial_t v, \nabla v) = A(\xi + v, \partial_t v, \nabla v)_{ij}\partial_i v \cdot \partial_j v - |\partial_t v|^2 \in \mathbb{R} \quad (3.2.18)$$

and consider the $(n+2) \times (n+2)$ matrices

$$M^0(v, \partial_t v, \nabla v) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n1} & \cdots & a_{nn} \end{pmatrix}, \tag{3.2.19}$$

$$M^j(v, \partial_t v, \nabla v) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -a_{1j} & \cdots & -a_{nj} \\ 0 & -a_{1j} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -a_{nj} & 0 & \cdots & 0 \end{pmatrix}, \tag{3.2.20}$$

for $j = 1, \ldots, n$, and

$$N(v, \partial_t v, \nabla v) = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ -\Psi & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.2.21}$$

We then define $A^0, A^1, \ldots, A^n, B$ to be the $(m+1)(n+2) \times (m+1)(n+2)$ matrices given in block form by

$$A^0(v, \partial_t v, \nabla v) = \begin{pmatrix} M^0 & 0 & \cdots & 0 \\ 0 & M^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M^0 \end{pmatrix}, A^j(v, \partial_t v, \nabla v) = \begin{pmatrix} M^j & 0 & \cdots & 0 \\ 0 & M^j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M^j \end{pmatrix}, \tag{3.2.22}$$

and

$$B(v, \partial_t v, \nabla v) = \begin{pmatrix} N & 0 & \cdots & 0 \\ 0 & N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N. \end{pmatrix}, \tag{3.2.23}$$

For $i = 1, \ldots, m+1$ set $F_i \in \mathbb{R}^{n+2}$ via

$$F_i(v, \partial_t v, \nabla v) = (0, \xi_i \Psi, 0, \ldots, 0) \tag{3.2.24}$$

and then define $\mathcal{F}$ to be the $(m+1)(n+2)$ vector

$$\mathcal{F}(v, \partial_t v, \nabla v) = (F_1(v, \partial_t v, \nabla v), F_2(v, \partial_t v, \nabla v), \ldots, F_{m+1}(v, \partial_t v, \nabla v)). \tag{3.2.25}$$

Then for $V = (v_1, \partial_t v_1, \nabla v_1, \ldots, v_{m+1}, \partial_t v_{m+1}, \nabla v_{m+1}) \in \mathbb{R}^{(m+1)(n+2)}$ we find that the wave map problem is equivalent to

$$A^0(V)\partial_t V + A^j(V)\partial_j V + B(V)V = \mathcal{F}(V), \tag{3.2.26}$$

which is a symmetric hyperbolic system.

We may then easily modify our analysis of the scalar quasilinear wave equation to prove the following theorem.

Theorem 3.2.1. *Let $k > 1 + n/2$ and assume that*

$$A \in C^k(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \times \mathbb{R}^{(m+1)\times n}; \mathbb{R}^{n\times n}) \tag{3.2.27}$$

is such that

$$A(z, w, p) = A^T(z, w, p) \text{ for all } (z, w, p) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \times \mathbb{R}^{(m+1)\times n} \tag{3.2.28}$$

and there exists $\theta > 0$ such that

$$A(z, w, p) \geq \theta I \text{ for all } (z, w, p) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \times \mathbb{R}^{(m+1)\times n}. \tag{3.2.29}$$

Let $\xi \in \mathbb{S}^m$, $\tilde{g} \in H^{k+1}(\mathbb{R}^n; \mathbb{R}^{m+1})$, $h \in H^k(\mathbb{R}^n; \mathbb{R}^{m+1})$ be such that

$$|\xi + \tilde{g}(x)| = 1 \text{ and } (\xi + \tilde{g}(x)) \cdot h(x) = 0 \text{ for all } x \in \mathbb{R}^n. \tag{3.2.30}$$

Then there exists $T \in (0, \infty]$ and $u \in C^2_b(\mathbb{R}^n \times [0, T]; \mathbb{R}^{m+1})$ such that the following hold.

1. *$u(x, t) \in \mathbb{S}^m$ for all $x \in \mathbb{R}^n$ and $t \in [0, T]$.*

2. *$u = \xi + v$, where*

$$\begin{aligned} v &\in L^\infty([0, T]; H^{k+1}(\mathbb{R}^n; \mathbb{R}^{m+1})), \\ \partial_t v &\in L^\infty([0, T]; H^k(\mathbb{R}^n; \mathbb{R}^{m+1})), \\ \partial_t^2 v &\in L^\infty([0, T]; H^{k-1}(\mathbb{R}^3; \mathbb{R}^{m+1})). \end{aligned} \tag{3.2.31}$$

3. *u solves*

$$\begin{cases} \partial_t^2 u - A(u, \partial_t u, \nabla u) : D^2 u = u(A(u, \partial_t u, \nabla u)_{ij} \partial_i u \cdot \partial_j u - |\partial_t u|^2) \\ u(\cdot, 0) = \xi + \tilde{g} \text{ and } \partial_t u(\cdot, 0) = h. \end{cases} \tag{3.2.32}$$

