A crash course in interpolation theory

Ian Tice
Department of Mathematical Sciences
Carnegie Mellon University

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0 Overview

0.1 Introduction

To say that the Lebesgue spaces, which arise as natural generalizations of the spaces of integrable and essentially bounded functions on a measure space, are useful in analysis is a ridiculous understatement, evident to anyone with even minor experience in the subject. A terse explanation of the ubiquity of these spaces is that they possess a host of functional analytic properties that provide powerful tools for working with them. Interpolation theory has its genesis in the study of a particularly useful set these properties related to the mutual embeddings of the Lebesgue spaces and the implications of these on the study of operators between Lebesgue spaces. Arguably, it is the latter, which centers on the theorems of Marcinkiewicz and Riesz-Thorin, that is the most useful in practice, as it allows one to study the mapping properties of operators on a wide range of Lebesgue spaces (the interpolation spaces) by first understanding the properties in a restricted setting (the endpoints of the interpolation). Abstract interpolation theory seeks to extend these ideas and tools into a more general setting.

The purpose of these notes is to provide a brief introduction to this theory and to highlight some of the powerful tools it provides for use in applications, in particular the idea of fractional regularity. As a warning, the title should be taken seriously: the notes are by no means a complete study of this topic, and huge portions of the theory and many important results are completely ignored. The reasons for this are two-fold. First, these notes originated in the course Function Spaces and Generalized Regularity, taught jointly by the author and Giovanni Leoni at the Scuola Matematica Interuniversitaria Summer School in Cortona, Italy from July 5th to 16th of 2021 (though, it should be noted that, due to the pandemic, the course was actually taught through Zoom from the author's basement). This school ran only for two weeks with fifteen hours of lectures on interpolation theory and fifteen more on fractional Sobolev spaces (taught by Leoni), and the material recorded here is already a strict superset of what it was possible to cover in the interpolation lectures. Second, there are many very good and very thorough books on the subject, and a deeper study should begin by looking at these. In particular, the texts of Bennett and Sharpley [1], Bergh and Löfström [2], Brudnyĭ and Krugljak [3], Lunardi [5], and Triebel [7] are all recommended.

The notes are organized as follows. In the remainder of this overview we review notation used throughout the notes and provide a quick motivation for our study by examining some interpolation properties of Lebesgue spaces In Section 1 we develop the classical theory of interpolation in Lebesgue spaces mentioned above. To properly frame this, we first need a number of tools from advanced real analysis: the distribution function, rearrangements, and Lorentz spaces. These are developed in Section 1.1. In Section 1.2 we prove the main theorems on interpolation in Lebesgue spaces: the classical theorems of Marcinkiewicz and Riesz-Thorin. In Section 1.3 we provide some examples of applications of these theorems by studying the Hardy-Littlewood maximal function, various estimates of the Fourier transform (the Hausdorff-Young theorem and its variants), and certain integral operators, including those of convolution and Riesz potential type.

In Section 2 we introduce the abstract interpolation theory that arises as a generalization of the Marcinkiewicz theorem. This is known as the real method of interpolation, as it relies entirely on real variable techniques, in contrast with a second known method that relies crucially on complex variables and holomorphic functions, which is entirely ignored in these notes. In the abstract framework we first need to build a scale of spaces analogous to the scale of spaces obtained by varying p in the Lebesgue context. The preliminary functional analysis for this is carried out in Section 2.1. Then in Section 2.2 we construct interpolation spaces from given pairs of compatible Banach spaces. In Section 2.3 we prove a couple important theorems about abstract interpolation. In Section 2.4 we then study three concrete examples of the spaces obtained through real interpolation and demonstrate some applications.

0.2 Notation

We will employ the following notational conventions throughout these notes.

1. We use $\mathbb{F}$ to denote either of the fields $\mathbb{R}$ or $\mathbb{C}$. All vector spaces are over $\mathbb{F}$, where $\mathbb{F}$ is allowed to be either. The natural numbers, $\mathbb{N}$, include 0.

2. We will often write $\mathbb{R}_+ = (0, \infty) = \{x \in \mathbb{R} \mid 0 < x\}$.

3. Given a metric space X, $x \in X$, and $r > 0$, we define the open balls $B(x, r) = \{y \in X \mid d(x, y) < r\}$ and the closed balls $B[x, r] = \{y \in X \mid d(x, y) \leq r\}$.

4. Given normed vector spaces X and Y over a common field, we write $\mathcal{L}(X; Y) = \{T : X \to Y \mid T \text{ is bounded and linear}\}$ and endow it with the usual operator norm
$$\|T\|_{\mathcal{L}(X;Y)} = \sup\{\|Tx\|_Y \mid \|x\|_X \leq 1\}. \tag{0.2.1}$$

5. Given a measure space $(X, \mathfrak{M}, \mu)$ and $1 \leq p \leq \infty$ we will write $L^p(X; \mathbb{F})$ for the space of $\mathbb{F}-$valued $p-$integrable functions. When $\mathbb{F} = \mathbb{R}$ we will typically abbreviate $L^p(X) = L^p(X; \mathbb{F})$. If we want to emphasize the measure we will write $L^p_\mu(X; \mathbb{F})$.

6. We will write $\ell^p(\mathbb{Z}; \mathbb{F}) = L^p_\mu(\mathbb{Z}; \mathbb{F})$ for μ the counting measure.

7. Given a measure space $(X, \mathfrak{M}, \mu)$ we will write $S(X; \mathbb{F})$ for the space of simple functions on X and $S_{fin}(X; \mathbb{F})$ for the space of simple functions with support of finite measure.

8. For $1 \leq n \in \mathbb{N}$ we write $\omega_n = \mathcal{L}^n(B(0, 1))$ for the $n-$dimensional Lebesgue measure (written $\mathcal{L}^n$) of the unit ball in $\mathbb{R}^n$, and we write $\alpha_n = \mathcal{H}^{n-1}(\partial B(0, 1))$ for the $(n-1)-$dimensional Hausdorff measure (written $\mathcal{H}^{n-1}$) of the unit sphere in $\mathbb{R}^n$. These are related via
$$\omega_n = \int_{B(0,1)} dx = \alpha_n \int_0^1 r^{n-1} dr = \frac{\alpha_n}{n}. \tag{0.2.2}$$

9. Given topological vector spaces X and Y, we write $X \hookrightarrow Y$ if $X \subseteq Y$ is a vector subspace and the inclusion map $I : X \to Y$ is continuous.

0.3 Motivation

Let $(X, \mathfrak{M}, \mu)$ be a measure space and $1 \le p \le \infty$. We recall two basic facts about Lebesgue spaces that serve as the initial motivation for developing the theory of interpolation. For the first consider $f \in L^p(X;\mathbb{F})\backslash\{0\}$ for $1 < p < \infty$ and let $t \in \mathbb{R}_+$. Then we can write

$$f = f\chi_{\{|f|>t\}} + f\chi_{\{|f|\le t\}} =: f_1 + f_2. \tag{0.3.1}$$

Then $f_2 \in L^\infty(X;\mathbb{F})$ and $\|f_2\|_{L^\infty} \le t$, while

$$\|f_1\|_{L^1} = \int_X |f_1|\, d\mu = \int_{\{|f|>t\}} |f|\, d\mu \le \int_{\{|f|>t\}} |f| \frac{|f|^{p-1}}{t^{p-1}} d\mu = \frac{1}{t^{p-1}} \|f\|_{L^p}^p , \tag{0.3.2}$$

which in particular means that $f_1 \in L^1(X;\mathbb{F})$. This shows that we can always decompose a generic $f \in L^p(X;\mathbb{F})$ into a sum of elements of $L^1(X;\mathbb{F})$ and $L^\infty(X;\mathbb{F})$, which is noteworthy since 1 and ∞ are the extreme points of the set $[1,\infty]$. In fact, we can take this a bit further by realizing that the parameter $t \in \mathbb{R}_+$ can be tuned. Indeed, we can optimize the right side of the bound

$$\|f_1\|_{L^1} + \|f_2\|_{L^\infty} \le \frac{1}{t^{p-1}} \|f\|_{L^p}^p + t \tag{0.3.3}$$

over $t \in \mathbb{R}_+$ to see that the minimal value occurs when

$$1 - (p-1)t^{-p} \|f\|_{L^p}^p = 0 \Leftrightarrow t = (p-1)^{1/p} \|f\|_{L^p} . \tag{0.3.4}$$

Using this choice of t in the estimate above shows that

$$\begin{aligned} \|f\|_{L^1+L^\infty} &:= \inf\{\|a\|_{L^1} + \|b\|_{L^\infty} \mid f = a+b, a \in L^1, b \in L^\infty\} \\ &\le \left((p-1)^{-1/p'} + (p-1)^{1/p}\right) \|f\|_{L^p} , \end{aligned} \tag{0.3.5}$$

which actually shows the stronger result that $L^p(X;\mathbb{F}) \hookrightarrow L^1(X;\mathbb{F}) + L^\infty(X;\mathbb{F})$, where the latter space is endowed with the infimum norm written above.

The second fact we wish to recall from Lebesgue theory has to do with multiple inclusions. Suppose that $1 \le p_0 < p_1 \le \infty$ and let $p_0 < p < p_1$ be given by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ for } \theta \in (0,1). \tag{0.3.6}$$

In other words, $1/p$ is the convex interpolation between $1/p_0$ and $1/p_1$. Suppose $f \in L^{p_0}(X;\mathbb{F}) \cap L^{p_1}(X;\mathbb{F})$. If $p_1 < \infty$, then Hölder's inequality allows us to estimate

$$\int_X |f|^p\, d\mu = \int_X |f|^{(1-\theta)p} |f|^{\theta p}\, d\mu \le \left(\int_X |f|^{p_0}\, d\mu\right)^{p(1-\theta)/p_0} \left(\int_X |f|^{p_1}\, d\mu\right)^{p\theta/p_1} , \tag{0.3.7}$$

which implies that $f \in L^p(X;\mathbb{F})$ and

$$\|f\|_{L^p} \le \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^{\theta} . \tag{0.3.8}$$

The same is true if $p_1 = \infty$, and we leave it as an exercise to verify this. In fact, we can take this a bit further by recalling that we can endow the space $L^{p_0}(X;\mathbb{F}) \cap L^{p_1}(X;\mathbb{F})$ with the norm

$\|f\|_{L^{p_0}\cap L^{p_1}} = \max\{\|f\|_{L^{p_0}}, \|f\|_{L^{p_1}}\}$. Then the previous estimate shows that $\|f\|_{L^p} \le \|f\|_{L^{p_0}\cap L^{p_1}}$, and so $L^{p_0}(X;\mathbb{F}) \cap L^{p_1}(X;\mathbb{F}) \hookrightarrow L^p(X;\mathbb{F})$.

Note that if we combine these results, we deduce that for all $1 < p < \infty$ we have the embeddings

$$L^1(X;\mathbb{F}) \cap L^\infty(X;\mathbb{F}) \hookrightarrow L^p(X;\mathbb{F}) \hookrightarrow L^1(X;\mathbb{F}) + L^\infty(X;\mathbb{F}), \tag{0.3.9}$$

which we can think of as telling us that the $L^p(X;\mathbb{F})$ spaces somehow interpolate between the spaces on the left and right. This and the above results show that the Lebesgue spaces have very interesting interpolation properties. The goal of these notes is to further develop these ideas, first in the context of Lebesgue spaces and their natural generalization, Lorentz spaces, and second in the completely abstract setting of "compatible" Banach spaces. This leads to a set of powerful tools that have many applications in analysis and PDE, for example.

1 Classical interpolation theory

In this section we develop the interpolation theory of Lebesgue and Lorentz spaces. First we develop the necessary analytic tools. Then we prove the theorems of Marcinkiewicz and Riesz-Thorin. We conclude the section with some applications of these theorems.

1.1 Analysis tools

Here we collect a number of analytic tools that are useful in interpolation theory.

1.1.1 The Minkowski and Hardy inequalities

We begin by proving a version of Minkowski's inequality for integrals.

Theorem 1.1.1 (Minkowski's inequality, integral form). *Suppose that $(X, \mathfrak{M}_X, \mu)$ and $(Y, \mathfrak{M}_Y, \nu)$ are $\sigma-$finite measure spaces and let $(X \times Y, \mathfrak{M}_x \otimes \mathfrak{M}_Y, \mu \otimes \nu)$ be the associated product space. Let $1 \le p < \infty$ and $f : X \times Y \to [0,\infty]$ be a $\mu \otimes \nu-$measurable function. Then*

$$\left(\int_X \left(\int_Y f(x,y) d\nu(y)\right)^p d\mu(x)\right)^{1/p} \le \int_Y \left(\int_X (f(x,y))^p d\mu(x)\right)^{1/p} d\nu(y). \tag{1.1.1}$$

Proof. We begin by making a reduction. We claim that it suffices to prove the result under the extra assumptions that X and Y are finite and that f is bounded. Indeed, if the result is proved in this case then the monotone convergence theorem allows us to extend it to the case of X and Y $\sigma-$finite with f bounded by using decompositions of X and Y into countably many finite-measure subsets. In turn, the monotone convergence theorem again allows us to extend to general f by passing to the limit with finite truncations of f. This proves the claim, so we henceforth assume that X and Y are finite and f is bounded.

When $p = 1$ Tonelli's theorem tells us that (1.1.1) actually holds as an equality. Suppose then that $1 < p < \infty$ and set $p' = p/(p-1) \in (1,\infty)$. We may assume without loss of generality that the left side of (1.1.1) is non-zero, as otherwise the result is trivially true. Note also that the right side of (1.1.1) is finite due to assumptions of X, Y and f.

Define the measurable function $F : X \to [0,\infty)$ via

$$F(x) = \int_Y f(x,y) d\nu(y). \tag{1.1.2}$$

If $F = 0$ for μ–a.e. $x \in X$, then there's nothing to prove since the left side of (1.1.1) vanishes in this case. We may assume, then that this is not the case. Then by Tonelli's theorem and Hölder's inequality we can bound

$$\|F\|_{L^p(X)}^p = \int_X \left(\int_Y f(x,y) d\nu(y) \right)^p d\mu(x) = \int_X \left(\int_Y f(x,y) d\nu(y) \right) (F(x))^{p-1} d\mu(x)$$
$$= \int_Y \int_X f(x,y)(F(x))^{p-1} d\mu(x) d\nu(y) \le \int_Y \left(\int_X (f(x,y))^p d\mu(x) \right)^{1/p} \left(\int_X (F(x))^p d\mu(x) \right)^{1/p'} d\nu(y)$$
$$= \|F\|_{L^p(X)}^{p-1} \int_Y \left(\int_X (f(x,y))^p d\mu(x) \right)^{1/p} d\nu(y). \quad (1.1.3)$$

By assumption, $0 < \|F\|_{L^p(X)} < \infty$, so we can divide both sides by by $\|F\|_{L^p(X)}^{p-1}$ to deduce (1.1.1). □

Remark 1.1.2. *Minkowski's integral inequality can also be proved using duality arguments.*

Next we prove another very useful pair of inequalities, due to G.H. Hardy.

Theorem 1.1.3 (Hardy's inequalities)**.** *If $f : (0,\infty) \to [0,\infty)$ is Lebesgue measurable, $s > 0$, and $1 \le p < \infty$, then*

$$\left(\int_0^\infty \frac{1}{x^{s+1}} \left(\int_0^x f(t) dt \right)^p dx \right)^{1/p} \le \frac{p}{s} \left(\int_0^\infty x^{p-s-1} f(x)^p dx \right)^{1/p}. \quad (1.1.4)$$

and

$$\left(\int_0^\infty x^{s-1} \left(\int_x^\infty f(t) dt \right)^p dx \right)^{1/p} \le \frac{p}{s} \left(\int_0^\infty x^{p+s-1} f(x)^p dx \right)^{1/p}. \quad (1.1.5)$$

Proof. Using the change of variable $t = xr$ shows that

$$\frac{1}{x^{(s+1)/p}} \int_0^x f(t) dt = \int_0^1 \frac{f(rx)}{x^{(s+1)/p-1}} dr. \quad (1.1.6)$$

Then

$$\left(\int_0^\infty \frac{1}{x^{s+1}} \left(\int_0^x f(t) dt \right)^p dx \right)^{1/p} = \left(\int_0^\infty \left(\int_0^1 \frac{f(rx)}{x^{(s+1)/p-1}} dr \right)^p dx \right)^{1/p}. \quad (1.1.7)$$

Minkowski's integral inequality then shows that

$$\left(\int_0^\infty \left(\int_0^1 \frac{f(rx)}{x^{(s+1)/p-1}} dr \right)^p dx \right)^{1/p} \le \int_0^1 \left(\int_0^\infty \frac{f(rx)^p}{x^{s+1-p}} dx \right)^{1/p} dr, \quad (1.1.8)$$

but we may again change variables to write

$$\int_0^\infty \frac{f(rx)^p}{x^{s+1-p}} dx = r^{s-p} \int_0^\infty \frac{f(x)^p}{x^{s+1-p}} dx. \quad (1.1.9)$$

Thus

$$\int_0^1 \left(\int_0^\infty \frac{f(rx)^p}{x^{s+1-p}} dx \right)^{1/p} dr = \left(\int_0^\infty \frac{f(x)^p}{x^{s+1-p}} dx \right)^{1/p} \int_0^1 r^{s/p-1} dr = \frac{p}{s} \left(\int_0^\infty x^{p-s-1} f(x)^p dx \right)^{1/p}. \quad (1.1.10)$$

Chaining these together gives the first inequality.

