Monstrous Functions

Ian Tice
Department of Mathematical Sciences
Carnegie Mellon University

CMU Math Club: October 12, 2018

0 Overview

In mathematics we are often interested in the question of extremes:

- What is the shortest distance between two points?
- What are the maximum and minimum values of a function?
- What is the minimum number of colors needed to draw a map so that no adjacent districts have the same color?
- What is the largest sporadic group?

These are all examples of quantitative extremes, but mathematicians are also interested in qualitative extremes. A common question of this type is:

- How wild or badly behaved can a particular mathematical object be?

This issue is often referred to as *pathological behavior*, i.e. mathematical objects that behave abnormally or in unexpected ways. Pathological constructions in mathematics are important (and fun!) because while mathematics is deeply intertwined with human experience and intuition, it is ultimately not bound by it. Pathological constructions highlight the extremes of this unboundedness.

The purpose of this talk is to construct a pathological function of a real variable that defies our intuition about how nice a continuous function should be. The result is originally due to Karl Weierstrass and was first published in 1872, but we will present a variant of Weierstrass' original argument.

1 Continuity and differentiability

Let us recall what it means for a real function to be continuous and differentiable.

Definition 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$.*

1. *We say that f is continuous at $x \in \mathbb{R}$ if*

$$\lim_{y \to x} f(y) = f(x). \tag{1.1}$$

2. *We say that f is continuous if it is continuous at every point $x \in \mathbb{R}$.*

3. *We say that f is differentiable at $x \in \mathbb{R}$ if*

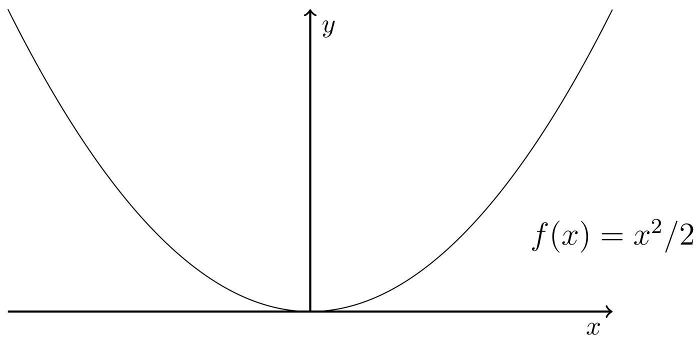
$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} \text{ exists,} \tag{1.2}$$

in which case we write

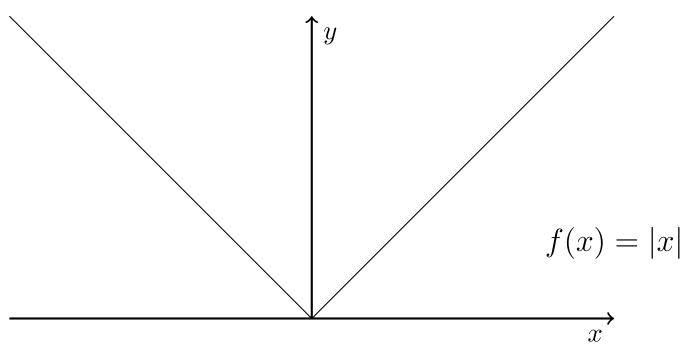
$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}. \tag{1.3}$$

4. *We say that f is differentiable if it is differentiable at every point $x \in \mathbb{R}$.*

Let's consider two basic examples. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2/2$ is continuous and differentiable with $f'(x) = x$.



The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is continuous but not differentiable at $x = 0$, though it is differentiable at every point $x \neq 0$ with $f'(x) = x/|x|$. Informally, we can see this in the graph because there is no unique tangent line at $x = 0$.



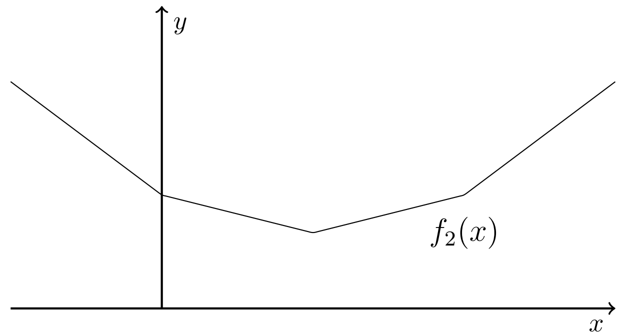
A fundamental result from the calculus is the following theorem.