4. *If $T < \infty$ then*

$$\limsup_{t \to T^-} \left(\|v(\cdot, t)\|_{H^{k+1}} + \|\partial_t v(\cdot, t)\|_{H^k} \right) = \infty. \tag{3.2.33}$$

Remark 3.2.2. *An alternate approach to perturbing $u = \xi + v$ would be to take the data $g \in H^{k+1}(\mathbb{R}^n; \mathbb{R}^{m+1})$ such that $|g(x)| = 1$ for $x \in B(0, R)$ for some $R > 0$. Assume also that $h \in H^k(\mathbb{R}^n; \mathbb{R}^{m+1})$ is such that $h \cdot g = 0$ in $B(0, R)$. We could then use our theory of quasilinear symmetric hyperbolic systems to produce local-in-time solutions to (3.2.15). Then, rather than deducing that $\mu = 1$ in all of $\mathbb{R}^n \times [0, T]$ as before, we would have to employ the finite speed of propagation to deduce that $\mu = 1$ in some space-time truncated cone with base $B(0, R)$. This then yields a solution on $B(0, R/2) \times [0, T_0]$ for some $T_0 < T$, which is therefore local in space and in time.*

3.3 Compressible Euler equations

Consider a compressible inviscid fluid evolving in $\mathbb{R}^3$. We describe the fluid with the velocity field $u : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$, the density $\rho : \mathbb{R}^3 \times [0, T] \to [0, \infty)$, and the entropy function $S : \mathbb{R}^3 \times [0, T] \to [0, \infty)$. The fluid also experiences a pressure P, which we will discuss more in a moment. The basic laws of fluid mechanics (to be taken for granted here) read as follows for any open set U with smooth boundary. Mass is conserved:

$$\frac{d}{dt}\int_U \rho = -\int_{\partial U} \rho u \cdot \nu = -\int_U \operatorname{div}(\rho u). \tag{3.3.1}$$

The change in momentum is given by the force acting on the fluid, which is a contact force due to pressure:

$$\frac{d}{dt}\int_U \rho u = -\int_{\partial U} (\rho u) u \cdot \nu - \int_{\partial U} P\nu = \int_U -\operatorname{div}(\rho u \otimes u) - \nabla P. \tag{3.3.2}$$

Finally, we will consider only isentropic fluids, for which entropy is conserved along the flow:

$$\frac{d}{dt}\int_U \rho S = -\int_{\partial U} \rho S u \cdot \nu = -\int_U \operatorname{div}(\rho S u). \tag{3.3.3}$$

Since $U \subseteq \mathbb{R}^3$ was an arbitrary open set with smooth boundary, we deduce the compressible Euler equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = 0 \\ \partial_t(\rho S) + \operatorname{div}(\rho S u) = 0. \end{cases} \tag{3.3.4}$$

Note that

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = (\partial_t \rho + \operatorname{div}(\rho u))u + \rho(\partial_t u + u \cdot \nabla u) = \rho(\partial_t u + u \cdot \nabla u) \tag{3.3.5}$$

and

$$\partial_t(\rho S) + \operatorname{div}(\rho S u) = S(\partial_t \rho + \operatorname{div}(\rho u)) + \rho(\partial_t S + u \cdot \nabla S). \tag{3.3.6}$$

Thus, the system (3.3.4) is equivalent (at least when solutions are C^1 and $\rho > 0$) to the system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla P = 0 \\ \partial_t S + u \cdot \nabla S = 0. \end{cases} \tag{3.3.7}$$

Note that we have six scalar unknowns: ρ, u_1, u_2, u_3, S, and P, but only five equations in (3.3.7). In order to close the system we must specify an "equation of state" that relates P to the variables S and ρ. This cannot be done arbitrarily, but instead must be done in a way consistent with thermodynamics. We will avoid this issue entirely and simply state that for a large class of gases, known as perfect gases, this is possible and leads to the equation of state

$$P = K\rho^\gamma e^{\beta S} \tag{3.3.8}$$

for $K > 0$, $\gamma > 1$, and $\beta > 0$ physical constants. Note that in this framework we can also solve for ρ in terms of P and S:

$$\rho(P, S) = \left(\frac{P}{K}\right)^{1/\gamma} e^{-\beta S/\gamma}. \tag{3.3.9}$$

We leave it as an exercise to verify that the system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t S + u \cdot \nabla S = 0 \\ P = K\rho^\gamma e^{\beta S} \end{cases} \tag{3.3.10}$$

is equivalent to the system

$$\begin{cases} \partial_t P + u \cdot \nabla P + \gamma P \operatorname{div} u = 0 \\ \partial_t S + u \cdot \nabla S = 0 \\ \rho = \left(\frac{P}{K}\right)^{1/\gamma} e^{-\beta S/\gamma}. \end{cases} \tag{3.3.11}$$

In other words, we're free to work with either the couple (ρ, S) or else the couple (P, S). It turns out that the latter is more convenient for the symmetric hyperbolic system framework. Thus (3.3.7) is equivalent to

$$\begin{cases} \partial_t P + u \cdot \nabla P + \gamma P \operatorname{div} u = 0 \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla P = 0 \\ \partial_t S + u \cdot \nabla S = 0 \\ \rho = \left(\frac{P}{K}\right)^{1/\gamma} e^{-\beta S/\gamma}, \end{cases} \tag{3.3.12}$$

which is now a closed system of equations.