The second identity follows similarly:

$$\left(\int_0^\infty x^{s-1}\left(\int_x^\infty f(t)dt\right)^p dx\right)^{1/p} = \left(\int_0^\infty \left(\int_1^\infty x^{(s-1)/p+1} f(rx)dr\right)^p dx\right)^{1/p}$$
$$\leq \int_1^\infty \left(\int_0^\infty x^{s-1+p} f(rx)^p dx\right)^p dr = \int_1^\infty \frac{1}{r^{1+s/p}} \left(\int_0^\infty y^{s-1+p} f(y)^p dy\right)^p dr$$
$$= \frac{p}{s}\left(\int_0^\infty y^{s-1+p} f(y)^p dy\right)^p. \quad (1.1.11)$$

□

Remark 1.1.4. *One particularly interesting case of Hardy's inequalities is occurs when $p > 1$ and $s = p - 1$, in which case*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right)^{1/p} \leq \frac{p}{p-1}\left(\int_0^\infty f(x)^p dx\right)^{1/p}. \quad (1.1.12)$$

This tells us that if $f \in L^p((0,\infty))$ then the average function $A : (0,\infty) \to [0,\infty)$ defined by

$$A(x) = \frac{1}{x}\int_0^x f(t)dt \quad (1.1.13)$$

is such that $A \in L^p((0,\infty))$.

1.1.2 The distribution function

We now establish an important relationship between the integral of a function (or rather the integral of its p^{th} power), and the size of the sets where $|f|$ is large.

Theorem 1.1.5 (Chebyshev's inequality)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space, $1 \leq p < \infty$, and $f : X \to \mathbb{F}$ be measurable. Then for each $t \in (0,\infty)$ we have that*

$$\mu(\{x \in X \mid |f(x)| > t\}) \leq \frac{1}{t^p}\int_X |f|^p \, d\mu. \quad (1.1.14)$$

Proof. Let $t \in (0,\infty)$. We compute

$$\mu(\{x \in X \mid |f(x)| > t\}) = \int_{\{|f|>t\}} d\mu \leq \int_{\{|f|>t\}} \frac{|f|^p}{t^p} d\mu \leq \frac{1}{t^p}\int_X |f|^p \, d\mu. \quad (1.1.15)$$

□

This suggests that we introduce some notation.

Definition 1.1.6. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f : X \to \mathbb{F}$ be measurable. We define the distribution function of f to be $d_f : [0,\infty) \to [0,\infty]$ given by*

$$d_f(t) = \mu(\{x \in X \mid |f(x)| > t\}). \quad (1.1.16)$$

Let's consider some examples.

Example 1.1.7. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $E \in \mathfrak{M}$ and $f = \chi_E$. Then for $t \geq 1$ we have that $\{x \in X \mid \chi_E(x) > t\} = \varnothing$, while for $t \in [0, 1)$ we have that $\{x \in X \mid \chi_E(x) > t\} = E$. Thus

$$d_{\chi_E}(t) = \begin{cases} \mu(E) & \text{if } t \in [0,1) \\ 0 & \text{if } t \in [1, \infty). \end{cases} \tag{1.1.17}$$

When $\mu(E) < \infty$ we can write this as $d_{\chi_E} = \mu(E)\chi_{[0,1)}$. △

Example 1.1.8. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $f : X \to [0, \infty)$ is a finite simple function given by

$$f = \sum_{i=1}^{n} a_i \chi_{E_i} \tag{1.1.18}$$

where $E_1, \ldots, E_n \in \mathfrak{M}$ are finite measure sets that are pairwise disjoint, and $0 < a_n < \cdots < a_1 < \infty$. Set $a_{n+1} = 0$. For $j = 1, \ldots, n$ set

$$b_j = \sum_{i=1}^{j} \mu(E_i). \tag{1.1.19}$$

Let $t \in [0, \infty)$. If $t \geq a_1$ then clearly $d_f(t) = \mu(\varnothing) = 0$. If $t \in [a_2, a_1)$, then $|f(x)| > t$ if and only if $x \in E_1$, so $d_f(t) = \mu(E_1)$. More generally, if $t \in [a_{j+1}, a_j)$ then

$$|f(x)| > t \Leftrightarrow x \in \bigcup_{i=1}^{j} E_i \tag{1.1.20}$$

and hence $d_f(t) = b_j$. Assembling this information shows that

$$d_f(t) = \sum_{j=1}^{n} b_j \chi_{[a_{j+1}, a_j)}(t). \tag{1.1.21}$$

Thus the distribution function of a finite simple function is again a finite simple function. △

Example 1.1.9. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $f : X \to [0, \infty)$ is a simple function that is not finite. We may then write

$$f = \sum_{i=1}^{n} a_i \chi_{E_i} \tag{1.1.22}$$

where $E_1, \ldots, E_n \in \mathfrak{M}$ are pairwise disjoint, and $0 < a_n < \cdots < a_1 < \infty$. Since f is not a finite simple function, there exists $1 \leq i \leq n$ such that $\mu(E_i) = \infty$. We then define

$$m = \min\{1 \leq i \leq n \mid \mu(E_i) = \infty\}. \tag{1.1.23}$$

If $m = 1$ then it's clear that

$$d_f(t) = \begin{cases} \infty & \text{for } 0 \leq t < a_1 \\ 0 & \text{for } a_1 \leq t. \end{cases} \tag{1.1.24}$$

Assume then that $1 < m \leq n$. For $j = 1, \ldots, m-1$ set

$$b_j = \sum_{i=1}^{j} \mu(E_i). \tag{1.1.25}$$

If $t \in [a_{j+1}, a_j)$ for $1 \le j \le m-1$ then

$$|f(x)| > t \Leftrightarrow x \in \bigcup_{i=1}^{j} E_i \tag{1.1.26}$$

and hence $d_f(t) = b_j$. On the other hand, if $0 \le t < a_m$ then $d_f(t) = \infty$. Assembling this information shows that

$$d_f(t) = \begin{cases} \infty & \text{for } 0 \le t < a_m \\ \sum_{j=1}^{m-1} b_j \chi_{[a_{j+1}, a_j)}(t) & \text{for } a_m \le t. \end{cases} \tag{1.1.27}$$

From this analysis we can write

$$d_f(t) = \sum_{j=1}^{n} b_j \chi_{[a_{j+1}, a_j)}(t) \tag{1.1.28}$$

with the understanding that $b_j = \infty$ for $j \ge m$. Thus the distribution function of a simple function is again a simple function. △

Example 1.1.10. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f : X \to \mathbb{F}$ be measurable. Then

$$d_f(t) = \mu(\{x \in X \mid |f(x)| > t\}) = d_{|f|}(t). \tag{1.1.29}$$

This shows that f and $|f| : X \to [0, \infty)$ have the same distribution functions. △

The next result establishes some basic properties of the distribution function.

Proposition 1.1.11. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f, g : X \to \mathbb{F}$ be measurable. Then the following hold.*

1. *If $|g| \le |f|$ a.e. then $d_g \le d_f$ on $[0, \infty)$.*
2. *If $c \in \mathbb{F}\backslash\{0\}$, then $d_{cf}(t) = d_f(t/|c|)$ for all $t \in [0, \infty)$.*
3. *If $t, s \in [0, \infty)$, then $d_{f+g}(t+s) \le d_f(t) + d_g(s)$.*
4. *If $t, s \in [0, \infty)$, then $d_{fg}(ts) \le d_f(t) + d_g(s)$.*
5. *d_f is nonincreasing and right continuous.*

Proof. The first two items are trivial. For the third item note that if $|f(x) + g(x)| > t + s$ then either $|f(x)| > t$ or $|g(x)| > s$ since otherwise

$$|f(x)| + |g(x)| \le t + s < |f(x) + g(x)| \le |f(x)| + |g(x)|, \tag{1.1.30}$$

a contradiction. Thus

$$\{x \in X \mid |f(x) + g(x)| > t + s\} \subseteq \{x \in X \mid |f(x)| > t\} \cup \{x \in X \mid |g(x)| > s\}, \tag{1.1.31}$$

which immediately implies that $d_{f+g}(t+s) \le d_f(t) + d_g(s)$. This proves the third item. The fourth item follows from a similar argument, which we leave as an exercise.

We now turn to the proof of the fifth item. It's obvious that d_f is nonincreasing. Fix $t \in [0,\infty)$ and suppose that $\{t_n\}_{n=\ell}^\infty \subset (t,\infty)$ is such that $t_n \to t$. Extract a decreasing subsequence $\{t_{n_m}\}_{m=\ell}^\infty$. Note that

$$\bigcup_{m=1}^\infty \{x \in X \mid |f(x)| > t_{n_m}\} = \{x \in X \mid |f(x)| > t\} \tag{1.1.32}$$

and that

$$\{x \in X \mid |f(x)| > t_{n_m}\} \subseteq \{x \in X \mid |f(x)| > t_{n_{m+1}}\}. \tag{1.1.33}$$

Hence

$$d_f(t) = \mu\left(\bigcup_{m=1}^\infty \{x \in X \mid |f(x)| > t_{n_m}\}\right) = \lim_{m\to\infty} \mu(\{x \in X \mid |f(x)| > t_{n_m}\}) = \lim_{m\to\infty} d_f(t_{n_m}). \tag{1.1.34}$$

On the other hand, since d_f is nondecreasing we have that

$$\lim_{m\to\infty} d_f(t_{n_m}) = \lim_{n\to\infty} d_f(t_n), \tag{1.1.35}$$

and so

$$d_f(t) = \lim_{n\to\infty} d_f(t_n). \tag{1.1.36}$$

This proves that d_f is right continuous, completing the proof of the fifth item.

□

Now we examine how the distribution function behaves with respect to sequences of functions. The next result should be thought of as analogous to the monotone convergence theorem and Fatou's lemma.

Proposition 1.1.12. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that for each $n \ge \ell \in \mathbb{Z}$ the function $f_n : X \to \mathbb{F}$ is measurable. Further suppose that $f : X \to \mathbb{F}$ is measurable. Then the following hold.*

1. *If $\{|f_n|\}_{n=\ell}^\infty$ is a.e. nondecreasing and $|f_n| \to |f|$ a.e. as $n \to \infty$, then $\{d_{f_n}\}_{n=\ell}^\infty$ is nondecreasing and $d_{f_n} \to d_f$ pointwise as $n \to \infty$.*
2. *If*
$$|f| \le \liminf_{n\to\infty} |f_n| \text{ a.e. in } X, \tag{1.1.37}$$
then
$$d_f \le \liminf_{n\to\infty} d_{f_n} \text{ in } [0,\infty). \tag{1.1.38}$$

Proof. We first prove the first item. Proposition 1.1.11 shows that $\{d_{f_n}\}_{n=\ell}^\infty$ is nondecreasing, so it suffices to prove that $d_{f_n} \to d_f$ pointwise as $n \to \infty$. Fix $t \in [0,\infty)$. Modifying the functions f_n on null sets if necessary, we may assume without loss of generality that $\{|f_n|\}_{n=\ell}^\infty$ is pointwise nondecreasing. Then

$$\{x \in X \mid |f_n(x)| > t\} \subseteq \{x \in X \mid |f_{n+1}(x)| > t\} \tag{1.1.39}$$

for each $n \ge \ell$, and

$$\bigcup_{n=1}^\infty \{x \in X \mid |f_n(x)| > t\} = \{x \in X \mid |f(x)| > t\}. \tag{1.1.40}$$

Thus

$$d_f(t) = \mu\left(\bigcup_{n=1}^{\infty}\{x \in X \mid |f_n(x)| > t\}\right) = \lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x)| > t\}) = \lim_{n\to\infty} d_{f_n}(t), \quad (1.1.41)$$

which completes the proof of the first item.

Next we prove the second item. For $n \geq \ell$ define the measurable function $g_n : X \to [0,\infty)$ via $g_n(x) = \inf_{m\geq n} |f_m(x)|$. By assumption we then have that $\{|g_n|\}_{n=\ell}^{\infty}$ is a.e. nondecreasing and $g_n = |g_n| \to g := \liminf_{n\to\infty} |f_n|$ a.e. as $n \to \infty$. Consequently, the first item tells us that

$$d_g = \lim_{n\to\infty} d_{g_n}. \quad (1.1.42)$$

On the other hand, by assumption we have that $|f| \leq |g| = g$ a.e. in X, and by construction we have that $|g_n| \leq |f_n|$ a.e. in X. Consequently, Proposition 1.1.11 implies that

$$d_f \leq d_g = \lim_{n\to\infty} d_{g_n} \leq \liminf_{n\to\infty} d_{f_n}. \quad (1.1.43)$$

This proves the second item.

□

Next we record a useful result that allows us to compute the distribution function for some map given that we know the distribution function of the map restricted to essentially disjoint sets.

Proposition 1.1.13. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $I \neq \varnothing$ is countable and $\{X_i\}_{i\in I} \subseteq \mathfrak{M}$ is such that $X = \bigcup_{i\in I} X_i$ and $\mu(X_i \cap X_j) = 0$ for $i, j \in I$ with $i \neq j$. Let $f : X \to \mathbb{F}$ be measurable, and for each $i \in I$ define the measurable function $f_i = f\chi_{X_i}$. Then*

$$d_f(t) = \sum_{i\in I} d_{f_i}(t) \text{ for all } t > 0. \quad (1.1.44)$$

Proof. Exercise. □

1.1.3 The layer cake theorem and its consequences

The following result establishes a deep connection between the distribution function of a map and integrals related to the map.

Theorem 1.1.14 (Layer cake theorem)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let ν be a Borel measure on $[0,\infty)$ such that the map $\varphi : [0,\infty) \to [0,\infty]$ given by $\varphi(t) = \nu([0,t))$ is finite for every $t \in [0,\infty)$. Let $f : X \to [0,\infty)$ be measurable. Then $\varphi \circ f$ is measurable and*

$$\int_X \varphi \circ f d\mu = \int_0^{\infty} d_f d\nu. \quad (1.1.45)$$

Proof. First note that φ is nondecreasing and $\varphi(0) = \nu(\varnothing) = 0$, so $\varphi \circ f$ is $\mu-$measurable.

Since f is measurable we can pick a sequence $\{f_n\}_{n=0}^{\infty} \subset S(X; [0,\infty))$ such that $f_n \leq f_{n+1}$ on X and $f_n \to f$ pointwise as $n \to \infty$. For each $n \in \mathbb{N}$ write

$$f_n = \sum_{i=1}^{N_n} a_{i,n}\chi_{E_{i,n}} \quad (1.1.46)$$

such that $0 < a_{N_n,n} < \cdots < a_{1,n}$. Define $a_{N_n+1,n} = 0$ as well as

$$b_{j,n} = \sum_{i=1}^{j} \mu(E_{i,n}) \in [0,\infty] \text{ for } 1 \le j \le N_n. \tag{1.1.47}$$

According to Examples 1.1.8 and 1.1.9, we have that $d_{f_n}(t) = 0$ for $t \ge a_{1,n}$ and

$$d_{f_n}(t) = b_{j,n} \text{ for } t \in [a_{j+1,n}, a_{j,n}). \tag{1.1.48}$$

Hence,

$$\int_0^\infty d_{f_n} d\nu = \sum_{j \in J_n} b_{j,n} \nu([a_{j+1,n}, a_{j,n})), \tag{1.1.49}$$

where

$$J_n = \{1 \le j \le N_n \mid \nu([a_{j+1,n}, a_{j,n})) > 0\}. \tag{1.1.50}$$

Note that

$$\varphi(a_{j,n}) - \varphi(a_{j+1,n}) = \nu([a_{j+1,n}, a_{j,n})) > 0 \text{ for } j \in J_n. \tag{1.1.51}$$

On the other hand,

$$\int_X \varphi \circ f_n d\mu = \sum_{i \in I_n} \varphi(a_{i,n}) \mu(E_{i,n}), \tag{1.1.52}$$

where

$$I_n = \{1 \le i \le N_n \mid \varphi(a_{i,n}) > 0\}. \tag{1.1.53}$$

According to (1.1.51) we have that $J_n \subseteq I_n$. Note also that since φ is nondecreasing the set I_n has the property that if $i \in I_n$ then $\{1, \ldots, i\} \subseteq I_n$, and so

$$j \in J_n \Rightarrow \{1, \ldots, j\} \subseteq I_n. \tag{1.1.54}$$

Now, from (1.1.52)and (1.1.54) we can compute (employing the convention that $\infty \cdot 0 = 0$ here)

$$\begin{aligned}
\int_X \varphi \circ f_n d\mu &= \sum_{i \in I_n} \varphi(a_{i,n}) \mu(E_{i,n}) = \sum_{i \in I_n} \mu(E_{i,n}) \sum_{j=i}^{N_n} \nu([a_{j+1,n}, a_{j,n})) \\
&= \sum_{i \in I_n} \sum_{j \in J_n} \mu(E_{i,n}) \nu([a_{j+1,n}, a_{j,n})) \chi_{\{i,\ldots,N_n\}}(j) = \sum_{j \in J_n} \sum_{i \in I_n} \mu(E_{i,n}) \nu([a_{j+1,n}, a_{j,n})) \chi_{\{i,\ldots,N_n\}}(j) \\
&= \sum_{j \in J_n} \nu([a_{j+1,n}, a_{j,n})) \sum_{i \in I_n \cap \{1,\ldots,j\}} \mu(E_{i,n}) = \sum_{j \in J_n} \nu([a_{j+1,n}, a_{j,n})) \sum_{i=1}^{j} \mu(E_{i,n}) \\
&= \sum_{j \in J_n} \nu([a_{j+1,n}, a_{j,n})) b_j,
\end{aligned} \tag{1.1.55}$$

and upon combining this with (1.1.49) we deduce that

$$\int_X \varphi \circ f_n d\mu = \int_0^\infty d_{f_n} d\nu. \tag{1.1.56}$$

Since φ is nondecreasing we know that $\{\varphi \circ f_n\}_{n=0}^\infty$ is also nondecreasing and converges pointwise to $\varphi \circ f$ as $n \to \infty$. Proposition 1.1.12 implies that $\{d_{f_n}\}_{n=0}^\infty$ is nondecreasing and $d_{f_n} \to d_f$ pointwise as $n \to \infty$. Combining these facts with the monotone convergence theorem, we may send $n \to \infty$ in (1.1.56) to conclude that (1.1.45) holds.