Theorem 1.2. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f is continuous.*

However, the examples given above show that the converse fails: there are continuous functions that are not differentiable. In our example $f(x) = |x|$ the function fails to be differentiable at exactly one point, namely $x = 0$. We can easily build functions that fail to be differentiable at exactly n points for any $n \geq 1$. For instance the function $f_n : \mathbb{R} \to \mathbb{R}$ given by

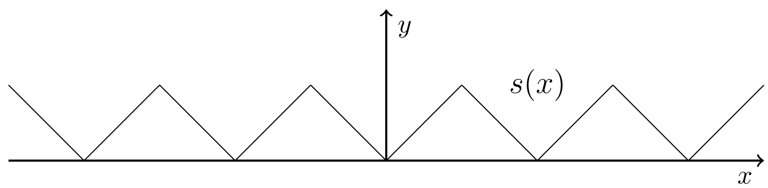
$$f_n(x) = \frac{1}{n} \sum_{m=0}^{n-1} |x - m| \tag{1.4}$$

is continuous but fails to be differentiable at the points $x = 0, \ldots, n - 1$.



In fact, it's not hard to find a continuous function that fails to be differentiable at infinitely many points. Consider the sawtooth function $g : \mathbb{R} \to \mathbb{R}$ given by

$$s(x) = 1 - |1 - \text{mod}(x, 2)|. \tag{1.5}$$



The sawtooth function is continuous but fails to be differentiable at each $x \in \mathbb{Z}$. We have thus found a very simple example a continuous function that fails to be differentiable on the countably infinite set of integers, $\mathbb{Z}$.

The above analysis already establishes some pretty pathological behavior: for any countable cardinal we've found a continuous function that fails to be differentiable on a set of that cardinality. However, all points in $\mathbb{Z}$ are isolated and in fact "most" (in the sense of cardinality or Lebesgue measure, for instance) points in $\mathbb{R}$ do not belong to $\mathbb{Z}$, so our sawtooth example is still differentiable at most points of $\mathbb{R}$. This begs the question:

- How big can the set of points at which a continuous function fails to be differentiable actually be?

We will answer this question, but first we need to recall some other tools.

2 Uniform convergence

We now recall some basic ideas from the theory of sequences of functions. We don't have the time to present the proofs in this talk, but the proofs can be found in most introductory books on Real Analysis.

Definition 2.1. *Suppose that for every $n \in \mathbb{N}$ we have a function $f_n : \mathbb{R} \to \mathbb{R}$.*

1. *We say that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$ if*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = 0. \tag{2.1}$$

   In this case we write $f_n \to f$ uniformly as $n \to \infty$.

2. We say that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy if

$$\lim_{m,n\to\infty} \sup_{x\in\mathbb{R}} |f_n(x) - f_m(x)| = 0. \tag{2.2}$$

We now recall two results of fundamental importance in the calculus. The first tells us that the concept of uniformly convergent is equivalent to the concept of uniformly Cauchy. This is important, as it allows us to deduce that a sequence is convergent even when we can't explicitly identify what its limit is.

Theorem 2.2. *Suppose that for every $n \in \mathbb{N}$ we have a function $f_n : \mathbb{R} \to \mathbb{R}$. Then the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent if and only if it is uniformly Cauchy.*

The second result establishes that the uniform limit of a sequence of continuous function is continuous as well. This can be thought of as a transference principle: continuity is transferred from the sequence to its uniform limit.

Theorem 2.3. *Suppose that for every $n \in \mathbb{N}$ we have a continuous function $f_n : \mathbb{R} \to \mathbb{R}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $f_n \to f$ uniformly as $n \to \infty$. Then f is continuous.*

Let's consider an important example. For $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ via $f_n(x) = \sqrt{4^{-n} + x^2}$. For any $n \in \mathbb{N}$ we can estimate

$$|x| = \sqrt{x^2} \le \sqrt{4^{-n} + x^2} = f_n(x) \tag{2.3}$$

and

$$(2^{-n} + |x|)^2 = 4^{-n} + 2^{-n+1} |x| + x^2 \ge 4^{-n} + x^2, \tag{2.4}$$

which implies that

$$f_n(x) = \sqrt{4^{-n} + x^2} \le \sqrt{(2^{-n} + |x|)^2} = 2^{-n} + |x|\,. \tag{2.5}$$