We now make the assumptions that P and S are given as perturbations of constant states $P_0, S_0 \in (0, \infty)$. That is, we postulate that

$$P(x,t) = P_0 + p(x,t) \text{ and } S = S_0 + s(x,t). \tag{3.3.13}$$

This will play an essential role in allowing us to prove that the A^0 matrix is elliptic. With this assumption we may then rewrite (3.3.12) as

$$\begin{cases} \partial_t p + u \cdot \nabla p + \gamma(P_0 + p) \operatorname{div} u = 0 \\ \rho(p,s)(\partial_t u + u \cdot \nabla u) + \nabla p = 0 \\ \partial_t s + u \cdot \nabla s = 0 \\ \rho(p,s) = \left(\frac{P_0+p}{K}\right)^{1/\gamma} e^{-\beta(S_0+s)/\gamma}. \end{cases} \tag{3.3.14}$$

We now rewrite (3.3.14) in matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho(p,s) & 0 & 0 & 0 \\ 0 & 0 & \rho(p,s) & 0 & 0 \\ 0 & 0 & 0 & \rho(p,s) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_t p \\ \partial_t u_1 \\ \partial_t u_2 \\ \partial_t u_3 \\ \partial_t s \end{pmatrix} + \begin{pmatrix} u_1 & \gamma(P_0+p) & 0 & 0 & 0 \\ 1 & \rho(p,s)u_1 & 0 & 0 & 0 \\ 0 & 0 & \rho(p,s)u_1 & 0 & 0 \\ 0 & 0 & 0 & \rho(p,s)u_1 & 0 \\ 0 & 0 & 0 & 0 & u_1 \end{pmatrix} \begin{pmatrix} \partial_1 p \\ \partial_1 u_1 \\ \partial_1 u_2 \\ \partial_1 u_3 \\ \partial_1 s \end{pmatrix}$$
$$+ \begin{pmatrix} u_2 & 0 & \gamma(P_0+p) & 0 & 0 \\ 0 & \rho(p,s)u_2 & 0 & 0 & 0 \\ 1 & 0 & \rho(p,s)u_2 & 0 & 0 \\ 0 & 0 & 0 & \rho(p,s)u_2 & 0 \\ 0 & 0 & 0 & 0 & u_2 \end{pmatrix} \begin{pmatrix} \partial_2 p \\ \partial_2 u_1 \\ \partial_2 u_2 \\ \partial_2 u_3 \\ \partial_2 s \end{pmatrix}$$
$$+ \begin{pmatrix} u_3 & 0 & 0 & \gamma(P_0+p) & 0 \\ 0 & \rho(p,s)u_3 & 0 & 0 & 0 \\ 0 & 0 & \rho(p,s)u_3 & 0 & 0 \\ 1 & 0 & 0 & \rho(p,s)u_3 & 0 \\ 0 & 0 & 0 & 0 & u_3 \end{pmatrix} \begin{pmatrix} \partial_3 p \\ \partial_3 u_1 \\ \partial_3 u_2 \\ \partial_3 u_3 \\ \partial_3 s \end{pmatrix} = 0. \tag{3.3.15}$$

This is, unfortunately, not a symmetric problem. However, it can be symmetrized by multiplying by the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \gamma(P_0+p) & 0 & 0 & 0 \\ 0 & 0 & \gamma(P_0+p) & 0 & 0 \\ 0 & 0 & 0 & \gamma(P_0+p) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3.16}$$

This turns (3.3.15) into

$$A^0(V)\partial_t V + A^j(V)\partial_j V = 0 \tag{3.3.17}$$

where

$$V = \begin{pmatrix} p \\ u_1 \\ u_2 \\ u_3 \\ s \end{pmatrix}, \tag{3.3.18}$$

$$A^0(V) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \gamma(P_0+p)\rho(p,s) & 0 & 0 & 0 \\ 0 & 0 & \gamma(P_0+p)\rho(p,s) & 0 & 0 \\ 0 & 0 & 0 & \gamma(P_0+p)\rho(p,s) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.3.19}$$

and for $j = 1, 2, 3$,

$$A^j(V) = \begin{pmatrix} 1 & & \gamma(P_0+p)e_j & & 0 \\ & \gamma(P_0+p)\rho(p,s)u_j & 0 & 0 & 0 \\ \gamma(P_0+p)e_j^T & 0 & \gamma(P_0+p)\rho(p,s)u_j & 0 & 0 \\ & 0 & 0 & \gamma(P_0+p)\rho(p,s)u_j & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3.20}$$

Thus (3.3.17) is a symmetric hyperbolic system with initial data

$$V_0 = \begin{pmatrix} p_0 \\ u_{0,1} \\ u_{0,2} \\ u_{0,3} \\ s_0 \end{pmatrix}. \tag{3.3.21}$$

Note that

$$A^0, A^1, \ldots, A^n \in C^\infty((-P_0,\infty) \times \mathbb{R}^3 \times \mathbb{R}). \tag{3.3.22}$$

We can now state our main theorem about the compressible Euler system.