□

The most important consequence of the Layer Cake theorem is the following corollary.

Corollary 1.1.15. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f : X \to \mathbb{F}$ be measurable. Then for $1 \le p < \infty$ we have that*

$$\int_X |f|^p \, d\mu = \int_0^\infty pt^{p-1} d_f(t) dt. \tag{1.1.57}$$

Proof. Let ν_p be the Borel measure on $[0, \infty)$ given by

$$\nu_p(E) = \int_E pt^{p-1} dt, \tag{1.1.58}$$

where here dt is Lebesgue measure on $[0, \infty)$. Then $\nu_p([0, t)) = t^p$ and we deduce from the layer cake representation and Example 1.1.10 that

$$\int_X |f|^p \, d\mu = \int_0^\infty pt^{p-1} d_{|f|}(t) dt = \int_0^\infty pt^{p-1} d_f(t) dt. \tag{1.1.59}$$

□

1.1.4 Decreasing rearrangements

We know that the distribution function is nonincreasing and right continuous. This does allow for the possibility of jump discontinuities. If none existed, then we would be free to invert the distribution function. Since we can't do this in general, we do the next best thing. The resulting function, which we define now, will play a key role in our subsequent analysis.

Definition 1.1.16. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f : X \to \mathbb{F}$ be measurable. We define the decreasing rearrangement of f to be the function $f^\# : [0, \infty) \to [0, \infty]$ given by*

$$f^\#(t) = \inf\{s \in [0, \infty) \mid d_f(s) \le t\}. \tag{1.1.60}$$

We will prove momentarily that $f^\#$ is nonincreasing, so it is a bit of an abuse of notation to call $f^\#$ the decreasing rearrangement. Nevertheless, this is the standard notation, so we adopt it here.

Let's consider some examples.

Example 1.1.17. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $E \in \mathfrak{M}$ and $f = \chi_E$. We know from Example 1.1.7 that

$$d_{\chi_E}(t) = \begin{cases} \mu(E) & \text{if } t \in [0, 1) \\ 0 & \text{if } t \in [1, \infty). \end{cases} \tag{1.1.61}$$

In particular this implies that $d_{\chi_E}(s) \le \mu(E)$ for all $s \in [0, \infty)$, and thus $(\chi_E)^\#(t) = 0$ for all $t \ge \mu(E)$. Also, if $t < \mu(E)$ then $d_{\chi_E}(s) \le t$ if and only if $s \in [1, \infty)$, which implies that $(\chi_E)^\#(t) = 1$. Hence

$$(\chi_E)^\#(t) = \begin{cases} 1 & \text{if } t \in [0, \mu(E)) \\ 0 & \text{if } t \in [\mu(E), \infty). \end{cases} \tag{1.1.62}$$

△

Example 1.1.18. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $f : X \to [0, \infty)$ is a finite simple function given by

$$f = \sum_{i=1}^{n} a_i \chi_{E_i} \tag{1.1.63}$$

where $E_1, \ldots, E_n \in \mathfrak{M}$ are finite measure sets that are pairwise disjoint, and $0 < a_n < \cdots < a_1 < \infty$. Set $a_{n+1} = 0$. We saw in Example 1.1.8 that

$$d_f(t) = \sum_{j=1}^{n} b_j \chi_{[a_{j+1}, a_j)}(t), \tag{1.1.64}$$

where for $1 \le j \le n$ we set

$$b_j = \sum_{i=1}^{j} \mu(E_i) \in (0, \infty). \tag{1.1.65}$$

Set $b_0 = 0$. Then an argument in the same spirit as Example 1.1.8, which we leave as an exercise, shows that

$$f^{\#}(t) = \sum_{j=1}^{n} a_j \chi_{[b_{j-1}, b_j)}(t). \tag{1.1.66}$$

This proves that the rearrangement of a finite simple function is a finite simple function.

△

Example 1.1.19. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $f : X \to [0, \infty)$ is a simple function that is not finite given by

$$f = \sum_{i=1}^{n} a_i \chi_{E_i} \tag{1.1.67}$$

where $E_1, \ldots, E_n \in \mathfrak{M}$ are finite measure sets that are pairwise disjoint, and $0 = a_{n+1} < a_n < \cdots < a_1 < \infty$. We saw in Example 1.1.9 that

$$d_f(t) = \begin{cases} \infty & \text{for } 0 \le t < a_m \\ \sum_{1 \le j \le m-1} b_j \chi_{[a_{j+1}, a_j)}(t) & \text{for } a_m \le t, \end{cases} \tag{1.1.68}$$

where

$$m = \min\{1 \le i \le n \mid \mu(E_i) = \infty\}, \tag{1.1.69}$$

$$b_j = \sum_{i=1}^{j} \mu(E_i) \in (0, \infty] \text{ if } 1 \le j \le m, \tag{1.1.70}$$

and we have employed the convention that the sum over an empty set of indices is 0. From this we find that if we set $b_0 = 0$ then we can write

$$f^{\#}(t) = \sum_{j=1}^{m} a_j \chi_{[b_{j-1}, b_j)}(t). \tag{1.1.71}$$

This proves that the rearrangement of a non-finite simple function is a non-finite simple function.

△

Example 1.1.20. Let $(X, \mathfrak{M}, \mu)$ be a measure space, $f : X \to \mathbb{F}$ be measurable and define $|f| : X \to [0, \infty)$ via $|f|(x) = |f(x)|$. We know from Example 1.1.10 that

$$d_f(t) = \mu(\{x \in X \mid |f(x)| > t\}) = d_{|f|}(t). \tag{1.1.72}$$

Hence $f^{\#} = |f|^{\#}$. △

Our next result establishes some of the basic properties of the decreasing rearrangement.

Proposition 1.1.21. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and suppose that $f, g : X \to \mathbb{F}$ are measurable. Then the following hold.*

1. *$f^{\#}$ is nonincreasing.*
2. *If $|g| \le |f|$ a.e. then $g^{\#} \le f^{\#}$ on $[0, \infty)$.*
3. *If $c \in \mathbb{F}$ then $(cf)^{\#} = |c| f^{\#}$.*
4. *If $t \in [0, \infty)$ and $d_f(t) < \infty$, then $f^{\#}(d_f(t)) \le t$.*
5. *If $t \in [0, \infty)$ and $f^{\#}(t) < \infty$, then $d_f(f^{\#}(t)) \le t$.*
6. *If $t, s \in [0, \infty)$, then $(f + g)^{\#}(t + s) \le f^{\#}(t) + g^{\#}(s)$.*
7. *If $t, s \in [0, \infty)$, then $(fg)^{\#}(t + s) \le f^{\#}(t)g^{\#}(s)$.*

Proof. The first item follows directly from the fact that d_f is nonincreasing.

To prove the second item we note that Proposition 1.1.11 tells us that $d_g \le d_f$, and the inequality $g^{\#} \le f^{\#}$ then follows easily from the definition. To prove the third item we first note that it suffices to prove the result when $c \neq 0$, as the equality is trivial when $c = 0$. For $c \in \mathbb{C}\backslash\{0\}$ and $t \in [0, \infty)$ the second item of Proposition 1.1.11 implies that

$$\{s \ge 0 \mid d_{cf}(s) \le t\} = \{s \ge 0 \mid d_f(s/|c|) \le t\} = |c| \{s \ge 0 \mid d_f(s) \le t\}, \tag{1.1.73}$$

which then implies that $(cf)^{\#}(t) = |c| f^{\#}(t)$, completing the proof of the third item.

The fourth item follows trivially since if $d_f(t) < \infty$ then

$$f^{\#}(d_f(t)) = \inf\{s \ge 0 \mid d_f(s) \le d_f(t)\} \le t. \tag{1.1.74}$$

For the fifth item we note that if $f^{\#}(t) < \infty$ then by definition for each $n \in \mathbb{N}$ we can find $r_n \in [f^{\#}(t), f^{\#}(t) + 2^{-n})$ such that $d_f(r_n) \le t$. Clearly $r_n \to f^{\#}(t)$ as $n \to \infty$, so the right continuity of d_f implies that

$$d_f(f^{\#}(t)) = \lim_{n\to\infty} d_f(r_n) \le t. \tag{1.1.75}$$

This proves the fifth item.

We now turn to the proof of the sixth item. Fix $t, s \in [0, \infty)$. Without loss of generality we may assume that $f^{\#}(t)+g^{\#}(s) < \infty$, in which case we can set $a = f^{\#}(t) < \infty$ and $b = g^{\#}(s) < \infty$. Then the fifth item tells us that $d_f(a) \le t$ and $d_g(b) \le s$. Consequently, the third item of Proposition 1.1.11 implies that

$$d_{f+g}(a + b) \le d_f(a) + d_g(b) \le t + s \tag{1.1.76}$$

and so by definition $(f + g)^{\#}(t + s) \le a + b = f^{\#}(t) + f^{\#}(s)$. The sixth item is proved.

To prove the seventh item we assume without loss of generality that $f^{\#}(t) \cdot g^{\#}(s) < \infty$ and again set $a = f^{\#}(t) < \infty$ and $b = g^{\#}(s) < \infty$. Then the third item of Proposition 1.1.11 tells us that

$$d_{fg}(ab) \leq d_f(a) + d_g(b) \leq t + s \tag{1.1.77}$$

and so $(fg)^{\#}(t+s) \leq ab = f^{\#}(t)g^{\#}(s)$. The proves the seventh item.

□

Now we record the most important features of decreasing rearrangements.

Theorem 1.1.22. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and suppose that $f : X \to \mathbb{F}$ is measurable. Let $f^{\#} : [0, \infty) \to [0, \infty]$ be its decreasing rearrangement. Then the following hold.*

1. *For $s, t \in [0, \infty)$ we have that $s < f^{\#}(t)$ if and only if $t < d_f(s)$.*
2. *$f^{\#}$ is right continuous.*
3. *If $s \in [0, \infty)$, then*
$$d_f(s) = \mu(\{x \in X \mid |f(x)| > s\}) = \mathcal{L}^1(\{t \in [0, \infty) \mid f^{\#}(t) > s\}), \tag{1.1.78}$$
where $\mathcal{L}^1$ denotes Lebesgue measure on $\mathbb{R}$. In particular $d_f = d_{f^{\#}}$.
4. *For every $1 \leq p < \infty$ we have that*
$$\int_X |f|^p \, d\mu = \int_0^\infty (f^{\#}(t))^p dt \tag{1.1.79}$$
and we also have that
$$\operatorname{esssup} |f| = \inf\{C > 0 \mid |f| \leq C \text{ a.e. in } X\} = \sup_{t>0} f^{\#}(t) = f^{\#}(0). \tag{1.1.80}$$

Proof. To prove the first item fix $s, t \in [0, \infty)$. If $s < f^{\#}(t)$ then $s \notin \{r \mid d_f(r) \leq t\}$, so we must have that $t < d_f(s)$. Now suppose that $t < d_f(s)$. Proposition 1.1.11 guarantees that d_f is right continuous and nonincreasing, so we can pick $\varepsilon > 0$ such that $t < d_f(r)$ for all $0 \leq r < s + \varepsilon$. This means that $[0, s+\varepsilon] \cap \{r \mid d_f(r) \leq t\} = \varnothing$, and hence $s < s + \varepsilon \leq f^{\#}(t)$. This completes the proof of the first item.

To prove the second item we fix $T \in [0, \infty)$. If $f^{\#}(T) = 0$ then $f^{\#}(r) = 0$ for all $r \geq T$, and so $f^{\#}$ is trivially right continuous at T. Suppose then, that $f^{\#}(T) > 0$ and pick $0 < s < f^{\#}(T)$. The first item then implies that $T < d_f(s)$, which then allows us to choose $T < r < d_f(s)$. In turn, the first item implies that $s < f^{\#}(t)$ for all $t \in [T, r]$. Then since $f^{\#}$ is nonincreasing we deduce that

$$s \leq \lim_{t \to T^+} f^{\#}(t) \leq f^{\#}(T). \tag{1.1.81}$$

This inequality holds for arbitrary $0 < s < f^{\#}(T)$, and hence

$$f^{\#}(T) = \lim_{t \to T^+} f^{\#}(t), \tag{1.1.82}$$

which proves that $f^{\#}$ is right continuous at T. This proves the second item.

To prove the third item fix $s \in [0, \infty)$. The first item then implies that

$$\mathcal{L}^1(\{t \in [0, \infty) \mid f^{\#}(t) > s\}) = \mathcal{L}^1(\{t \in [0, \infty) \mid d_f(s) > t\}) = \mathcal{L}^1([0, d_f(s))) = d_f(s), \tag{1.1.83}$$

which proves the third item.

Finally, we prove the fourth item. The third item tells us that $d_f = d_{f^\#}$, so Corollary 1.1.15 implies the fourth item for $1 \le p < \infty$. Finally, we have that

$$\operatorname{esssup} |f| = \inf\{s \ge 0 \mid d_f(s) = 0\} = \inf\{s \ge 0 \mid d_{f^\#}(s) = 0\} = \sup_{t>0} f^\#(t), \tag{1.1.84}$$

which completes the proof of the fourth item. □

The following result serves as an analog of the monotone convergence theorem and Fatou's lemma for decreasing rearrangements.

Proposition 1.1.23. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, and suppose that for each $n \ge \ell \in \mathbb{Z}$ the function $f_n : X \to \mathbb{F}$ is measurable. Further suppose that $f : X \to \mathbb{F}$ is measurable. Then the following hold.*

1. *If $\{|f_n|\}_{n=\ell}^\infty$ is a.e. nondecreasing and $|f_n| \to |f|$ a.e. as $n \to \infty$, then $\{f_n^\#\}_{n=\ell}^\infty$ is nondecreasing and $f_n^\# \to f^\#$ pointwise as $n \to \infty$.*
2. *If*
$$|f| \le \liminf_{n\to\infty} |f_n| \text{ a.e. in } X, \tag{1.1.85}$$
then
$$f^\# \le \liminf_{n\to\infty} f_n^\# \text{ in } [0,\infty). \tag{1.1.86}$$

Proof. We begin with the proof of the first item. According to Proposition 1.1.21 we have that $f_n^\# \le f_{n+1}^\# \le f^\#$ for all $n \ge \ell$, and so

$$\lim_{n\to\infty} f_n^\#(t) \le f^\#(t) \text{ for all } t \in [0,\infty). \tag{1.1.87}$$

Suppose, by way of contradiction, that there exists $t \in [0,\infty)$ such that

$$\lim_{n\to\infty} f_n^\#(t) < f^\#(t) \tag{1.1.88}$$

and choose $s \in [0,\infty)$ such that

$$\lim_{n\to\infty} f_n^\#(t) < s < f^\#(t). \tag{1.1.89}$$

By the first item of Theorem 1.1.22 we then have that $t < d_f(s)$. Proposition 1.1.12 tells us that $d_{f_n} \to d_f$ as $n \to \infty$, so we can find $N \ge \ell$ such that $n \ge N$ implies that $t < d_{f_n}(s) \le d_f(s)$, which particular implies that $s < f_N^\#(t)$, a contradiction since $\{f_n^\#\}_{n=\ell}^\infty$ is nondecreasing. This proves the first item.

To prove the second let us suppose, again by way of contradiction, that

$$\liminf_{n\to\infty} f_n^\#(t) < f^\#(t) \tag{1.1.90}$$

for some $t \in [0,\infty)$. In particular we can pick $s \in [0,\infty)$ such that

$$\liminf_{n\to\infty} f_n^\#(t) < s < f^\#(t). \tag{1.1.91}$$

Then again by Theorem 1.1.22 and Proposition 1.1.12 we have that

$$t < d_f(s) \le \liminf_{n\to\infty} d_{f_n}(s), \tag{1.1.92}$$

which implies that there exists $N \geq \ell$ such that

$$t < d_{f_n}(s) \text{ for all } n \geq N, \tag{1.1.93}$$

and hence that

$$s < f_n^\#(t) \text{ for all } n \geq N. \tag{1.1.94}$$

Consequently,

$$s \leq \liminf_{n\to\infty} f_n^\#(t) < s, \tag{1.1.95}$$

a contradiction. Hence

$$f^\#(t) \leq \liminf_{n\to\infty} f_n^\#(t) \text{ for all } t \in [0,\infty), \tag{1.1.96}$$

which proves the second item. □

1.1.5 The Hardy-Littlewood rearrangement inequality

Our goal now is to prove a key inequality related to rearrangements. We begin with a lemma.