We combine these to see that

$$0 \le f_n(x) - |x| \le 2^{-n} \text{ for all } n \in \mathbb{N}, \tag{2.6}$$

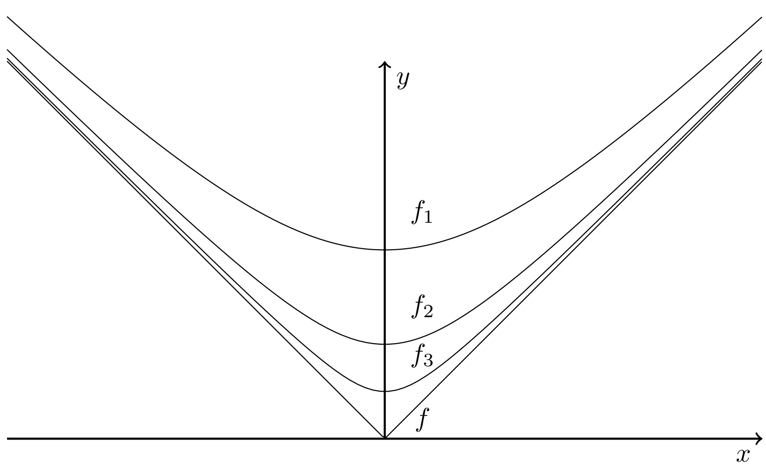
which implies that

$$\sup_{x\in\mathbb{R}} |f_n(x) - |x|| \le 2^{-n}. \tag{2.7}$$

Hence

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_n(x) - |x|| = 0 \tag{2.8}$$

and we deduce the uniform convergence $f_n \to f$ as $n \to \infty$, where $f(x) = |x|$. The functions are depicted in the following graphic.



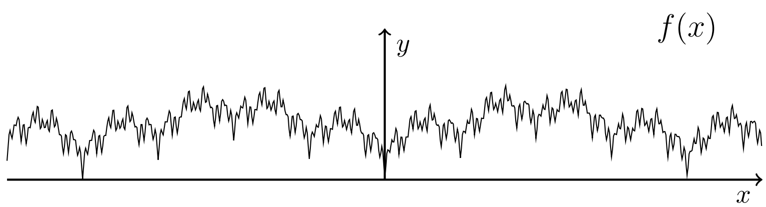
The uniform convergence $f_n \to f$ as $n \to \infty$

Now, Theorem 2.2 tells us that $\{f_n\}_n$ is also uniformly Cauchy, though this could also have been deduced by hand. On the other than Theorem 2.3 tells us that the limiting function has to be continuous, but this is evident already since we were able to explicitly identify the limiting function. Note that each f_n is actually differentiable but the limit f is not. This tells us something important: continuity is preserved by uniform limits, but differentiability is not.

3 Weierstrass' monster

We now answer the question posed at the end of Section 1 by constructing a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable *nowhere*! More precisely, we will construct a continuous function that fails to be differentiable at every point in $\mathbb{R}$. The contemporaries of Weierstrass denounced this construction and called the resulting function a "monster." Hermite famously wrote, "I turn with terror and horror from this lamentable scourge of functions with no derivatives."

The idea behind the construction is simple. We will use a linear combination of sawtooth functions at different scales in order to construct a sequence of functions that fail to be differentiable at more and more points. When combined with a uniform limiting argument we will find the desired function f, which is represented in the following graphic.



Weierstrass' monster

Theorem 3.1 (Weierstrass' monster)**.** *There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is nowhere differentiable.*

Proof. We will construct a sequence of continuous functions $f_N : \mathbb{R} \to \mathbb{R}$ such that $|f_N(x)| \leq 1$ for all $x \in \mathbb{R}$ and $f_N \to f$ uniformly as $N \to \infty$, where $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x)| \leq 1$ for all $x \in \mathbb{R}$, and for each $x \in \mathbb{R}$ and $1 \leq m \in \mathbb{N}$ there exists $\delta_m \in \mathbb{R}$ with $|\delta_m| = 4^{-m}/2$ such that

$$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| \geq \frac{1}{8}(3^m + 1). \tag{3.1}$$

The uniform convergence and Theorem 2.3 show that f is continuous, and (3.1) shows that

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} \text{ does not exist for any } x \in \mathbb{R}, \tag{3.2}$$

and hence f is nowhere differentiable.