Theorem 3.3.1. *Let $k > 5/2 = 1 + 3/2$. Suppose that $p_0, s_0 \in H^k(\mathbb{R}^3;\mathbb{R})$ and $u_0 \in H^k(\mathbb{R}^3;\mathbb{R}^3)$ and that*

$$p_0(\mathbb{R}^3) \subset\subset (-P_0,\infty). \tag{3.3.23}$$

Then there exists $T \in (0,\infty]$ and functions

$$\begin{aligned} p &\in C_b^1(\mathbb{R}^3 \times [0,T];\mathbb{R}) \\ s &\in C_b^1(\mathbb{R}^3 \times [0,T];\mathbb{R}) \\ u &\in C_b^1(\mathbb{R}^3 \times [0,T];\mathbb{R}^3) \end{aligned} \tag{3.3.24}$$

such that the following hold.

1. $p \in L^\infty([0,T]; H^k(\mathbb{R}^3;\mathbb{R}))$, $\partial_t p \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^3;\mathbb{R}))$, *and there exists a compact set* $K_1 \subset (-P_0, \infty)$ *such that*
$$p(\mathbb{R}^3, t) \subseteq K_1 \text{ for all } t \in [0,T]. \tag{3.3.25}$$

2. $s \in L^\infty([0,T]; H^k(\mathbb{R}^3;\mathbb{R}))$, $\partial_t s \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^3;\mathbb{R}))$.

3. $u \in L^\infty([0,T]; H^k(\mathbb{R}^3;\mathbb{R}^3))$, $\partial_t u \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^3;\mathbb{R}^3))$.

4. *The triple* (p, u, s) *solve*
$$\begin{cases} \partial_t p + u \cdot \nabla p + \gamma(P_0 + p) \operatorname{div} u = 0 \\ \rho(p,s)(\partial_t u + u \cdot \nabla u) + \nabla p = 0 \\ \partial_t s + u \cdot \nabla s = 0 \end{cases} \tag{3.3.26}$$
in $\mathbb{R}^3 \times [0,T]$, *where* $\rho(p,s) = \left(\frac{P_0+p}{K}\right)^{1/\gamma} e^{-\beta(S_0+s)/\gamma}$. *Moreover,*
$$p(\cdot, 0) = p_0, s(\cdot, 0) = s_0, \text{ and } u(\cdot, 0) = u_0 \text{ in } \mathbb{R}^3. \tag{3.3.27}$$

5. *If* $T < \infty$ *then either*
$$\liminf_{t \to T^-} \inf_{x \in \mathbb{R}^3} p(x,t) = -P_0 \tag{3.3.28}$$
or else
$$\limsup_{t \to T^-} \left(\|u(\cdot,t)\|_{H^k} + \|p(\cdot,t)\|_{H^k} + \|s(\cdot,t)\|_{H^k} \right) = \infty. \tag{3.3.29}$$

Proof. The assumptions on the data allow us to find Ω and r such that Assumptions 2.3.1 hold. We may then apply Theorem 2.4.2 to produce solutions on a time interval $[0, T_0]$. We may then iterate as long as neither of the two conditions in the fifth item hold, which extends the solution to $[T_0, T_1]$. We continue this process until either the solution exists on $[0, \infty]$ or else one of the conditions in the fifth item hold. □

Remark 3.3.2. *The first condition corresponds to vacuum formation, as in this setting ρ vanishes. The second condition in the fifth item corresponds to "blow-up" of the H^k norm of the solution. This can happen, for instance, if shockwaves form. Both of these are real possibilities in gases.*

3.4 Compressible magnetohydrodynamics

A plasma is an ionized gas in which the flow of electrical charge plays a serious role in the dynamics. The equations of magnetohydrodynamics offer a good model of plasmas. They couple the compressible Euler system to equations of motion for the magnetic field, which are derived from the Maxwell system and some simplifying approximations. We will not attempt to derive the system here (for the derivation see Chapter 37 of Chandrasekhar's book [2]), but simply state it:

$$\begin{cases} \partial_t P + u \cdot \nabla P + \gamma P \operatorname{div} u = 0 \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla P = \operatorname{curl}(B) \times B \\ \partial_t S + u \cdot \nabla S = 0 \\ \partial_t B = \operatorname{curl}(u \times B) \\ \operatorname{div} B = 0. \end{cases} \tag{3.4.1}$$