Lemma 1.1.24. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $f : X \to \mathbb{F}$ is measurable, and let $E \in \mathfrak{M}$. Then for every $t \in [0,\infty)$ we have that*

$$d_{f\chi_E}(t) \leq \min\{d_f(t), \mu(E)\} \tag{1.1.97}$$

and

$$(f\chi_E)^\#(t) \leq f^\#(t)\chi_{[0,\mu(E))}(t). \tag{1.1.98}$$

Moreover,

$$\int_E |f|\, d\mu \leq \int_0^{\mu(E)} f^\#(t)dt. \tag{1.1.99}$$

Proof. Fix $t \in [0,\infty)$. Since $E \subseteq X$ we may then bound

$$d_{f\chi_E}(t) = \mu(\{x \in E \mid |f(x)| > t\}) \leq \min\{d_f(t), \mu(E)\}. \tag{1.1.100}$$

With this inequality established we then see that

$$\{s \in [0,\infty) \mid d_f(s) \leq t\} \subseteq \{s \in [0,\infty) \mid d_{f\chi_E}(s) \leq t\}, \tag{1.1.101}$$

which implies that $(f\chi_E)^\#(t) \leq f^\#(t)$. Similarly, if $\mu(E) \leq t$ then $d_{f\chi_E}(s) \leq \mu(E) \leq t$ for all $s \in [0,\infty)$ and hence $(f\chi_E)^\#(t) = 0$. Thus

$$(f\chi_E)^\#(t) \leq f^\#(t)\chi_{[0,\mu(E))}(t). \tag{1.1.102}$$

The inequality (1.1.99) then follows directly from the last bound and Corollary 1.1.15.

□

We now have the tools needed to prove the aforementioned inequality. It serves as a sort of intermediate inequality that nestles between the terms encountered in the standard Hölder inequality.

Theorem 1.1.25 (Hardy-Littlewood rearrangement inequality)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space, and suppose that $f, g : X \to [0,\infty]$ are measurable. Then*

$$\int_X fg d\mu \leq \int_0^\infty f^\#(t)g^\#(t)dt. \tag{1.1.103}$$

Proof. We begin by making two reductions. First, it suffices to prove the result under the assumption that $f \neq 0$ and $g \neq 0$, i.e. neither f nor g is trivial, as in this case the inequality is trivially satisfied. Second, we claim that it suffices to prove the result when f is simple. Indeed, suppose the result is proved whenever $f, g : X \to [0, \infty)$ and f is simple. We then choose a sequence $\{f_n\}_{n=0}^{\infty}$ of non-negative simple functions such that $f_n \nearrow f$ a.e. as $n \to \infty$. Applying the result, we find that

$$\int_X f_n g d\mu \leq \int_0^\infty f_n^\#(t) g^\#(t) dt \tag{1.1.104}$$

According to Proposition 1.1.23 we have that $f_n^\# \nearrow f^\#$ on $[0, \infty)$ as $n \to \infty$, so the monotone convergence theorem allows us to send $n \to \infty$ in (1.1.104) to deduce that (1.1.103) holds. This proves the claim.

Assume then that $f, g : X \to [0, \infty)$ and that f is simple. We may write

$$f = \sum_{i=1}^n a_i \chi_{E_i}, \tag{1.1.105}$$

where $E_1, \ldots, E_n \in \mathfrak{M}$ are pairwise disjoint, and $0 := a_{n+1} < a_n < \cdots < a_1 < \infty$. We saw in Examples 1.1.18 and 1.1.19 that

$$f^\#(t) = \sum_{j=1}^m a_j \chi_{[b_{j-1}, b_j)}(t), \tag{1.1.106}$$

where

$$m = \begin{cases} n & \text{if } f \text{ is finite} \\ \min\{1 \leq i \leq n \mid \mu(E_i) = \infty\} & \text{otherwise ,} \end{cases} \tag{1.1.107}$$

$b_0 = 0$, and

$$b_j = \sum_{i=1}^j \mu(E_i) \in (0, \infty] \text{ if } 1 \leq j \leq m. \tag{1.1.108}$$

Write $F_0 = \varnothing$ and for $1 \leq j \leq n$ set $F_j = \bigcup_{i=1}^j E_i$ and $d_j = a_j - a_{j+1}$. This allows us to rewrite

$$f = \sum_{j=1}^n d_j \chi_{F_j}. \tag{1.1.109}$$

Similarly, we have the identity

$$\sum_{j=1}^{m-1} d_j \chi_{F_j}(x) = f(x) - a_m \text{ for all } x \in F_{m-1}. \tag{1.1.110}$$

Now we use Theorem 1.1.22, Lemma 1.1.24, (1.1.106) and (1.1.110) to bound

$$\begin{aligned} \int_X fg d\mu &= \int_{F_n} fg d\mu = \int_{F_{m-1}} fg d\mu + \int_{F_n \setminus F_{m-1}} fg d\mu \leq \int_{F_{m-1}} fg d\mu + \int_{F_n \setminus F_{m-1}} a_m g d\mu \\ &= \int_{F_{m-1}} (f - a_m) g d\mu + \int_{F_n} a_m g d\mu = \sum_{j=1}^{m-1} d_j \int_X g \chi_{F_j} d\mu + a_m \int_X g \chi_{F_n} d\mu \\ &= \sum_{j=1}^{m-1} d_j \int_0^\infty (g\chi_{F_j})^\#(t) dt + a_m \int_0^\infty (g\chi_{F_n})^\#(t) dt \leq \sum_{j=1}^{m-1} d_j \int_0^{\mu(F_j)} g^\#(t) dt + a_m \int_0^{\mu(F_n)} g^\#(t) dt. \end{aligned} \tag{1.1.111}$$

We then compute

$$\sum_{j=1}^{m-1} d_j \int_0^{\mu(F_j)} g^{\#}(t)dt = \sum_{j=1}^{m-1}(a_j - a_{j+1}) \int_0^{\mu(F_j)} g^{\#}(t)dt$$
$$= \sum_{j=1}^{m-1} a_j \int_0^{\mu(F_j)} g^{\#}(t)dt - \sum_{j=1}^{m-1} a_{j+1} \int_0^{\mu(F_j)} g^{\#}(t)dt$$
$$= \sum_{j=1}^{m-1} a_j \int_0^{\mu(F_j)} g^{\#}(t)dt - \sum_{j=1}^{m} a_j \int_0^{\mu(F_{j-1})} g^{\#}(t)dt = -a_m \int_0^{\mu(F_{m-1})} g^{\#}(t)dt + \sum_{j=1}^{m-1} a_j \int_{\mu(F_{j-1})}^{\mu(F_j)} g^{\#}(t)dt. \tag{1.1.112}$$

Chaining the previous two estimates together then reveals that

$$\int_X fgd\mu \leq \sum_{j=1}^{m-1} a_j \int_{\mu(F_{j-1})}^{\mu(F_j)} g^{\#}(t)dt + a_m \int_{\mu(f_{m-1})}^{\mu(F_n)} g^{\#}(t)dt$$
$$= \int_0^{\infty} \sum_{j=1}^{m-1} a_j \chi_{[b_{j-1},b_j)}(t) g^{\#}(t)dt + a_m \int_{\mu(F_{m-1})}^{\mu(F_n)} g^{\#}(t)dt. \tag{1.1.113}$$

We now split to cases. If f is finite, then $m-1 = n-1$ and $\mu(F_n) < \infty$, so (1.1.106) allows us to compute

$$\int_0^{\infty} \sum_{j=1}^{m-1} a_j \chi_{[b_{j-1},b_j)}(t) g^{\#}(t)dt + a_m \int_{\mu(F_{m-1})}^{\mu(F_n)} g^{\#}(t)dt = \int_0^{\infty} \sum_{j=1}^{n} a_j \chi_{[b_{j-1},b_j)}(t) g^{\#}(t)dt$$
$$= \int_0^{\infty} f^{\#}(t) g^{\#}(t)dt. \tag{1.1.114}$$

On the other hand, if f is not finite, them $\mu(F_n) = \mu(F_m) = b_m = \mu(E_m) = \infty$, so (1.1.106) allows us to compute

$$\int_0^{\infty} \sum_{j=1}^{m-1} a_j \chi_{[b_{j-1},b_j)}(t) g^{\#}(t)dt + a_m \int_{\mu(F_{m-1})}^{\mu(F_n)} g^{\#}(t)dt$$
$$= \int_0^{\infty} \sum_{j=1}^{m-1} a_j \chi_{[b_{j-1},b_j)}(t) g^{\#}(t)dt + a_m \int_{b_{m-1}}^{b_m} g^{\#}(t)dt = \int_0^{\infty} \sum_{j=1}^{m} a_j \chi_{[b_{j-1},b_j)}(t) g^{\#}(t)dt$$
$$= \int_0^{\infty} f^{\#}(t) g^{\#}(t)dt. \tag{1.1.115}$$

Combining (1.1.113), (1.1.114), and (1.1.115) then proves (1.1.103).

□

We proved the Hardy-Littlewood inequality for non-negative functions, but it also works just as well for more general maps.

Corollary 1.1.26. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f, g : X \to \mathbb{F}$ be measurable. Then*

$$\left| \int_X fgd\mu \right| \leq \int_X |fg| \, d\mu \leq \int_0^{\infty} f^{\#}(t) g^{\#}(t)dt. \tag{1.1.116}$$

Proof. Exercise. □

1.1.6 Lorentz spaces

Our aim now is to define a larger class of functions than the Lebesgue spaces. To motivate this we begin with the following simple result.

Proposition 1.1.27. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, $1 \le p < \infty$, and $f : X \to \mathbb{F}$ be measurable. Then*

$$\sup_{t>0} t(d_f(t))^{1/p} = \sup_{t>0} t^{1/p} f^{\#}(t). \tag{1.1.117}$$

Proof. Write

$$A = \sup_{t\ge 0} t(d_f(t))^{1/p} \text{ and } B = \sup_{t\ge 0} t^{1/p} f^{\#}(t). \tag{1.1.118}$$

We will prove that $B \le A \le B$.

Suppose, by way of contradiction, that $A < B$ and pick $s \in (0,\infty)$ such that $A < s < B$. By the definition of B we may then choose $t > 0$ such that $s < t^{1/p} f^{\#}(t)$. By Theorem 1.1.22 we then see that

$$\frac{s}{t^{1/p}} < f^{\#}(t) \Rightarrow t < d_f\left(\frac{s}{t^{1/p}}\right) \Rightarrow s < \frac{s}{t^{1/p}}\left[d_f\left(\frac{s}{t^{1/p}}\right)\right]^{1/p} \le A, \tag{1.1.119}$$

which is a contradiction. Thus $B \le A$.

Now suppose, again by way of contradiction, that $B < A$ and pick $B < s < A$. Choose $t > 0$ such that $s < t(d_f(t))^{1/p}$. Then Theorem 1.1.22 again tells us that

$$\frac{s^p}{t^p} < d_f(t) \Rightarrow t < f^{\#}\left(\frac{s^p}{t^p}\right) \Rightarrow s < \frac{s}{t} f^{\#}\left(\frac{s^p}{t^p}\right) \le B, \tag{1.1.120}$$

which is a contradiction. Thus $A \le B$. □

Consider a function $f \in L^p(X;\mathbb{F})$. We know from Theorem 1.1.22 that

$$\|f\|_{L^p} = \left(\int_0^\infty (f^{\#}(t))^p dt\right)^{1/p} = \left(\int_0^\infty (t^{1/p} f^{\#}(t))^p \frac{dt}{t}\right)^{1/p} = \left\|(\cdot)^{1/p} f^{\#}\right\|_{L^p_\mu((0,\infty)}, \tag{1.1.121}$$

where we have defined the measure $\mu = dt/t$ on $(0,\infty)$. On the other hand, Chebyshev's inequality and Proposition 1.1.27 tell us that

$$\sup_{t>0} t^{1/p} f^{\#}(t) = \sup_{t>0} t(d_f(t))^{1/p} \le \|f\|_{L^p}. \tag{1.1.122}$$

The null sets of μ and $\mathcal{L}^1$ (standard Lebesgue measure) agree, and so $L^\infty_\mu((0,\infty)) = L^\infty_{\mathcal{L}^1}((0,\infty))$. Thus

$$\sup_{t>0} t^{1/p} f^{\#}(t) = \left\|(\cdot)^{1/p} f^{\#}(\cdot)\right\|_{L^\infty \mu((0,\infty))}. \tag{1.1.123}$$

These computations suggest that there is interesting information encoded in the quantities

$$\left\|(\cdot)^{1/p} f^{\#}(\cdot)\right\|_{L^q_\mu((0,\infty))} \text{ for } 1 \le q \le \infty. \tag{1.1.124}$$

Indeed, there is! We begin our exploration of this by defining some new spaces.

Definition 1.1.28. *Let $(X, \mathfrak{M}, \mu)$ be a measure space.*

1. *Let $f : X \to \mathbb{F}$ is measurable, $1 \le p \le \infty$, and $1 \le q \le \infty$. For $q < \infty$ we define*

$$|||f|||_{L^{p,q}} = \left(\int_0^\infty (t^{1/p} f^\#(t))^q \frac{dt}{t} \right)^{1/q} \in [0, \infty], \tag{1.1.125}$$

and for $q = \infty$ we define

$$|||f|||_{L^{p,\infty}} = \sup\{t^{1/p} f^\#(t) \mid t > 0\} \in [0, \infty]. \tag{1.1.126}$$

2. *For $1 \le p, q \le \infty$ we define the Lorentz space $L^{p,q}$ to be*

$$L^{p,q}(X; \mathbb{F}) = L^{p,q}(X, \mu; \mathbb{F}) = \{[f]_\simeq \mid f : X \to \mathbb{F} \text{ is measurable and } |||f|||_{L^{p,q}} < \infty\} \tag{1.1.127}$$

where once more $[\cdot]_\simeq$ is the equivalence class generated by a.e. equality. We will write $L^{p,q}(X) = L^{p,q}(X; \mathbb{R})$.

Let's consider some examples. The first shows that $S_{fin}(X; \mathbb{F}) \subseteq L^{p,q}(X; \mathbb{F})$ when $p, q < \infty$.

Example 1.1.29. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $f : X \to [0, \infty)$ is a finite simple function given by

$$f = \sum_{i=1}^n a_i \chi_{E_i} \tag{1.1.128}$$

where $E_1, \dots, E_n \in \mathfrak{M}$ are finite measure sets that are pairwise disjoint, and $0 < a_n < \cdots < a_1 < \infty$. Set $a_{n+1} = 0$. We saw in Example 1.1.18 that if we set

$$b_j = \sum_{i=1}^j \mu(E_i). \tag{1.1.129}$$

for $1 \le j \le n$, and $b_0 = 0$, then

$$f^\#(t) = \sum_{j=1}^n a_j \chi_{[b_{j-1}, b_j)}(t). \tag{1.1.130}$$

For $1 \le p, q < \infty$ we then compute

$$|||f|||^q_{L^{p,q}} = \int_0^\infty (t^{1/p} f^\#(t))^q \frac{dt}{t} = \sum_{j=1}^n a_j^q \int_{b_{j-1}}^{b_j} t^{q/p-1} dt = \frac{p}{q} \sum_{j=1}^n a_j^q \left(b_j^{q/p} - b_{j-1}^{q/p} \right). \tag{1.1.131}$$

△

The next example shows just how inclusive the Lorentz space $L^{p,\infty}$ can be.

Example 1.1.30. Let $1 \le p < \infty$ and consider $X = \mathbb{R}^n$ with Lebesgue measure. For $\alpha > 0$ define $f : \mathbb{R}^n \to \mathbb{R}$ via $f(0) = 0$ and $f(x) = |x|^{-\alpha}$. Then

$$t < \frac{1}{|x|^\alpha} \Leftrightarrow |x| < \frac{1}{t^{1/\alpha}} \tag{1.1.132}$$

and so

$$d_f(t) = \omega_n t^{-n/\alpha} \tag{1.1.133}$$

where $\omega_n = \mathcal{L}^n(B(0, 1))$. Consequently, $f \in L^{p,\infty}$ if and only if $p = n/\alpha$, in which case

$$|||f|||_{L^{p,\infty}} = \omega_n^{1/p} = \omega_n^{\alpha/n}. \tag{1.1.134}$$

The bound $1 \le p$ requires that

$$0 < \alpha \le n. \tag{1.1.135}$$

In particular, if $\alpha = n/p$ for a given $1 \le p$, then $f \in L^{p,\infty}(\mathbb{R}^n)$. △

We used Proposition 1.1.27 to justify our choice of the form of $|||\cdot|||_{L^{p,q}}$, and this result showed that $|||f|||_{L^{p,\infty}} = \sup_{t>0} t(d_f(t))^{1/p}$, i.e. we can recharacterize this quantity in terms of the distribution function. We can do something similar for the integral quantities as well.

Proposition 1.1.31. *Let $(X,\mathfrak{M},\mu)$ be a measure space, V be a finite dimensional normed vector space, $1 \le q < \infty$, and $0 < \alpha < \infty$. Then*

$$\int_0^\infty t^\alpha (f^\#(t))^q \frac{dt}{t} = \frac{q}{\alpha} \int_0^\infty s^q (d_f(s))^\alpha \frac{ds}{s}. \tag{1.1.136}$$

In particular, for $1 \le p < \infty$ we have that

$$|||f|||_{L^{p,q}} = p^{1/q} \left(\int_0^\infty s^p (d_f(s))^{q/p} \frac{ds}{s} \right)^{1/q}. \tag{1.1.137}$$

Proof. The second assertion follows from the first by setting $\alpha = q/p$. To prove the first we compute, using Tonelli's theorem and the fundamental theorem of calculus:

$$\begin{aligned} \int_0^\infty t^\alpha (f^\#(t))^q \frac{dt}{t} &= \int_0^\infty t^{\alpha-1} \int_0^{f^\#(t)} q s^{q-1} ds dt = \int_0^\infty t^{\alpha-1} \int_0^\infty q s^{q-1} \chi_{(0,f^\#(t))}(s) ds dt \\ &= q \int_0^\infty s^{q-1} \int_0^\infty t^{\alpha-1} \chi_{(0,f^\#(t))}(s) dt ds = q \int_0^\infty s^{q-1} \int_{\{t>0 \mid f^\#(t)>s\}} t^{\alpha-1} dt ds. \end{aligned} \tag{1.1.138}$$

However, from Theorem 1.1.22 we know that

$$\{t > 0 \mid s < f^\#(t)\} = \{t > 0 \mid t < d_f(s)\} = (0, d_f(s)). \tag{1.1.139}$$

Thus,

$$\int_0^\infty t^\alpha (f^\#(t))^q \frac{dt}{t} = q \int_0^\infty s^{q-1} \int_0^{d_f(s)} t^{\alpha-1} dt ds = \frac{q}{\alpha} \int_0^\infty s^{q-1} (d_f(s))^\alpha ds, \tag{1.1.140}$$

which is the desired equality. □

Remark 1.1.32. *The identity used here can be significantly generalized. Indeed, if μ, ν are Radon measures on $[0,\infty)$ define $\varphi_\mu, \varphi_\nu : [0,\infty) \to [0,\infty)$ via $\varphi_\mu(t) = \mu([0,t))$ and $\varphi_\nu(t) = \nu([0,t))$. Then for Radon measures μ and ν on $[0,\infty)$ we have that*

$$\int_0^\infty \varphi_\mu \circ f^\#(t) d\nu(t) = \int_0^\infty \varphi_\nu \circ d_f(s) d\mu(s). \tag{1.1.141}$$

We leave it as an exercise to verify this.