Let $s : \mathbb{R} \to \mathbb{R}$ be the sawtooth function defined in Section 1. By construction s satisfies the following properties:

$$|s(x) - s(y)| \leq |x - y| \text{ for all } x, y \in \mathbb{R}, \tag{3.3}$$

$$0 \leq s(x) \leq 1 \text{ for all } x \in \mathbb{R}, \tag{3.4}$$

and

$$s(x + 2) = s(x) \text{ for all } x \in \mathbb{R}. \tag{3.5}$$

For $N \in \mathbb{N}$ we define $f_N : \mathbb{R} \to \mathbb{R}$ via

$$f_N(x) = \frac{1}{4}\sum_{n=0}^{N} \frac{3^n}{4^n} s(4^n x). \tag{3.6}$$

Since linear combinations of continuous functions are continuous, we have that each f_N is continuous. Notice also that

$$\sup_{x\in\mathbb{R}} |f_N(x)| \le \frac{1}{4}\sum_{n=0}^{N} \frac{3^n}{4^n} \le \frac{1}{4}\frac{1}{1-3/4} = 1. \tag{3.7}$$

For $M > N \ge 0$ we have the estimate

$$|f_M(x) - f_N(x)| = \frac{1}{4}\sum_{n=N+1}^{M} \frac{3^n}{4^n} s(4^n x) \le \frac{1}{4}\sum_{n=N+1}^{M} \frac{3^n}{4^n} \tag{3.8}$$

and so

$$\sup_{x\in\mathbb{R}} |f_M(x) - f_N(x)| \le \frac{1}{4}\sum_{n=N+1}^{M} \left(\frac{3}{4}\right)^n, \tag{3.9}$$

which implies that $\{f_N\}_N$ is uniformly Cauchy. Theorems 2.2 and 2.3 then guarantee that there exists a continuous $f : \mathbb{R} \to \mathbb{R}$ such that $f_N \to f$ uniformly as $N \to \infty$. Moreover, (3.7) shows that $0 \le f(x) \le 1$ for all $x \in \mathbb{R}$.

Fix $x \in \mathbb{R}$ and $m \in \mathbb{N}$ positive. Notice that $[4^m x - 1/2, 4^m x + 1/2) \cap \mathbb{Z} = \{z\}$ for some unique $z \in \mathbb{Z}$. We set

$$\delta_m = \begin{cases} \frac{1}{2}4^{-m} & \text{if } z \in [4^m x - 1/2, 4^m x) \\ -\frac{1}{2}4^{-m} & \text{if } z \in [4^m x, 4^m x + 1/2). \end{cases} \tag{3.10}$$

We also set

$$\gamma_n = \frac{s(4^n(x+\delta_m)) - s(4^n x)}{\delta_m} \text{ for } n \in \mathbb{N}. \tag{3.11}$$

When $0 \le n < m$ we use (3.3) to estimate

$$|\gamma_n| \le \frac{|4^n \delta_m|}{|\delta_m|} = 4^n, \tag{3.12}$$

and when $n = m$ the choice of δ_m allows us to compute

$$|\gamma_m| = 4^m. \tag{3.13}$$

If $n > m$, then $4^n \delta_m$ is an even integer, and hence the periodicity of s implies that $\gamma_n = 0$. Consequently, we find that if $N \ge m$, then

$$4\frac{|f_N(x+\delta_m) - f_N(x)|}{|\delta_m|} = \left|\sum_{n=0}^{m} \frac{3^n}{4^n}\gamma_n\right| \ge \left|\frac{3^m}{4^m}\gamma_m\right| - \left|\sum_{n=0}^{m-1} \frac{3^n}{4^n}\gamma_n\right|$$
$$\ge 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1). \tag{3.14}$$

Sending $N \to \infty$ in (3.14), we find that f satisfies (3.1). □

Remark 3.2. *In fact, it can be shown that "most" continuous functions are similarly monstrous, where "most" must be understood in the sense of the Baire category theorem. This means that in the set of continuous functions, it is only the rare gem that is differentiable.*