Here P, ρ, u, S are as before and $B : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ is the magnetic field in the plasma. The term $\operatorname{curl} B \times B$ is the second equation is the Lorentz force on the fluid due to the magnetic field. The term on the right of the fourth equation may be rewritten as

$$\operatorname{curl}(u \times B) = u \operatorname{div} B - B \operatorname{div} u + B \cdot \nabla u - u \cdot \nabla B = -B \operatorname{div} u + B \cdot \nabla u - u \cdot \nabla B, \tag{3.4.2}$$

and so the fourth equation may be rewritten as

$$\partial_t B + u \cdot \nabla B = B \cdot \nabla u - B \operatorname{div} u. \tag{3.4.3}$$

On the other hand, we can compute

$$\operatorname{curl}(B) \times B = B \cdot \nabla B - \frac{1}{2} \nabla |B|^2 = B \cdot \nabla B - DB^T B. \tag{3.4.4}$$

Of course, we could cancel the last term, but we leave it for the sake of symmetry. The equations then read, after again perturbing $P = P_0 + p$ and $S = S_0 + s$,

$$\begin{cases} \partial_t p + u \cdot \nabla p + \gamma(P_0 + p) \operatorname{div} u = 0 \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla p = B \cdot \nabla B - DB^T B \\ \partial_t s + u \cdot \nabla s = 0 \\ \partial_t B + u \cdot \nabla B = B \cdot \nabla u - B \operatorname{div} u \\ \operatorname{div} B = 0. \end{cases} \tag{3.4.5}$$

Next we note that if

$$\operatorname{div}(B(\cdot, 0)) = 0, \tag{3.4.6}$$

then

$$\operatorname{div}(\partial_t B) = \operatorname{div} \operatorname{curl}(u \times B) = 0 \tag{3.4.7}$$

and hence

$$\operatorname{div} B(\cdot, t) = \operatorname{div} B(\cdot, 0) = 0 \text{ for all } t \geq 0. \tag{3.4.8}$$

Thus we may enforce the fifth equation in (3.4.5) by imposing the compatibility condition for the initial data: $\operatorname{div} B_0 = 0$. We thus reduce to

$$\begin{cases} \partial_t p + u \cdot \nabla s + \gamma(P_0 + p) \operatorname{div} u = 0 \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla p = B \cdot \nabla B - DB^T B \\ \partial_t s + u \cdot \nabla s = 0 \\ \partial_t B + u \cdot \nabla B = B \cdot \nabla u - B \operatorname{div} u. \end{cases} \tag{3.4.9}$$

Write $I \in \mathbb{R}^{3\times 3}$ for the identity matrix. We may then write a system of equations for (3.4.9) as the following in $\mathbb{R}^8$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho(p,s)I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_t p \\ \partial_t u \\ \partial_t B \\ \partial_t S \end{pmatrix} + \begin{pmatrix} u_j & \gamma(P_0 + p) e_j & 0 & 0 \\ e_j^T & \rho(p,s) u_j I & -B_j I + e_j \otimes B & 0 \\ 0 & -B_j I + B \otimes e_j & u_j I & 0 \\ 0 & 0 & 0 & u_j \end{pmatrix} \begin{pmatrix} \partial_j p \\ \partial_j u \\ \partial_j B \\ \partial_j S \end{pmatrix} = 0. \tag{3.4.10}$$

Again this is not symmetric, but we may symmetrize by multiplying by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma(P_0+p) & 0 & 0 \\ 0 & 0 & \gamma(P_0+p) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{8\times 8}. \tag{3.4.11}$$

This results in

$$A^0(p,u,B,s)\begin{pmatrix} \partial_t p \\ \partial_t u \\ \partial_t B \\ \partial_t S \end{pmatrix} + A^j(p,u,B,s)\begin{pmatrix} \partial_j p \\ \partial_j u \\ \partial_j B \\ \partial_j S \end{pmatrix} = 0, \tag{3.4.12}$$

where

$$A^0(p,u,B,s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma\rho(p,s)(P_0+p)I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.4.13}$$

and

$$A^j(p,u,B,s) = \begin{pmatrix} u_j & \gamma(P_0+p)e_j & 0 & 0 \\ \gamma(P_0+p)e_j^T & \gamma\rho(p,s)(P_0+p)u_jI & \gamma(P_0+p)(-B_jI+e_j\otimes B) & 0 \\ 0 & \gamma(P_0+p)(-B_jI+B\otimes e_j) & \gamma(P_0+p)u_jI & 0 \\ 0 & 0 & 0 & u_j \end{pmatrix}, \tag{3.4.14}$$

which then is a symmetric hyperbolic system. We augment this with initial data

$$(p_0,u_0,B_0,s_0) \in H^k(\mathbb{R}^3;\mathbb{R}^8) \text{ with } \operatorname{div} B_0 = 0. \tag{3.4.15}$$

Note that

$$A^0, A^1, \ldots, A^n \in C^\infty((-P_0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}). \tag{3.4.16}$$

We can now state our main theorem about the MHD system.