Now we record some very basic properties of Lorentz spaces.

Proposition 1.1.33. *Let $(X,\mathfrak{M},\mu)$ be a measure space and $1 \le p,q \le \infty$. Then the following hold.*

1. *$L^{p,p}(X;\mathbb{F}) = L^p(X;\mathbb{F})$ and $|||\cdot|||_{L^{p,p}} = \|\cdot\|_{L^p}$.*
2. *If $1 \le q < \infty$ and $\mu(X) > 0$ then $L^{\infty,q}(X;\mathbb{F}) = \{0\}$.*

3. *If $1 \le p < \infty$ and $1 \le q \le \infty$ or $p = q = \infty$, then*

$$S_{fin}(X;\mathbb{F}) \subseteq L^{p,q}(X;\mathbb{F}), \tag{1.1.142}$$

where we recall that $S_{fin}(X;\mathbb{F})$ denotes the set of simple functions of finite support.

Proof. Exercise. □

At this point it is not clear why we have used the symbol $|||\cdot|||$ in place of the usual norm symbol $\|\cdot\|$. The reason, unfortunately, is that $|||\cdot|||_{L^{p,q}}$ is not actually a norm in general. We investigate this now.

Proposition 1.1.34. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $1 \le p, q \le \infty$. Let $f, g : X \to \mathbb{F}$ be measurable and suppose that $|||f|||_{L^{p,q}}, |||g|||_{L^{p,q}} < \infty$. Then the following hold.*

1. *$|||f|||_{L^{p,q}} \ge 0$ and $|||f|||_{L^{p,q}} = 0$ if and only if $f = 0$ a.e.*
2. *If $\alpha \in \mathbb{F}$ then $|||\alpha f|||_{L^{p,q}} = |\alpha| \, |||f|||_{L^{p,q}}$.*
3. *We have that $|||f + g|||_{L^{p,q}} \le 2^{1/p} \, |||f|||_{L^{p,q}} + 2^{1/p} \, |||f|||_{L^{p,q}}$.*
4. *$L^{p,q}(X;\mathbb{F})$ is a vector space.*

Proof. The first and second items are trivial. We know from Proposition 1.1.21 that

$$(f+g)^{\#}(t) \le f^{\#}(t/2) + g^{\#}(t/2). \tag{1.1.143}$$

If $p, q < \infty$, then Minkowski's inequality provides us with the bound

$$\begin{aligned} |||f+g|||_{L^{p,q}} &= \left(\int_0^\infty (t^{1/p}(f+g)^{\#}(t))^q \frac{dt}{t} \right)^{1/q} \le \left(\int_0^\infty (t^{1/p} f^{\#}(t/2))^q \frac{dt}{t} \right)^{1/q} \\ &+ \left(\int_0^\infty (t^{1/p} g^{\#}(t/2))^q \frac{dt}{t} \right)^{1/q} = 2^{1/p} \left(\int_0^\infty (t^{1/p} f^{\#}(t))^q \frac{dt}{t} \right)^{1/q} + 2^{1/p} \left(\int_0^\infty (t^{1/p} g^{\#}(t/2))^q \frac{dt}{t} \right)^{1/q} \\ &= 2^{1/p} \, |||f|||_{L^{p,q}} + 2^{1/p} \, |||g|||_{L^{p,q}}. \end{aligned} \tag{1.1.144}$$

A similar argument, which we leave as an exercise, shows that (1.1.144) also holds when either $p = \infty$ or $q = \infty$. This completes the proof of the third item. The fourth item then follows from the second and third. □

In general the bound in the third item cannot be improved, as we show in the following example.

Example 1.1.35. Let $1 \le p < \infty$ and define $f, g : (0,1) \to \mathbb{R}$ via $f(x) = x^{-1/p}$ and $g(x) = (1-x)^{-1/p}$. Note that for $t > 0$

$$f(x) > t \Leftrightarrow \frac{1}{t^p} > x \tag{1.1.145}$$

and so

$$d_f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 1/t^p & \text{if } 1 < t. \end{cases} \tag{1.1.146}$$

which together with Proposition 1.1.27 implies that

$$|||f|||_{L^{p,\infty}} = \sup_{t>0} t(d_f(t))^{1/p} = 1. \tag{1.1.147}$$

Similarly, $|||g|||_{L^{p,\infty}} = 1$.

Next write $h = f + g$ and note that

$$0 = h'(x) = -\frac{1}{p}x^{-1/p-1} + \frac{1}{p}(1-x)^{-1/p-1} \Leftrightarrow x = 1 - x \Leftrightarrow x = 1/2. \tag{1.1.148}$$

From this and the fact that h diverges at 0 and 1 we deduce that f achieves its minimum at $x = 1/2$ and

$$\min_{0<x<1} h(x) = h(1/2) = 2^{1/p} + 2^{1/p} = 2^{1+1/p}. \tag{1.1.149}$$

Thus,

$$d_{f+g}(t) = d_h(t) = 1 \text{ for } 0 < t < 2^{1+1/p}, \tag{1.1.150}$$

and in turn this and Proposition 1.1.27 imply that

$$|||f + g|||_{L^{p,\infty}} \geq \lim_{t \to 2^{1+1/p}} t1^{1/p} = 2^{1+1/p}. \tag{1.1.151}$$

From these calculations we then find that

$$\frac{|||f + g|||_{L^{p,\infty}}}{|||f|||_{L^{p,\infty}} + |||g|||_{L^{p,\infty}}} \geq 2^{1/p} > 1. \tag{1.1.152}$$

When combined with the bound from the previous proposition, we deduce that this bound is actually an equality.

△

We now give a name to the norm variant we have discovered.

Definition 1.1.36. *Let V be a vector space. A function $|||\cdot||| : V \times V \to [0, \infty)$ is called a quasinorm if*

1. *$|||v||| = 0$ if and only if $v = 0$,*
2. *$|||\alpha v||| = |\alpha| \, |||v|||$ for every $v \in V$ and $\alpha \in \mathbb{F}$,*
3. *There exists a constant $C \geq 1$ such that $|||v + w||| \leq C(|||v||| + |||w|||)$ for all $v, w \in V$.*

We say V is a quasinormed space if it is equipped with a quasinorm $|||\cdot|||$. Given a quasinorm we can define convergent and Cauchy sequences in the obvious way. With these notions in hand we can say that V is quasi-Banach if every Cauchy sequence is convergent.

Example 1.1.37. If $(X, \mathfrak{M}, \mu)$ is a measure space, then $L^{p,q}(X; \mathbb{F})$ is a quasinormed space. △

Quasinorms are almost as good as norms, but there are certain technical results that they don't provide. For example, the following version of the generalized triangle inequality holds in quasinormed spaces.

Proposition 1.1.38. *Let V be a vector space equipped with a quasinorm $|||\cdot|||$ with constant $C \geq 1$. If $v_1, \ldots, v_m \in V$ for $m \geq 2$, then*

$$|||v_1 + \cdots + v_m||| \leq C^{m-1} \sum_{i=1}^{m} |||v_i||| \,. \tag{1.1.153}$$

Proof. Exercise. Hint: induct on $m \geq 2$. □

Next we show an essential inclusion result for the Lorentz spaces: $L^{p,q}(X;\mathbb{F}) \subseteq L^{p,r}(X;\mathbb{F})$ when $q < r$.

Theorem 1.1.39. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$, and $1 \leq q < r \leq \infty$. If $f \in L^{p,q}(X;\mathbb{F})$ then $f \in L^{p,r}(X;\mathbb{F})$ and*

$$|||f|||_{L^{p,r}} \leq \left(\frac{q}{p}\right)^{1/q-1/r} |||f|||_{L^{p,q}}. \tag{1.1.154}$$

In particular, we have the subspace inclusions

$$L^{p,1}(X;\mathbb{F}) \subseteq L^{p,q}(X;\mathbb{F}) \subseteq L^{p,r}(X;\mathbb{F}) \subseteq L^{p,\infty}(X;\mathbb{F}). \tag{1.1.155}$$

Proof. We first consider the case $r = \infty$, in which case $q < \infty$ by assumption. Fix $t > 0$ and write

$$t^{1/p} = \left(\frac{q}{p}\int_0^t (s^{1/p})^q \frac{ds}{s}\right)^{1/q}. \tag{1.1.156}$$

Then since $f^\#$ is nonincreasing we can bound

$$t^{1/p} f^\#(t) = \left(\frac{q}{p}\int_0^t (s^{1/p} f^\#(t))^q \frac{ds}{s}\right)^{1/q} \leq \left(\frac{q}{p}\int_0^t (s^{1/p} f^\#(s))^q \frac{ds}{s}\right)^{1/q} \leq \left(\frac{q}{p}\right)^{1/q} |||f|||_{L^{p,q}}. \tag{1.1.157}$$

Since $t > 0$ was arbitrary, we deduce that $f \in L^{p,\infty}(X;\mathbb{F})$ and

$$|||f|||_{L^{p,\infty}} = \sup_{t>0} t^{1/p} f^\#(t) \leq \left(\frac{q}{p}\right)^{1/q} |||f|||_{L^{p,q}}. \tag{1.1.158}$$

Now suppose that $1 \leq q < r < \infty$. We compute

$$\begin{aligned}\int_0^\infty (t^{1/p} f^\#(t))^r \frac{dt}{t} = \int_0^\infty (t^{1/p} f^\#(t))^{r-q} (t^{1/p} f^\#(t))^q \frac{dt}{t} &\leq |||f|||_{L^{p,\infty}}^{r-q} \int_0^\infty (t^{1/p} f^\#(t))^q \frac{dt}{t} \\ &= |||f|||_{L^{p,\infty}}^{r-q} |||f|||_{L^{p,q}}^q,\end{aligned} \tag{1.1.159}$$

which then implies that

$$|||f|||_{L^{p,r}} \leq |||f|||_{L^{p,\infty}}^{1-q/r} |||f|||_{L^{p,q}}^{q/r} \leq \left(\frac{q}{p}\right)^{1/q-1/r} |||f|||_{L^{p,q}}. \tag{1.1.160}$$

□

Next we establish a version of Hölder's inequality. We begin with a version for functions taking values in a field $\mathbb{F}$.

Theorem 1.1.40 (Hölder's inequality for Lorentz spaces)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $1 \leq p, q \leq \infty$. Suppose that $f \in L^{p,q}(X;\mathbb{F})$ and $g \in L^{p',q'}(X;\mathbb{F})$, where*

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ and } \frac{1}{q} + \frac{1}{q'} = 1. \tag{1.1.161}$$

Then $fg \in L^1(X;\mathbb{F})$ and

$$\left|\int_X fg d\mu\right| \leq \int_X |f|\,|g|\, d\mu \leq |||f|||_{L^{p,q}} |||g|||_{L^{p',q'}}. \tag{1.1.162}$$

Proof. It clearly suffices to prove only the second bound in (1.1.162). To prove it we use Theorem 1.1.25 and the standard Hölder inequality to bound

$$\int_X |f|\,|g|\,d\mu \le \int_0^\infty f^{\#}(t)g^{\#}(t)dt = \int_0^\infty \frac{t^{1/p}}{t^{1/q}} f^{\#}(t) \frac{t^{1/p'}}{t^{1/q'}} g^{\#}(t)dt \le |||f|||_{L^{p,q}}\, |||g|||_{L^{p',q'}}\,. \tag{1.1.163}$$

This is the desired estimate.

□

A typical exercise in Lebesgue theory shows that the Lebesgue spaces satisfy a nice interpolation property: inclusion in L^{p_0} and L^{p_1} for $p_0 < p_1$ implies inclusion in L^p for $p_0 < p < p_1$. The same is true in Lorentz spaces, though the inclusion is actually a bit stronger than we might expect. We conclude our initial discussion of Lorentz spaces by exploring this now.

Theorem 1.1.41. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, $1 \le p_0 < p_1 \le \infty$, and $1 \le q_0, q_1 \le \infty$. If $f \in L^{p_0,q_0}(X;\mathbb{F}) \cap L^{p_1,q_1}(X;\mathbb{F})$, then $f \in L^{p,q}(X;\mathbb{F})$ for every $p_0 < p < p_1$ and $1 \le q \le \infty$, and*

$$|||f|||_{L^{p,q}} \le \left(\frac{p}{q}\right)^{1/q} \left(\frac{q_0}{p_0}\right)^{\theta/q_0} \left(\frac{q_1}{p_1}\right)^{(1-\theta)/q_1} \left(\frac{p_0}{p-p_0} + \frac{p_1}{p_1-p}\right)^{1/q} |||f|||^{\theta}_{L^{p_0,q_0}}\, |||f|||^{1-\theta}_{L^{p_1,q_1}} \tag{1.1.164}$$

where

$$\theta = \frac{1/p - 1/p_1}{1/p_0 - 1/p_1} \in (0,1). \tag{1.1.165}$$

Proof. We will prove the result when $q < \infty$ and leave the case $q = \infty$ as an exercise. The result is trivial if $f = 0$, so we may assume that $f \neq 0$.

Fix $T \in (0,\infty)$. Theorem 1.1.39 guarantees that $f \in L^{p_0,\infty}(X;\mathbb{F}) \cap L^{p_1,\infty}(X;\mathbb{F})$, so we have the estimate

$$f^{\#}(t) \le \min\{t^{-1/p_0}\, |||f|||_{L^{p_0,\infty}}\,, t^{-1/p_1}\, |||f|||_{L^{p_1,\infty}}\}. \tag{1.1.166}$$

We may then bound

$$\begin{aligned} |||f|||^q_{L^{p,q}} = \int_0^\infty (t^{1/p} f^{\#}(t))^q \frac{dt}{t} &= \int_0^T (t^{1/p} f^{\#}(t))^q \frac{dt}{t} + \int_T^\infty (t^{1/p} f^{\#}(t))^q \frac{dt}{t} \\ &\le |||f|||^q_{L^{p_1,\infty}} \int_0^T (t^{1/p-1/p_1})^q \frac{dt}{t} + |||f|||^q_{L^{p_0,\infty}} \int_T^\infty (t^{1/p-1/p_0})^q \frac{dt}{t} \\ &= \frac{|||f|||^q_{L^{p_1,\infty}}}{\alpha} T^{\alpha} + \frac{|||f|||^q_{L^{p_0,\infty}}}{\beta} T^{-\beta} \end{aligned} \tag{1.1.167}$$

for

$$\alpha = \frac{q}{p} - \frac{q}{p_1} > 0 \text{ and } \beta = \frac{q}{p_0} - \frac{q}{p} > 0. \tag{1.1.168}$$

If $a, b > 0$, then the map $T \mapsto a\alpha^{-1}T^{\alpha} + b\beta^{-1}T^{-\beta}$ achieves its global minimum at

$$T_{min} = \left(\frac{b}{a}\right)^{1/(\alpha+\beta)}, \tag{1.1.169}$$

and the minimal value is

$$\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) a^{\beta/(\alpha+\beta)} b^{\alpha/(\alpha+\beta)}. \tag{1.1.170}$$

Since $f \neq 0$ we may then minimize the right side of (1.1.41) as a function of T. Doing so, we arrive at the bound

$$\left\|\left| f \right\|\right|_{L^{p,q}} \le \left(\frac{p}{q}\right)^{1/q} \left(\frac{p_0}{p-p_0} + \frac{p_1}{p_1 - p}\right)^{1/q} \left\|\left| f \right\|\right|^{\theta}_{L^{p_0,\infty}} \left\|\left| f \right\|\right|^{1-\theta}_{L^{p_1,\infty}} . \tag{1.1.171}$$

Then (1.1.164) follows by chaining this bound together with the bound of Theorem 1.1.39. □

Finally, we turn to the issue of completeness of Lorentz spaces, establishing that they are indeed quasi-Banach.

Theorem 1.1.42. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $1 \le p, q \le \infty$. Then $L^{p,q}(X;\mathbb{F})$ is a quasi-Banach space.*

Proof. We will prove the result when $\mathbb{F} = \mathbb{R}$ and leave the details of how to extend this to $\mathbb{F} = \mathbb{C}$ as an exercise.

If $p = \infty$ and $q < \infty$ then $L^{p,q}(X) = L^{\infty,q}(X) = \{0\}$ and is thus trivially complete. If $p = q = \infty$ then $L^{p,q}(X) = L^{\infty}(X)$ and is thus complete due to the completeness of the Lebesgue spaces. It remains to handle the case $1 \le p < \infty$ and $1 \le q \le \infty$. We will prove the result under the extra assumption that $q < \infty$ and leave the case $q = \infty$ as an exercise.