Theorem 3.4.1. *Let $k > 5/2 = 1 + 3/2$. Suppose that $p_0, s_0 \in H^k(\mathbb{R}^3;\mathbb{R})$ and $u_0, B_0 \in H^k(\mathbb{R}^3;\mathbb{R}^3)$ and that*

$$p_0(\mathbb{R}^3) \subset\subset (-P_0,\infty) \text{ and } \operatorname{div} B_0 = 0. \tag{3.4.17}$$

Then there exists $T \in (0,\infty]$ and functions

$$\begin{aligned} p &\in C_b^1(\mathbb{R}^3\times[0,T];\mathbb{R}) \\ s &\in C_b^1(\mathbb{R}^3\times[0,T];\mathbb{R}) \\ u &\in C_b^1(\mathbb{R}^3\times[0,T];\mathbb{R}^3) \\ B &\in C_b^1(\mathbb{R}^3\times[0,T];\mathbb{R}^3) \end{aligned} \tag{3.4.18}$$

such that the following hold.

1. *$p \in L^\infty([0,T];H^k(\mathbb{R}^3;\mathbb{R}))$, $\partial_t p \in L^\infty([0,T];H^{k-1}(\mathbb{R}^3;\mathbb{R}))$, and there exists a compact set $K_1 \subset (-P_0,\infty)$ such that*

$$p(\mathbb{R}^3,t) \subseteq K_1 \text{ for all } t \in [0,T]. \tag{3.4.19}$$

2. *$s \in L^\infty([0,T];H^k(\mathbb{R}^3;\mathbb{R}))$, $\partial_t s \in L^\infty([0,T];H^{k-1}(\mathbb{R}^3;\mathbb{R}))$.*

3. $u \in L^\infty([0,T]; H^k(\mathbb{R}^3;\mathbb{R}^3))$, $\partial_t u \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^3;\mathbb{R}^3))$.

4. $B \in L^\infty([0,T]; H^k(\mathbb{R}^3;\mathbb{R}^3))$, $\partial_t B \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^3;\mathbb{R}^3))$, *and* $\operatorname{div} B(x,0) = 0$ *for all* $x \in \mathbb{R}^n$ *and* $t \in [0,T]$.

5. *The quadruple* (p,u,B,s) *solve*

$$\begin{cases} \partial_t p + u \cdot \nabla p + \gamma(P_0 + p) \operatorname{div} u = 0 \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla p = \operatorname{curl} B \times B \\ \partial_t s + u \cdot \nabla s = 0 \\ \partial_t B + u \cdot \nabla B = B \cdot \nabla u - B \operatorname{div} u \\ \operatorname{div} B = 0. \end{cases} \tag{3.4.20}$$

in $\mathbb{R}^3 \times [0,T]$, *where* $\rho(p,s) = \left(\frac{P_0+p}{K}\right)^{1/\gamma} e^{-\beta(S_0+s)/\gamma}$. *Moreover,*

$$p(\cdot,0) = p_0, s(\cdot,0) = s_0, B(\cdot,0) = B_0, \text{ and } u(\cdot,0) = u_0 \text{ in } \mathbb{R}^3. \tag{3.4.21}$$

6. *If* $T < \infty$ *then either*

$$\liminf_{t\to T^-} \inf_{x\in\mathbb{R}^3} p(x,t) = -P_0 \tag{3.4.22}$$

or else

$$\limsup_{t\to T^-} \left(\|u(\cdot,t)\|_{H^k} + \|B(\cdot,t)\|_{H^k} + \|p(\cdot,t)\|_{H^k} + \|s(\cdot,t)\|_{H^k}\right) = \infty. \tag{3.4.23}$$

Proof. The assumptions on the data allow us to find Ω and r such that Assumptions 2.3.1 hold. We may then apply Theorem 2.4.2 to produce solutions on a time interval $[0,T_0]$. We may then iterate as long as neither of the two conditions in the fifth item hold, which extends the solution to $[T_0,T_1]$. We continue this process until either the solution exists on $[0,\infty]$ or else one of the conditions in the fifth item hold. □

3.5 Shallow water equations

We now turn to a system of equations that is not exactly derived from first principles in physics, but instead is derived as an approximation to a problem often encountered in practice. Suppose that an incompressible fluid (the ocean, say) flows above a flat rigid interface and has a free boundary above. In other words, we assume that there is a height function $h : \mathbb{R}^2 \times [0,T] \to (0,\infty)$ such that the fluid occupies the body

$$\Omega(t) = \{(x,z) \in \mathbb{R}^3 \mid 0 < z < h(x,t)\} \subseteq \mathbb{R}^3 \tag{3.5.1}$$

for each $t \geq 0$. Let us also write

$$\Sigma(t) = \{(x,z) \in \mathbb{R}^3 \mid z = h(x,t)\} \text{ and } \Sigma_b = \{(x,0) \in \mathbb{R}^3\} \tag{3.5.2}$$

for the moving fluid interface and flat bottom, respectively. Let us further suppose that a uniform gravitational field acts on the fluid and points toward the flat bottom. Since we have split points in $\mathbb{R}^3$ as (x,z) is is reasonable to split the fluid velocity field as (u,θ) with $u \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ and to write the $\mathbb{R}^3$ gradient as (∇, ∂_z). The equations of motion then read