Suppose that $\{f_n\}_{n=\ell}^{\infty} \subseteq L^{p,q}(X)$ is Cauchy. According to Proposition 1.1.27 and Theorem 1.1.39 we have that

$$\sup_{t>0} t(d_g(t))^{1/p} = \sup_{t>0} t^{1/p} g^{\#}(t) \le \left(\frac{q}{p}\right)^{1/q} \left\|\left| g \right\|\right|_{L^{p,q}} \tag{1.1.172}$$

for every $g \in L^{p,q}(X)$. Applying this to $g = f_n - f_m$ shows that $\{f_n\}_{n=\ell}^{\infty}$ is Cauchy in measure, and so we can find a measurable function $f : X \to \mathbb{R}$ and a subsequence $\{f_{n_k}\}_{k=\ell}^{\infty}$ such that $f_{n_k} \to f$ a.e. in X as $k \to \infty$.

For $\ell \le j \in \mathbb{Z}$ we have that

$$\left| f - f_{n_j} \right| = \lim_{k\to\infty} \left| f_{n_k} - f_{n_j} \right| \text{ a.e. in } X, \tag{1.1.173}$$

and so Proposition 1.1.23 implies that

$$(f - f_{n_j})^{\#} \le \liminf_{k\to\infty} (f_{n_k} - f_{n_j})^{\#} \text{ in } [0,\infty). \tag{1.1.174}$$

This and Fatou's lemma then allow us to bound

$$\begin{aligned} \left\|\left| f - f_{n_j} \right\|\right|^q_{L^{p,q}} &= \int_0^{\infty} (t^{1/p}(f - f_{n_j})^{\#}(t))^q \frac{dt}{t} \le \int_0^{\infty} \liminf_{k\to\infty} (t^{1/p}(f_{n_k} - f_{n_j})^{\#}(t))^q \frac{dt}{t} \\ &\le \liminf_{k\to\infty} \int_0^{\infty} (t^{1/p}(f_{n_k} - f_{n_j})^{\#}(t))^q \frac{dt}{t} = \liminf_{k\to\infty} \left\|\left| f_{n_k} - f_{n_j} \right\|\right|^q_{L^{p,q}} . \end{aligned} \tag{1.1.175}$$

Upon sending $j \to \infty$ and utilizing the fact that $\{f_n\}_{n=\ell}^{\infty}$ is Cauchy in $L^{p,q}(X)$, we deduce that $f_{n_k} - f \to 0$ in $L^{p,q}(X)$ as $k \to \infty$. In turn, since quasi-Cauchy sequences are bounded (the proof for Cauchy sequences works), this and Proposition 1.1.23 imply that

$$\begin{aligned} \int_0^T (t^{1/p} f^{\#}(t))^q \frac{dt}{t} \le \liminf_{j\to\infty} \int_0^T (t^{1/p} f^{\#}_{n_j}(t))^q \frac{dt}{t} \le \liminf_{j\to\infty} \int_0^{\infty} (t^{1/p} f^{\#}_{n_j}(t))^q \frac{dt}{t} \\ = \liminf_{j\to\infty} \left\|\left| f_{n_j} \right\|\right|^q_{L^{p,q}} < \infty \end{aligned} \tag{1.1.176}$$

for all $T > 0$, and hence that $f \in L^{p,q}(X)$. We leave it as an exercise to check that Cauchy sequences in quasinormed spaces with convergent subsequences are actually convergent. Thus $f_n \to f$ in $L^{p,q}(X)$ as $n \to \infty$, which proves that $L^{p,q}(X)$ is complete.

□

1.2 Interpolation theorems

Our goal in this section is to exploit the properties of the Lebesgue and Lorentz spaces in order to prove some interpolation results for operators between them. The point of this is that it is often easy to check that an operator (say, linear) is bounded between certain pairs of spaces. We will look for results that allow us to extend from these special pairs to other pairs that interpolate between. We will prove two main results: the Marcinkiewicz and Riesz-Thorin interpolation theorems.

1.2.1 The real method of Marcinkiewicz

We begin with some definitions, starting with the notion of truncation.

Definition 1.2.1. *Let $(X, \mathfrak{M}, \mu)$ be a measure space.*

1. *We write*
$$L^0(X;\mathbb{F}) = \{[f]_\simeq \mid f : X \to \mathbb{F} \text{ is measurable}\}. \tag{1.2.1}$$
We know from measure theory that $L^0(X;\mathbb{F})$ is a vector space over $\mathbb{F}$.

2. *Given $f \in L^0(X;\mathbb{F})$ and $t \in [0,\infty)$ we define the truncation of f to be $f_t : X \to \mathbb{F}$ given by*
$$f_t(x) = \begin{cases} f(x) & \text{if } |f(x)| \le t \\ 0 & \text{if } |f(x)| > t. \end{cases} \tag{1.2.2}$$
Clearly $f_t \in L^0(X;\mathbb{F})$.

3. *Suppose that $E \subseteq L^0(X;\mathbb{F})$. We say that E is closed under truncation if $f \in E$ implies that $f_t \in E$ for every $t \in [0,\infty)$.*

Example 1.2.2. Simple functions are closed under truncation, as are finite simple functions. △

Example 1.2.3. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then $L^{p,q}(X;\mathbb{F})$ is closed under truncation for every $1 \le p, q \le \infty$. The same is true of $L^{p_0,q_0}(X;\mathbb{F}) \cap L^{p_1,q_1}(X;\mathbb{F})$ and $L^{p_0,q_0}(X;\mathbb{F}) + L^{p_1,q_1}(X;\mathbb{F})$. To see this note that if f_r is the truncation of the measurable function $f : X \to \mathbb{F}$, then
$$d_{f_r}(t) = \begin{cases} d_f(t) - d_f(r) & \text{for } 0 \le t < r \\ 0 & \text{for } t \ge r \end{cases} \tag{1.2.3}$$
and so
$$f_r^\#(t) = \inf\{s \mid d_{f_r}(s) \le t\} = \inf\{0 \le s < r \mid d_f(s) - d_f(r) \le t\} = f^\#(d_f(r) + t). \tag{1.2.4}$$
In particular, this means that
$$f_r^\#(t) \le \min\{f^\#(t), r\} \tag{1.2.5}$$
and so $f_r \in L^{p_i,q_i}(X;\mathbb{F})$ whenever $f \in L^{p_i,q_i}(X;\mathbb{F})$.

△

We now prove one of our two main interpolation results.

Theorem 1.2.4 (Marcinkiewicz). *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be measure spaces and let $1 \leq p_i, r_i, s_i \leq \infty$ for $i = 0, 1$ be such that $p_0 < p_1$, $r_0 \neq r_1$, $s_0 = 1$, and*

$$s_1 = \begin{cases} 1 & \text{if } p_1 < \infty \\ \infty & \text{if } p_1 = \infty. \end{cases} \tag{1.2.6}$$

Let $U \subseteq L^0(X;\mathbb{F})$ be a subspace closed under truncation, and suppose that $T : U \to L^0(Y;\mathbb{F})$ is such that there exist $A, C_0, C_1 > 0$ such that

$$|T(f+g)| \leq A\left(|T(f)| + |T(g)|\right) \nu - a.e. \text{ in } Y \tag{1.2.7}$$

for every $f, g \in U$, and

$$|||T(f)|||_{L^{r_i,\infty}(Y;\mathbb{F})} \leq C_i \, |||f|||_{L^{p_i,s_i}(X;\mathbb{F})} \tag{1.2.8}$$

for every $f \in U \cap L^{p_i,s_i}(X;\mathbb{F})$ and $i = 0, 1$. Then for every $\theta \in (0,1)$ there exists a constant $C = C(\theta, p_i, r_i, s_i, A, C_i) > 0$ such that

$$|||T(f)|||_{L^{r,q}(Y;\mathbb{F})} \leq C \, |||f|||_{L^{p,q}(X;\mathbb{F})} \tag{1.2.9}$$

for every $f \in U \cap L^{p,q}(X;\mathbb{F})$, where $1 \leq q \leq \infty$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}. \tag{1.2.10}$$

Proof. Note that $p_0 < p < p_1$. We will prove the result under the assumption that $1 \leq p_1, q < \infty$ and leave the remaining cases as an exercise. Throughout the proof we will employ the convention of writing $C > 0$ for a generic constant depending on θ, p_0, p_1, r_0, r_1, q, A, C_0, and C_1 that may change from line to line. Let $f \in U \cap L^{p,q}(X;\mathbb{F})$. We divide the rest of the proof into several steps.

Step 1 – Set up

The relations (1.2.10) and some simple algebra show that

$$\frac{1/r_0 - 1/r}{1/r_0 - 1/r_1} = \theta = \frac{1/p_0 - 1/p}{1/p_0 - 1/p_1} \text{ and } \frac{1/r - 1/r_1}{1/r_0 - 1/r_1} = 1 - \theta = \frac{1/p - 1/p_1}{1/p_0 - 1/p_1} \tag{1.2.11}$$

and hence we may define

$$\gamma := \frac{1/r_0 - 1/r}{1/p_0 - 1/p} = \frac{1/r_0 - 1/r_1}{1/p_0 - 1/p_1} = \frac{1/r - 1/r_1}{1/p - 1/p_1} > 0. \tag{1.2.12}$$

For $\tau > 0$ define $f_\tau, g_\tau : X \to \mathbb{F}$ via

$$f_\tau(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq f^{\#}(\tau^\gamma) \\ 0 & \text{otherwise} \end{cases} \tag{1.2.13}$$

and

$$g_\tau(x) = \begin{cases} f(x) & \text{if } |f(x)| > f^{\#}(\tau^\gamma) \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.14}$$

Since U is closed under truncation we have that $f_\tau \in U$, and since $g_\tau = f - f_\tau$ and U is a vector space we have that $g_\tau \in U$. By construction we have that $|f_\tau|, |g_\tau| \leq |f|$, and so $f_\tau^{\#}, g_\tau^{\#} \leq f^{\#}$. Additionally, a simple computation, which we leave as an exercise, reveals that

$$f_\tau^{\#}(t) \leq \begin{cases} f^{\#}(\tau^\gamma) & \text{if } 0 < t < \tau^\gamma \\ f^{\#}(t) & \text{if } \tau^\gamma \leq t \end{cases} \tag{1.2.15}$$

and

$$g_\tau^{\#}(t) \le \begin{cases} f^{\#}(t) & \text{if } 0 < t < \tau^\gamma \\ 0 & \text{if } \tau^\gamma \le t. \end{cases} \tag{1.2.16}$$

Step 2 – Estimates for f_τ and g_τ

Hölder's inequality and the bounds $1/p_1 < 1/p < 1/p_0$ imply that

$$\int_0^\infty t^{1/p_0} g_\tau^{\#}(t)\frac{dt}{t} \le \int_0^{\tau^\gamma} t^{1/p_0} f^{\#}(t)\frac{dt}{t} \le \left(\int_0^{\tau^\gamma} t^{q'/p_0 - q'/p}\frac{dt}{t}\right)^{1/q'} \left(\int_0^{\tau^\gamma} (t^{1/p} f^{\#}(t))^q \frac{dt}{t}\right)^{1/q}$$
$$\le C\tau^{\gamma(1/p_0-1/p)} \left(\int_0^{\tau^\gamma} (t^{1/p} f^{\#}(t))^q \frac{dt}{t}\right)^{1/q} < \infty \tag{1.2.17}$$

and

$$\int_0^\infty t^{1/p_1} f_\tau^{\#}(t)\frac{dt}{t} \le f^{\#}(\tau^\gamma)\int_0^{\tau^\gamma} t^{1/p_1}\frac{dt}{t} + \int_{\tau^\gamma}^\infty t^{1/p_1} f^{\#}(t)\frac{dt}{t}$$
$$\le p_1 f^{\#}(\tau^\gamma)\tau^{\gamma/p_1} + \left(\int_{\tau^\gamma}^\infty t^{q'/p_1 - q'/p}\frac{dt}{t}\right)^{1/q'} \left(\int_{\tau^\gamma}^\infty (t^{1/p} f^{\#}(t))^q \frac{dt}{t}\right)^{1/q}$$
$$\le C f^{\#}(\tau^\gamma)\tau^{\gamma/p_1} + C\tau^{\gamma(1/p_1-1/p)} \left(\int_{\tau^\gamma}^\infty (t^{1/p} f^{\#}(t))^q \frac{dt}{t}\right)^{1/q} < \infty \tag{1.2.18}$$

from which we deduce that

$$g_\tau \in L^{p_0,1}(X;\mathbb{F}) \text{ and } f_\tau \in L^{p_1,1}(X;\mathbb{F}) \tag{1.2.19}$$

for every $\tau > 0$.

We know from (1.2.8) that

$$|||T(h)|||_{L^{r_i,\infty}(Y;\mathbb{F})} \le C\,|||h|||_{L^{p_i,1}(X;\mathbb{F})} \text{ for } i = 0, 1 \tag{1.2.20}$$

whenever $h \in U \cap L^{p_i,1}(X;\mathbb{F})$. Applying this to f_τ and g_τ shows that

$$|||T(g_\tau)|||_{L^{r_0,\infty}(Y;\mathbb{F})} \le C\,|||g_\tau|||_{L^{p_0,1}(X;\mathbb{F})} \text{ and } |||T(f_\tau)|||_{L^{r_1,\infty}(Y;\mathbb{F})} \le C\,|||f_\tau|||_{L^{p_1,1}(X;\mathbb{F})}, \tag{1.2.21}$$

which, when combined with the estimates (1.2.17) and (1.2.18), implies that for each $\tau > 0$ we have the bounds

$$[T(g_\tau)]^{\#}(\tau) \le C\tau^{-1/r_0}\int_0^\infty t^{1/p_0} g_\tau^{\#}(t)\frac{dt}{t} \le C\tau^{-1/r_0}\int_0^{\tau^\gamma} t^{1/p_0} f^{\#}(t)\frac{dt}{t} \tag{1.2.22}$$

and

$$[T(f_\tau)]^{\#}(\tau) \le C\tau^{-1/r_1}\int_0^\infty t^{1/p_1} f_\tau^{\#}(t)\frac{dt}{t} \le C f^{\#}(\tau^\gamma)\tau^{\gamma/p_1 - 1/r_1} + C\tau^{-1/r_1}\int_{\tau^\gamma}^\infty t^{1/p_1} f^{\#}(t)\frac{dt}{t}. \tag{1.2.23}$$

Step 3 – Estimates from (1.2.7)

From the bound (1.2.7) we deduce that

$$|T(f)| = |T(f_\tau + g_\tau)| \le A\,|T(f_\tau)| + A\,|T(g_\tau)| \quad \nu - \text{a.e. in } Y, \tag{1.2.24}$$

and so Proposition 1.1.21 and Example 1.1.20 tell us that

$$[T(f)]^{\#}(t) \leq A[|T(f_\tau)| + |T(g_\tau)|]^{\#}(t) \leq A[|T(f_\tau)|]^{\#}(t/2) + A[|T(g_\tau)|]^{\#}(t/2)$$
$$= A[T(f_\tau)]^{\#}(t/2) + A[T(g_\tau)]^{\#}(t/2) \quad (1.2.25)$$

for all $t > 0$, and in particular for $t = 2\tau$. Consequently, a change of variable and Minkowski's inequality imply that

$$\frac{1}{2^{1/r}} \|\!|T(f)|\!\|_{L^{r,q}(Y;\mathbb{F})} = \left(\int_0^\infty (\tau^{1/r}[T(f)]^{\#}(2\tau))^q \frac{d\tau}{\tau}\right)^{1/q}$$
$$\leq C\left(\int_0^\infty (\tau^{1/r}[[T(f_\tau)]^{\#}(\tau) + [T(g_\tau)]^{\#}(\tau)])^q \frac{d\tau}{\tau}\right)^{1/q}$$
$$\leq C\left(\int_0^\infty (\tau^{1/r}[T(f_\tau)]^{\#}(\tau))^q \frac{d\tau}{\tau}\right)^{1/q} + C\left(\int_0^\infty (\tau^{1/r}[T(g_\tau)]^{\#}(\tau))^q \frac{d\tau}{\tau}\right)^{1/q}. \quad (1.2.26)$$

Step 4 – Synthesis and conclusion

Multiplying (1.2.22) and (1.2.23) by $\tau^{1/r}$ and employing the algebraic relations in (1.2.12) shows that

$$\tau^{1/r}(T(g_\tau))^{\#}(\tau) \leq C\tau^{1/r-1/r_0}\int_0^{\tau^\gamma} t^{1/p_0} f^{\#}(t)\frac{dt}{t} = C\tau^{\gamma(1/p-1/p_0)}\int_0^{\tau^\gamma} t^{1/p_0} f^{\#}(t)\frac{dt}{t} \quad (1.2.27)$$

and

$$\tau^{1/r}[T(f_\tau)]^{\#}(\tau) \leq Cf^{\#}(\tau^\gamma)\tau^{\gamma/p_1+1/r-1/r_1} + C\tau^{1/r-1/r_1}\int_{\tau^\gamma}^\infty t^{1/p_1} f^{\#}(t)\frac{dt}{t}$$
$$\leq Cf^{\#}(\tau^\gamma)\tau^{\gamma/p} + C\tau^{\gamma(1/p-1/p_1)}\int_{\tau^\gamma}^\infty t^{1/p_1} f^{\#}(t)\frac{dt}{t}. \quad (1.2.28)$$

From (1.2.27), the change of variable $z = \tau^\gamma$ (which implies $d\tau/\tau = \gamma^{-1}dz/z$), and Hardy's inequality we may then bound

$$\left(\int_0^\infty (\tau^{1/r}[T(g_\tau)]^{\#}(\tau))^q \frac{d\tau}{\tau}\right)^{1/q} \leq C\left(\int_0^\infty \tau^{\gamma q(1/p-1/p_0)}\left(\int_0^{\tau^\gamma} t^{1/p_0} f^{\#}(t)\frac{dt}{t}\right)^q \frac{d\tau}{\tau}\right)^{1/q}$$
$$= C\left(\int_0^\infty z^{q(1/p-1/p_0)}\left(\int_0^z t^{1/p_0} f^{\#}(t)\frac{dt}{t}\right)^q \frac{dz}{z}\right)^{1/q} \leq C\left(\int_0^\infty t^{q-q(1/p_0-1/p)-1}[t^{1/p_0} f^{\#}(t)]^q \frac{dt}{t^q}\right)^{1/q}$$
$$= C\left(\int_0^\infty [t^{1/p} f^{\#}(t)]^q \frac{dt}{t}\right)^{1/q} = C\,\|\!|f|\!\|_{L^{p,q}(X;V)}. \quad (1.2.29)$$

Similarly, (1.2.28) allows us to bound

$$\left(\int_0^\infty (\tau^{1/r}(T(f_\tau))^{\#}(\tau))^q \frac{d\tau}{\tau}\right)^{1/q} \leq C\left(\int_0^\infty \tau^{\gamma q/p}[f^{\#}(\tau^\gamma)]^q \frac{d\tau}{\tau}\right)^q$$
$$+ C\left(\int_0^\infty \tau^{\gamma q(1/p-1/p_1)}\left(\int_{\tau^\gamma}^\infty t^{1/p_1} f^{\#}(t)\frac{dt}{t}\right)^q \frac{d\tau}{\tau}\right)^q$$
$$= C\left(\int_0^\infty [z^{1/p} f^{\#}(z)]^q \frac{dz}{z}\right)^q + C\left(\int_0^\infty z^{q(1/p-1/p_1)}\left(\int_z^\infty t^{1/p_1} f^{\#}(t)\frac{dt}{t}\right)^q \frac{dz}{z}\right)^q$$
$$\leq C\left(\int_0^\infty [t^{1/p} f^{\#}(t)]^q \frac{dt}{t}\right)^q = C\,\|\!|f|\!\|_{L^{p,q}(X;\mathbb{F})}. \quad (1.2.30)$$

We now chain together the bounds (1.2.26), (1.2.29), and (1.2.30) to deduce that

$$|||T(f)|||_{L^{r,q}(Y;\mathbb{F})} \leq C\,|||f|||_{L^{p,q}(X;\mathbb{F})}\,. \tag{1.2.31}$$

This is (1.2.9). □

Some remarks are in order.

Remark 1.2.5. *With a bit of elementary but tedious work, the exact form of the constant $C = C(\theta, p_i, r_i, s_i, A, C_i) > 0$ that appears in the final estimate can be tracked in the proof of the Marcinkiewicz theorem. If we were to do so we would find that the constant blows up as $\theta \to 0$ and $\theta \to 1$. This is, of course, not surprising: we should not expect to be able to improve our assumed estimates for free as a byproduct of the theorem.*

Remark 1.2.6. *In the Marcinkiewicz theorem we do not assume that the map T is linear, and in particular we make no assumptions about how T behaves relative to scalar multiplication. There is then no good reason for us to require the functions on X and on Y take values in the same field. In fact, an examination of the proof shows that the exact same results hold if we replace $L^0(X;\mathbb{F})$ and $L^0(Y;\mathbb{F})$ with $L^0(X;\mathbb{F}_1)$ and $L^0(Y;\mathbb{F}_2)$. In other words, the fields we use as the target spaces for our functions can be different. This is often useful in practical applications of the theorem.*

Remark 1.2.7. *Probably the most common use of Theorem 1.2.4 occurs in showing that the map T is bounded between Lebesgue spaces. In order to get a Lebesgue estimate from the theorem we need $p \leq r$ and $q = r$:*

$$\|T(f)\|_{L^r(Y;\mathbb{F})} = |||T(f)|||_{L^{r,r}(Y;\mathbb{F})} \leq C\,|||f|||_{L^{p,r}(X;\mathbb{F})} \leq C'\,|||f|||_{L^{p,p}(X;\mathbb{F})} = C'\,\|f\|_{L^p(X;\mathbb{F})} \tag{1.2.32}$$

for every $f \in U \cap L^p(X;V)$, where C and C' are some constants depending on the parameters. In order to guarantee that $p \leq r$ we typically assume that $p_i \leq r_i$ for $i = 0, 1$, which then implies that $p \leq r$. This places a fairly serious restriction on the range of validity of the Marcinkiewicz theorem, though there are many uses of the result anyway.

In the event that $L^{p,q}(X;\mathbb{F}) \subseteq U$ in the Marcinkiewicz theorem, we immediately find that we can view $T : L^{p,q}(X;\mathbb{F}) \to L^{r,q}(Y;\mathbb{F})$ as a bounded map in the sense that $|||T(f)|||_{L^{r,q}} \leq C\,\|f\|_{L^{p,q}}$. However, in practice we often want to utilize a space U that is much smaller than $L^{p,q}(X;\mathbb{F})$ but possibly dense. In this case we can actually use Marcinkiewicz to extend T to $L^{p,q}(X;\mathbb{F})$, provided that T satisfies a slightly more restrictive condition than stated in the theorem. We explore this in the next example. We will return to more practical applications of the theorem soon.

Example 1.2.8. Assume the hypotheses of Theorem 1.2.4. Suppose that $T : U \to L^0(Y;\mathbb{F})$ is real-valued, i.e. $T(f)$ is real a.e. in Y for each $f \in U$. Further suppose that T is a sublinear operator, i.e. for $f, g \in U$ we have that

$$T(f+g) \leq T(f) + T(g) \text{ and } |T(-f)| = |T(f)| \text{ a.e. in } Y. \tag{1.2.33}$$

Consequently, for $f, g \in U$ we can estimate

$$T(f) \leq T(f-g) + T(g) \Rightarrow T(f) - T(g) \leq T(f-g) \leq |T(f-g)| \text{ a.e. in } Y, \tag{1.2.34}$$

and so upon reversing the roles of f and g we find that

$$T(g) - T(f) \leq |T(g-f)| \Rightarrow -|T(f-g)| = -|T(g-f)| \leq T(f) - T(g) \text{ a.e. in } Y. \tag{1.2.35}$$

Combining these, we deduce that

$$|T(f) - T(g)| \leq |T(f-g)| \text{ a.e. in } Y, \tag{1.2.36}$$

which implies that

$$|||T(f) - T(g)|||_{L^{r,s}(Y)} \leq |||T(f-g)|||_{L^{r,s}(Y)}. \tag{1.2.37}$$

Combining this estimate with the one provided by the Marcinkiewicz interpolation theorem then shows that

$$|||T(f) - T(g)|||_{L^{r,q}(Y)} \leq |||T(f-g)|||_{L^{r,q}(Y)} \leq C\,|||f-g|||_{L^{p,q}(X)}. \tag{1.2.38}$$

Now, if we know that U is dense in $L^{p,q}(X;V)$, then we may use the fact that $L^{r,s}(Y)$ is quasi-Banach (proved in Theorem 1.1.42) to uniquely extend T to a sublinear operator $T : L^{p,q}(X;\mathbb{F}) \to L^{r,q}(Y;\mathbb{F})$ satisfying

$$|||T(f)|||_{L^{r,q}} \leq C\,|||f|||_{L^{p,q}}. \tag{1.2.39}$$

△

1.2.2 The complex method of Riesz-Thorin

We now aim for the proof of the second interpolation result. It will be tied to the structure of the complex numbers. We begin by recalling the maximum modulus principle for holomorphic functions.

Theorem 1.2.9 (Maximum modulus principle). *Suppose that $\varnothing \neq U \subseteq \mathbb{C}$ is open and connected and that $f : U \to \mathbb{C}$ is holomorphic in U. If there exists $z \in U$ such that $|f(z)| = \max_U |f|$, then f is constant in U. In particular, if $\bar{U}$ is compact and f extends continuously to $\bar{U}$ then*

$$\max_{\bar{U}} |f| = \max_{\partial U} |f|. \tag{1.2.40}$$

Now we prove a very interesting estimate known as Hadamard's three lines lemma. It will be the workhorse for our interpolation result.

Lemma 1.2.10 (Hadamard's three lines lemma). *Let $R = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$, and suppose that $f \in C^0_b(R;\mathbb{C})$ is holomorphic in R°. Further suppose that $M_0, M_1 > 0$ are such that*

$$|f(0+iy)| \leq M_0 \text{ and } |f(1+iy)| \leq M_1 \text{ for all } y \in \mathbb{R}. \tag{1.2.41}$$

Then for every $x \in [0,1]$ we have that

$$|f(x+iy)| \leq M_0^{1-x} M_1^x. \tag{1.2.42}$$

Proof. Note first that for $z = x + iy \in R$ we have that

$$\left|M_0^{1-z} M_1^z\right| = M_0^{1-x} M_1^x \geq \min\{M_0, M_1\} > 0. \tag{1.2.43}$$

This allows us to define the functions $g, g_n \in C^0_b(R;\mathbb{C})$ (here $1 \leq n \in \mathbb{N}$) via

$$g(z) = \frac{f(z)}{M_0^{1-z} M_1^z} \text{ and } g_n(z) = g(z) e^{(z^2-1)/n}. \tag{1.2.44}$$

The boundedness of g follows from the boundedness of f and (1.2.43), while the boundedness of g_n follows since for $z = x + iy$ we have that

$$|g_n(z)| = \frac{|f(z)|}{M_0^{1-x} M_1^x} e^{(x^2-y^2-1)/n} \le \frac{\|f\|_{C_b^0(R)}}{\min\{M_0, M_1\}} e^{-y^2/n}. \tag{1.2.45}$$

This estimate also tells us that

$$|g_n(z)| \le 1 \text{ for } z \in \partial R \tag{1.2.46}$$

and that for each $n \ge 1$ there exists $r_n > 0$ such that

$$|\text{Im}(z)| \ge r_n \Rightarrow |g_n(z)| \le 1. \tag{1.2.47}$$

Clearly g and g_n for $n \ge 1$ are holomorphic in R°. Fix $n \ge 1$. For each $m \ge 1$ define the rectangle $R_m = \{z \in R^\circ \mid |\text{Im}(z)| < r_n + m\}$. According to the estimates (1.2.46) and (1.2.47) we have that for $n \in \mathbb{N}$

$$\max_{\partial R_m} |g_n| \le 1, \tag{1.2.48}$$

and so the modulus principle guarantees that

$$\max_{R_m} |g_n| \le 1. \tag{1.2.49}$$

Sending $m \to \infty$ then shows that

$$\sup_R |g_n| \le 1. \tag{1.2.50}$$

Finally, since $g_n(z) \to g(z)$ as $n \to \infty$ for each $z \in R$, we deduce that $|g(z)| \le 1$ on R, and hence that (1.2.42) holds.

□

We now have all the tools needed to prove the second interpolation result, the Riesz-Thorin interpolation theorem.

Theorem 1.2.11 (Riesz-Thorin)**.** *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be measure spaces and suppose that $1 \le p_0, p_1, q_0, q_1 \le \infty$. If $q_0 = q_1 = \infty$ then also suppose that Y is $\sigma-$finite. Write $U = S(X;\mathbb{C}) \cap L^{p_0}(X;\mathbb{C}) \cap L^{p_1}(X;\mathbb{C})$ and suppose that $T : U \to L^0(Y;\mathbb{C})$ is a linear map and that there exist $M_0, M_1 > 0$ such that*

$$\|Tf\|_{L^{q_0}(Y;\mathbb{C})} \le M_0 \|f\|_{L^{p_0}(X;\mathbb{C})} \text{ and } \|Tf\|_{L^{q_1}(Y;\mathbb{C})} \le M_1 \|f\|_{L^{p_1}(X;\mathbb{C})} \tag{1.2.51}$$

for all $f \in U$. Let $\theta \in [0,1]$ and $1 \le p, q \le \infty$ be given by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{1.2.52}$$

Then

$$\|Tf\|_{L^q(Y;\mathbb{C})} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(X;\mathbb{C})} \text{ for all } f \in U. \tag{1.2.53}$$

In particular, the map T extends to a bounded linear map $T : L^p(X;\mathbb{C}) \to L^q(Y;\mathbb{C})$, satisfying (1.2.53).

Proof. First note that once (1.2.53) is established, the extension of T to a bounded linear operator $T : L^p(X;\mathbb{C}) \to L^q(Y;\mathbb{C})$ follows immediately from the density of U in $L^p(X;\mathbb{C})$. Thus it suffices to prove (1.2.53). Throughout the proof we adopt the convention that if $c = d = \infty$, then $c/d = 1$.

Let $R = \{z \in \mathbb{C} \mid \operatorname{Re}(z) \in [0,1]\} \subset \mathbb{C}$. Define the functions $P, Q : R \to \mathbb{C}$ via

$$P(z) = (1-z)p/p_0 + zp/p_1 \tag{1.2.54}$$

$$Q(z) = (1-z)q'/q_0' + zq'/q_1'. \tag{1.2.55}$$

By construction we have that

$$\begin{aligned} 0 \le \min\{p/p_0, p/p_1\} \le \operatorname{Re}(P(z)) \le \max\{p/p_0, p/p_1\} < \infty \\ 0 \le \min\{q'/q_0', q'/q_1'\} \le \operatorname{Re}(Q(z)) \le \max\{q'/q_0', q'/q_1'\} < \infty. \end{aligned} \tag{1.2.56}$$

Write $f \in U$ and $g \in S_{fin}(Y;\mathbb{C})$ as

$$f = \sum_{j=1}^{J} r_j v_j \chi_{E_j} \text{ and } g = \sum_{k=1}^{K} s_k w_k \chi_{F_k} \tag{1.2.57}$$

for $\{E_j\}_{j=1}^J \subseteq \mathfrak{M}$ and $\{F_k\}_{k=1}^K \subseteq \mathfrak{N}$ pairwise disjoint and $r_j, s_k \in (0,\infty)$, and $|v_j| = |w_k| = 1$ for $1 \le j \le J$ and $1 \le k \le K$. Since g is a finite simple function we know that $\nu(F_k) < \infty$ for $1 \le k \le K$.

Note that for each $j = 1, \dots, J$ we have that $v_j \chi_{E_j} \in U$, so

$$T(v_j \chi_{E_j}) \in L^{q_0}(Y;\mathbb{C}) \cap L^{q_1}(Y;\mathbb{C}) \subseteq L^q(Y;\mathbb{C}) \tag{1.2.58}$$

and hence that $Tf \in L^q(Y;\mathbb{C})$. Moreover, since $\nu(F_k) < \infty$, Hölder's inequality implies that $T(v_j \chi_{E_j}) \in L^1(F_k;\mathbb{C})$, and hence

$$\int_{F_k} T(v_j \chi_{E_j}) d\nu \in \mathbb{C} \text{ for } 1 \le j \le J, 1 \le k \le K. \tag{1.2.59}$$

For $z \in R$ define

$$f_z = \sum_{j=1}^{J} r_j^{P(z)} v_j \chi_{E_j} \in S(X;\mathbb{C}) \text{ and } g_z = \sum_{k=1}^{K} s_k^{Q(z)} w_k \chi_{F_k} \in S_{fin}(Y;\mathbb{C}). \tag{1.2.60}$$

We then define the function $F \in C_b^0(R;\mathbb{C})$ via

$$F(z) = \int_Y g_z T f_z d\nu = \sum_{j=1}^{J} \sum_{k=1}^{K} r_j^{P(z)} s_k^{Q(z)} w_k \left(\int_{F_k} T(v_j \chi_{E_j}) d\nu \right) \in \mathbb{C}. \tag{1.2.61}$$

Note that the boundedness of F on R follows from the fact that $\operatorname{Re}(P(z))$ and $\operatorname{Re}(Q(z))$ satisfy (1.2.56). Furthermore, for each j, k the maps

$$R^\circ \ni z \mapsto r_j^{P(z)} \in \mathbb{C} \text{ and } R^\circ \ni z \mapsto s_k^{Q(z)} \in \mathbb{C} \tag{1.2.62}$$

are holomorphic, and thus F is holomorphic on R°.

Suppose that $z \in R$ is such that $\mathrm{Re}(z) = 0$. In this case, if $p_0, q_0' < \infty$ then

$$\left| r_j^{p_0 P(z)} \right| = r_j^p \text{ and } \left| s_k^{q_0' Q(z)} \right| = s_k^{q'}, \tag{1.2.63}$$

which implies that

$$\|f_z\|_{L^{p_0}(X;\mathbb{C})} = \|f\|_{L^p(X;\mathbb{C})}^{p/p_0} \text{ and } \|g_z\|_{L^{q_0'}(Y;\mathbb{C})} = \|g\|_{L^{q'}(Y;\mathbb{C})}^{q'/q_0'}. \tag{1.2.64}$$

We leave it as an exercise to verify that (1.2.64) also holds if either $p_0 = \infty$ or $q_0' = \infty$.

Now suppose that $z \in R$ is such that $\mathrm{Re}(z) = 1$. In this case, if $p_1, q_1' < \infty$ then

$$\left| r_j^{p_1 P(z)} \right| = r_j^p \text{ and } \left| s_k^{q_1' Q(z)} \right| = s_k^{q'}, \tag{1.2.65}$$

and so

$$\|f_z\|_{L^{p_1}(X;\mathbb{C})} = \|f\|_{L^p(X;\mathbb{C})}^{p/p_1} \text{ and } \|g_z\|_{L^{q_1'}(Y;\mathbb{C})} = \|g\|_{L^{q'}(Y;\mathbb{C})}^{q'/q_1'}. \tag{1.2.66}$$

We again leave it as an exercise to verify that (1.2.66) also holds if either $p_1 = \infty$ or $q_1' = \infty$.