$$\begin{cases} \rho(\partial_t u + u\cdot\nabla u + \theta\partial_z u) + \nabla p = 0 & \text{in } \Omega(t) \\ \rho(\partial_t \theta + u\cdot\nabla\theta + \theta\partial_z\theta) + \partial_z p = -g\rho & \text{in } \Omega(t) \\ \operatorname{div} u + \partial_z\theta = 0 & \text{in } \Omega(t) \\ \partial_t h + \partial_1 h u_1 + \partial_2 h u_2 = \theta & \text{on } \Sigma(t) \\ p = 0 & \text{on } \Sigma(t) \\ \theta = 0 & \text{on } \Sigma_b. \end{cases} \tag{3.5.3}$$

Here $\rho > 0$ is the constant density of the incompressible fluid and $g > 0$ is the gravitational field strength.

The shallow water approximation assumes that the vertical scale of the fluid (the characteristic height of h, for instance) is much, much smaller than the characteristic horizontal scale. This essentially corresponds to very long waves propagating through the fluid. The approximation further posits that the horizontal components of momentum do not vary much vertically. We implement these assumptions by making the following ansatz:

$$\begin{cases} u = u(x,t) \\ \rho(\partial_t\theta + u\cdot\nabla\theta + \theta\partial_z\theta) = 0. \end{cases} \tag{3.5.4}$$

The former condition clearly says that the horizontal velocity does not vary at all with the vertical variable, but the connection between the latter condition and the scaling is not immediately clear. We will ignore this issue here and simply say that it follows from a scaling argument. For further details we refer to Chapter 7 of Neu's book [6].

The ansatz immediately leads to two essential reductions. First, the second equation in (3.5.3) becomes

$$\partial_z p = -g\rho \Rightarrow p(x,z) = -g\rho z + \psi(x,t) \text{ for some } \psi(x,t). \tag{3.5.5}$$

To compute ψ we use the second to last equation in (3.5.3), which says that

$$p(x, h(x,t)) = 0, \tag{3.5.6}$$

and so

$$0 = -g\rho h(x,t) + \psi(x,t) \Rightarrow \psi(x,t) = g\rho h(x,t) \Rightarrow p(x,z) - -g\rho z + g\rho h(x,t). \tag{3.5.7}$$

Upon plugging this into the first equation in (3.5.3) we find that

$$\rho(\partial_t u + u\cdot\nabla u) + g\rho\nabla h = 0. \tag{3.5.8}$$

Next we use the third and sixth equations in (3.5.3) to compute

$$\theta = -z \operatorname{div} u. \tag{3.5.9}$$

We then plug this into the fourth equation in (3.5.3) to deduce that

$$\begin{aligned} \partial_t h(x,t) &+ \partial_1 h(x,t) u_1(x,t) + \partial_2 h(x,t) u_2(x,t) \\ &= \partial_t h(x,t) + \partial_1 h(x,t) u_1(x, h(x,t), t) + \partial_2 h(x,t) u_2(x, h(x,t), t) \\ &= \theta(x, h(x,t), t) = -h(x,t)(\partial_1 u_1(x,t) + \partial_2 u_2(x,t)), \end{aligned} \tag{3.5.10}$$

and thus

$$\partial_t h + \nabla h \cdot u + h \operatorname{div} u = 0. \tag{3.5.11}$$

On balance we then reduce to the following system for $u : \mathbb{R}^2 \times [0,T] \to \mathbb{R}^2$ and $h : \mathbb{R}^2 \times [0,T] \to (0,\infty)$:

$$\begin{cases} \partial_t h + u \cdot \nabla h + h \operatorname{div} u = 0 \\ \partial_t u + u \cdot \nabla u + g \nabla h = 0 \end{cases} \quad \text{in } \mathbb{R}^2 \times [0,T]. \tag{3.5.12}$$

Note that the gravitational term $g > 0$ is essential in producing the coupling in (3.5.12). Indeed, if we set $g = 0$ then (3.5.12) becomes

$$\begin{cases} \partial_t u + u \cdot \nabla u = 0 \\ \partial_t h + u \cdot \nabla h + h \operatorname{div} u = 0, \end{cases} \tag{3.5.13}$$

which is a decoupled system: we may first solve for u using the first equation and then use the resulting u to generate the coefficients in the second equation. In this case we can further reduce the complexity of the system by making the ansatz $u(x_1, x_2) = (v(x_1), 0)$. The first equation then reduces to

$$\partial_t v + v \partial_1 v = 0, \tag{3.5.14}$$

which is Burger's equation.

The system (3.5.12) is very similar to the compressible Euler system. We can write it as the first order problem

$$\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_t u \\ \partial_t h \end{pmatrix} + \begin{pmatrix} u_j I_{2\times 2} & g e_j \\ h e_j & u_j \end{pmatrix} \begin{pmatrix} \partial_j u \\ \partial_j h \end{pmatrix} = 0. \tag{3.5.15}$$

We can symmetrize by multiplying by

$$\begin{pmatrix} h I_{2\times 2} & 0 \\ 0 & g \end{pmatrix}. \tag{3.5.16}$$

This yields

$$\begin{pmatrix} h I_{2\times 2} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \partial_t u \\ \partial_t h \end{pmatrix} + \begin{pmatrix} h u_j I_{2\times 2} & g h e_j \\ g h e_j & g u_j \end{pmatrix} \begin{pmatrix} \partial_j u \\ \partial_j h \end{pmatrix} = 0. \tag{3.5.17}$$

In order to make the A^0 matrix elliptic we must introduce a perturbation formulation as for the compressible Euler problem. We postulate that $h(x,t) = H_0 + \eta(x,t)$ for $\eta : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ and $H_0 > 0$ is some constant. Then the symmetric problem can be rewritten as

$$\begin{pmatrix} (H_0 + \eta) I_{2\times 2} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \partial_t u \\ \partial_t \eta \end{pmatrix} + \begin{pmatrix} (H_0 + \eta) u_j I_{2\times 2} & g(H_0 + \eta) e_j \\ g(H_0 + \eta) e_j & g u_j \end{pmatrix} \begin{pmatrix} \partial_j u \\ \partial_j \eta \end{pmatrix} = 0. \tag{3.5.18}$$

Using this formulation we can then easily prove the following theorem.

Theorem 3.5.1. *Let $k > 2 = 1 + 2/2$. Suppose that $\eta_0 \in H^k(\mathbb{R}^2; \mathbb{R})$ and $u_0 \in H^k(\mathbb{R}^2; \mathbb{R}^2)$ and that*

$$\eta_0(\mathbb{R}^2) \subset\subset (-H_0, \infty). \tag{3.5.19}$$

Then there exists $T \in (0, \infty]$ and functions

$$\begin{aligned} \eta &\in C_b^1(\mathbb{R}^2 \times [0,T]; \mathbb{R}) \\ u &\in C_b^1(\mathbb{R}^2 \times [0,T]; \mathbb{R}^2) \end{aligned} \tag{3.5.20}$$

such that the following hold.

1. $\eta \in L^\infty([0,T]; H^k(\mathbb{R}^2;\mathbb{R}))$, $\partial_t p \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^2;\mathbb{R}))$, *and there exists a compact set* $K_1 \subset (-H_0, \infty)$ *such that*
$$\eta(\mathbb{R}^2, t) \subseteq K_1 \text{ for all } t \in [0,T]. \tag{3.5.21}$$
In particular $H_0 + \eta(x,t) > 0$ *for all* $x \in \mathbb{R}^2$ *and* $t \in [0,T]$.

2. $u \in L^\infty([0,T]; H^k(\mathbb{R}^2;\mathbb{R}^2))$, $\partial_t u \in L^\infty([0,T]; H^{k-1}(\mathbb{R}^2;\mathbb{R}^2))$.

3. *Let* $h = H_0 + \eta$. *Then the pair* (h,u) *solves*
$$\begin{cases} \partial_t h + u \cdot \nabla h + h \operatorname{div} u = 0 \\ \partial_t u + u \cdot \nabla u + g \nabla h = 0. \end{cases} \tag{3.5.22}$$
in $\mathbb{R}^2 \times [0,T]$. *Moreover,*
$$h(\cdot,0) = H_0 + \eta_0 \text{ and } u(\cdot,0) = u_0 \text{ in } \mathbb{R}^2. \tag{3.5.23}$$

4. *If* $T < \infty$ *then either*
$$\liminf_{t \to T^-} \inf_{x \in \mathbb{R}^2} \eta(x,t) = -H_0 \tag{3.5.24}$$
or else
$$\limsup_{t \to T^-} \left(\|u(\cdot,t)\|_{H^k} + \|\eta(\cdot,t)\|_{H^k} \right) = \infty. \tag{3.5.25}$$

Remark 3.5.2. *The fact that* $g > 0$ *is essential here. If* $g \leq 0$ *then the symmetric hyperbolic structure breaks.*

References

[1] R. Abraham, J. Marsden, T. Ratiu. *Manifolds, tensor analysis, and applications.* Second edition. Applied Mathematical Sciences, 75. Springer-Verlag, New York, 1988.

[2] S. Chandrasekhar. *Hydrodynamic and hydromagnetic stability.* The International Series of Monographs on Physics Clarendon Press, Oxford 1961

[3] A. Fischer, J. Marsden. The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system. I. *Comm. Math. Phys.* **28** (1972), 1–38.

[4] K. Friedrichs. Symmetric hyperbolic linear differential equations. *Comm. Pure Appl. Math.* **7**, (1954). 345–392.

[5] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.* **58** (1975), no. 3, 181–205.

[6] J. Neu. *Training manual on transport and fluids.* Graduate Studies in Mathematics, 109. American Mathematical Society, Providence, RI, 2010.