Now, if $z \in R$ is such that $\mathrm{Re}(z) = 0$ then (1.2.51), (1.2.64), and Hölder's inequality allow us to estimate

$$\begin{aligned} |F(z)| \le \|Tf_z\|_{L^{q_0}(Y;\mathbb{C})} \|g_z\|_{L^{q_0'}(Y;\mathbb{C})} &\le M_0 \|f_z\|_{L^{p_0}(X;V)} \|g_z\|_{L^{q_0'}(Y;\mathbb{C})} \\ &= M_0 \|f\|_{L^p(X;\mathbb{C})}^{p/p_0} \|g\|_{L^{q'}(Y;\mathbb{C})}^{q'/q_0'}. \end{aligned} \tag{1.2.67}$$

Similarly, if $z \in R$ is such that $\mathrm{Re}(z) = 1$ then (1.2.51), (1.2.66) and Hölder's inequality imply that

$$\begin{aligned} |F(z)| \le \|Tf_z\|_{L^{q_1}(Y;\mathbb{C})} \|g_z\|_{L^{q_1'}(Y;\mathbb{C})} &\le M_1 \|f_z\|_{L^{p_1}(X;\mathbb{C})} \|g_z\|_{L^{q_1'}(Y;\mathbb{C})} \\ &= M_1 \|f\|_{L^p(X;V)}^{p/p_1} \|g\|_{L^{q'}(Y;\mathbb{C})}^{q'/q_1'}. \end{aligned} \tag{1.2.68}$$

Lemma 1.2.10 and (1.2.52) then imply that

$$\begin{aligned} |F(\theta)| \le \left(M_0 \|f\|_{L^p(X;\mathbb{C})}^{p/p_0} \|g\|_{L^{q'}(Y;\mathbb{C})}^{q'/q_0'} \right)^{1-\theta} &\left(M_1 \|f\|_{L^p(X;\mathbb{C})}^{p/p_1} \|g\|_{L^{q'}(Y;\mathbb{C})}^{q'/q_1'} \right)^{\theta} \\ &= M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(X;\mathbb{C})} \|g\|_{L^{q'}(Y;\mathbb{C})}. \end{aligned} \tag{1.2.69}$$

Since $P(\theta) = Q(\theta) = 1$ by construction, we then have that $f_\theta = f$, $g_\theta = g$, and

$$\left| \int_Y gTf d\nu \right| = |F(\theta)| \le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(X;\mathbb{C})} \|g\|_{L^{q'}(Y;\mathbb{C})}. \tag{1.2.70}$$

Since we know from above that $Tf \in L^q(Y;\mathbb{C})$, we may then employ a standard result in Lebesgue theory to bound

$$\begin{aligned} \|Tf\|_{L^q(Y;\mathbb{C})} = \sup\{ \left| \int_Y gTf d\nu \right| \mid g \in S_{fin}(Y;\mathbb{C}) &\text{ and } \|g\|_{L^{q'}(Y;\mathbb{C})} \le 1\} \\ &\le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(X;\mathbb{C})}, \end{aligned} \tag{1.2.71}$$

which is (1.2.53).

□

1.2.3 Comparing and contrasting the two interpolation theorems

It is worth pointing out the differences between the Marcinkiewicz and Riesz-Thorin theorems. There is a trade-off between the two results. Marcinkiewicz let's us have a more general type of operator (not necessarily linear, but satisfying (1.2.7)) and works in more general spaces (the Lorentz scale) with weak estimates (bounds in $L^{r_i,\infty}$), but we pay a constant that depends on the p, q, r, s parameters and blows up as we approach the endpoints. For Lebesgue spaces it is also subject to the restrictions discussed in Remark 1.2.7. Riesz-Thorin requires a more restrictive operator (linear), and works only for a stricter class of spaces (stronger estimates required) over the complex field, but it gives better interpolated bounds (the constants don't blow up), and it is not subject to the restrictions of Remark 1.2.7. Note that neither theorem is stronger than the other: they are simply different.

1.3 Applications

Our goal now is to demonstrate various uses of the Marcinkiewicz and Riesz-Thorin theorems.

1.3.1 The Hardy-Littlewood maximal function

Let $f : \mathbb{R}^n \to \mathbb{F}$ be locally integrable. Then the Hardy-Littlewood maximal function $\mathcal{M}(f) : \mathbb{R}^n \to [0, \infty]$ is defined via

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{\omega_n r^n} \int_{B(x,r)} |f| \,. \tag{1.3.1}$$

Note that for a fixed $x \in \mathbb{R}^n$ the continuity of the map $\mathbb{R}_+ \ni r \mapsto \frac{1}{\omega_n r^n} \int_{B(x,r)} |f| \in \mathbb{R}$ shows that the supremum over $r > 0$ can be replaced by the supremum over $r \in \mathbb{Q} \cap \mathbb{R}_+$. The latter set is countable, so $\mathcal{M}(f)$ is measurable. It is then a trivial matter to verify that the Hardy-Littlewood maximal function, viewed as a map $\mathcal{M} : L^1_{loc}(\mathbb{R}^n; \mathbb{F}) \to L^0(\mathbb{R}^n; \mathbb{F})$, is sublinear (see Example 1.2.8).

Let $f \neq 0$ be locally integrable. We may trivially bound

$$\|\mathcal{M}(f)\|_{L^\infty} \leq \|f\|_{L^\infty} \,. \tag{1.3.2}$$

On the other hand, since $f \neq 0$ we can pick $\delta > 0$ and $0 < R < \infty$ such that

$$\int_{B(0,R)} |f| \geq \delta. \tag{1.3.3}$$

Then for $x \in \mathbb{R}^n$ with $|x| > R$ have that $B(0, R) \subseteq B(x, R + |x|)$ and $1/(2\,|x|) \leq 1/(|x| + R)$, so we can bound

$$\mathcal{M}(f)(x) \geq \frac{1}{\omega_n(|x| + R)^n} \int_{B(x,|x|+R)} |f| \geq \frac{1}{\omega_n(|x| + R)^n} \int_{B(0,R)} |f| \geq \frac{\delta}{2^n \omega_n |x|^n}. \tag{1.3.4}$$

Hence,

$$\int_{B(0,R)^c} \mathcal{M}(f)(x)dx \geq \frac{\delta}{2^n \omega_n} \int_{B(0,R)^c} \frac{dx}{|x|^n} = \infty, \tag{1.3.5}$$

and we deduce from this that $\mathcal{M}(f)$ is never integrable when f is nontrivial and locally integrable. In spite of this failure, we can get a weaker estimate that will be sufficient for using Marcinkiewicz.

Theorem 1.3.1. *Let $1 < p < \infty$ and $1 \le q \le \infty$. The Hardy-Littlewood maximal operator extends uniquely to a bounded sublinear operator $\mathcal{M} : L^{p,q}(\mathbb{R}^n;\mathbb{F}) \to L^{p,q}(\mathbb{R}^n;\mathbb{F})$, i.e. there exists a constant $C = C(p,q) > 0$ such that*

$$|||\mathcal{M}(f)|||_{L^{p,q}} \le C\, |||f|||_{L^{p,q}} \text{ for all } f \in L^{p,q}(\mathbb{R}^n;\mathbb{F}). \tag{1.3.6}$$

In particular, by taking $q = p$ we have that $\mathcal{M} : L^p(\mathbb{R}^n;\mathbb{F}) \to L^p(\mathbb{R}^n;\mathbb{F})$ is a bounded sublinear operator.

Proof. Let $f \in L^1(\mathbb{R}^n;\mathbb{F})$. If $x \in \mathbb{R}^n$, $t > 0$, and $\mathcal{M}(f)(x) > t$ then there exists $r_x > 0$ such that

$$\int_{B(x,r_x)} |f| > t\omega_n r_x^n, \tag{1.3.7}$$

and hence

$$E_t := \{x \in \mathbb{R}^n \mid \mathcal{M}(f)(x) > t\} \subseteq \bigcup_{x \in E_t} B(x, r_x). \tag{1.3.8}$$

Vitali's lemma allows us to extract a countable subcollection $\{B(x_i,r_i)\}_i$ that is pairwise disjoint and satisfies the bound

$$\sum_i \mathcal{L}^n(B(x_i,r_i)) \ge C\mathcal{L}^n(E_t) \tag{1.3.9}$$

for a constant $C = C(n)$ depending on the dimension n. Then

$$d_{\mathcal{M}(f)}(t) = \mathcal{L}^n(E_t) \le \frac{1}{C}\sum_i \omega_n r_i^n \le \frac{1}{Ct}\sum_i \int_{B(x_i,r_i)} |f| \le \frac{1}{Ct}\int_{\mathbb{R}^n} |f| = \frac{1}{Ct}\|f\|_{L^1}. \tag{1.3.10}$$

Consequently,

$$|||\mathcal{M}(f)|||_{L^{1,\infty}} \le C_1 \|f\|_{L^1} \tag{1.3.11}$$

for some constant $C_1 > 0$.

We now know that $\mathcal{M}$ maps $L^\infty(\mathbb{R}^n;\mathbb{F})$ to $L^\infty(\mathbb{R}^n;\mathbb{F})$ and $L^1(\mathbb{R}^n;\mathbb{F})$ to $L^{1,\infty}(\mathbb{R}^n;\mathbb{F})$ in a bounded way. The Marcinkiewicz interpolation theorem and Example 1.2.8 then tell us that for each $1 < p < \infty$ and $1 \le q \le \infty$ there exists $C > 0$ such that

$$|||\mathcal{M}(f)|||_{L^{p,q}} \le C\, |||f|||_{L^{p,q}} \tag{1.3.12}$$

for all $f \in L^{p,q}(\mathbb{R}^n;\mathbb{F})$. In particular, choosing $q = p$ shows that

$$\|\mathcal{M}(f)\|_{L^p} \le C\, \|f\|_{L^p} \text{ for all } 1 < p < \infty. \tag{1.3.13}$$

□

1.3.2 The Hausdorff-Young inequality and its variants

The Fourier transform is the map $\hat{\cdot} : L^1(\mathbb{R}^n;\mathbb{C}) \to L^\infty(\mathbb{R}^n;\mathbb{C})$ defined via

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx. \tag{1.3.14}$$

It's a simple matter to verify that $\hat{\cdot}$ is a bounded linear map,

$$\left\|\hat{f}\right\|_{L^\infty} \le \|f\|_{L^1}, \tag{1.3.15}$$

and that $\hat{f}$ is uniformly continuous. Actually, the Riemann-Lebesgue lemma (a proof of which can be found in any book on Fourier analysis) shows that

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0 \tag{1.3.16}$$

so $\hat{\cdot}$ actually maps into the space of uniformly continuous functions that decay to zero at infinity.

Remarkably, for $f, g \in L^1(\mathbb{R}^n;\mathbb{C}) \cap L^2(\mathbb{R}^n;\mathbb{C})$ we have that

$$\int_{\mathbb{R}^n} f\bar{g} = \int_{\mathbb{R}^n} \hat{f}\bar{\hat{g}} \text{ and } \|f\|_{L^2} = \left\|\hat{f}\right\|_{L^2}, \tag{1.3.17}$$

which allows us to uniquely extend $\hat{\cdot}$ to a unitary bijection $\hat{\cdot} : L^2(\mathbb{R}^n;\mathbb{C}) \to L^2(\mathbb{R}^n;\mathbb{C})$ satisfying the above for all $f, g \in L^2(\mathbb{R}^n;\mathbb{C})$. Again, a proof can be found in any book on Fourier analysis.

We will take these facts as given and use them as inputs in interpolation theory. The classic Hausdorff-Young inequality applies Riesz-Thorin to these estimates to deduce further boundedness properties of the Fourier transform.

Theorem 1.3.2 (Hausdorff-Young)**.** *Let $1 \le p \le 2$. Then the Fourier transform extends to a bounded linear map $\hat{\cdot} : L^p(\mathbb{R}^n;\mathbb{C}) \to L^{p'}(\mathbb{R}^n;\mathbb{C})$. In particular,*

$$\left\|\hat{f}\right\|_{L^{p'}} \le \|f\|_{L^p} \text{ for all } f \in L^p(\mathbb{R}^n;\mathbb{C}). \tag{1.3.18}$$

Proof. The above bounds tell us that

$$\left\|\hat{f}\right\|_{L^\infty} \le \|f\|_{L^1} \text{ for all } f \in L^1(\mathbb{R}^n;\mathbb{C}) \tag{1.3.19}$$

and

$$\left\|\hat{f}\right\|_{L^2} = \|f\|_{L^2} \text{ for all } f \in L^2(\mathbb{R}^n;\mathbb{C}). \tag{1.3.20}$$

The result then follows from these bounds and the Riesz-Thorin interpolation theorem after we note that if

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{1} \text{ and } \frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{\infty} \tag{1.3.21}$$

then

$$\frac{1}{q} + \frac{1}{p} = 1 \tag{1.3.22}$$

and hence $q = p'$. □

Note that we also could have used Marcinkiewicz to deduce the boundedness of $\hat{\cdot}$ from $L^p(\mathbb{R}^n;\mathbb{C})$ to $L^{p'}(\mathbb{R}^n;\mathbb{C})$ since $p \le p'$ for $1 \le p \le 2$. However, this would result in a worse estimate for the norm than that provided by Riesz-Thorin. We can still use Marcinkiewicz to learn something, though.

Theorem 1.3.3 (Hausdorff-Young, Lorentz variant)**.** *Let $1 < p < 2$ and $1 \le q \le \infty$. Then the Fourier transform extends to a bounded linear operator $\hat{\cdot} : L^{p,q}(\mathbb{R}^n;\mathbb{C}) \to L^{p',q}(\mathbb{R}^n;\mathbb{C})$.*

Proof. Exercise. □

Remarkably, this isn't the end of the story! We can derive another variant that employs certain weights on the Fourier side.

Theorem 1.3.4 (Hausdorff-Young, weighted variant). *Let $1 < p < 2$ and $p \le q \le p'$. Then there exists a constant $C > 0$ such that*

$$\left(\int_{\mathbb{R}^n} \left| \hat{f}(\xi) \right|^q \frac{d\xi}{|\xi|^{n(1-q/p')}} \right)^{1/q} \le C \|f\|_{L^p} \tag{1.3.23}$$

for every $f \in L^p(\mathbb{R}^n; \mathbb{C})$.

Proof. Define the measure μ on $\mathbb{R}^n$ via $\mu = d\xi / |\xi|^{2n}$. We will write $L^p_\mu(\mathbb{R}^n; \mathbb{C})$ and $L^{p,q}_\mu(\mathbb{R}^n; \mathbb{C})$ for the Lebesgue and Lorentz spaces relative to the measure μ, and when we omit the μ we mean the usual spaces relative to Lebesgue measure.

Define the map $T : L^1(\mathbb{R}^n; \mathbb{C}) + L^2(\mathbb{R}^n; \mathbb{C}) \to L^0_\mu(\mathbb{R}^n; \mathbb{C})$ via

$$Tf(\xi) = |\xi|^n \hat{f}(\xi). \tag{1.3.24}$$

Using (1.3.17), we may bound

$$\|Tf\|_{L^2_\mu} = \left(\int_{\mathbb{R}^n} |\xi|^{2n} \left| \hat{f}(\xi) \right|^2 \frac{d\xi}{|\xi|^{2n}} \right)^{1/2} = \left\| \hat{f} \right\|_{L^2} = \|f\|_{L^2}. \tag{1.3.25}$$

Next, we claim that

$$|||Tf|||_{L^{1,\infty}_\mu} \le \omega_n \|f\|_{L^1}, \tag{1.3.26}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ (see the Section 0.2). To see this first note that if $\xi \in \mathbb{R}^n$ is such that $t < |Tf(\xi)|$, then (1.3.15) implies that

$$t < |\xi|^n \left| \hat{f}(\xi) \right| \le |\xi|^n \|f\|_{L^1} \Rightarrow \left(\frac{t}{\|f\|_{L^1}} \right)^{1/n} < |\xi|, \tag{1.3.27}$$

and hence

$$\{\xi \in \mathbb{R}^n \mid |Tf(\xi)| > t\} \subset B(0, t^{1/n} / \|f\|_{L^1}^{1/n})^c. \tag{1.3.28}$$

Thus, for $t \in \mathbb{R}_+$ we may bound

$$\mu(\{\xi \in \mathbb{R}^n \mid |Tf(\xi)| > t\}) \le \int_{B(0,t^{1/n}/\|f\|_{L^1}^{1/n})^c} \frac{d\xi}{|\xi|^{2n}} = \alpha_n \int_{t^{1/n}/\|f\|_{L^1}^{1/n}}^\infty \frac{r^{n-1} dr}{r^{2n}} = \frac{\alpha_n}{n} \frac{\|f\|_{L^1}}{t}, \tag{1.3.29}$$

which implies (1.3.26) since $\omega_n = \alpha_n / n$.

With (1.3.25) and (1.3.26) in hand, we may apply the Marcinkiewicz interpolation theorem to deduce that for each $1 < p \le 2$ there exists $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} \left| \hat{f}(\xi) \right|^p \frac{d\xi}{|\xi|^{n(2-p)}} \right)^{1/p} = \|Tf\|_{L^p_\mu} \le C \|f\|_{L^p} \tag{1.3.30}$$

for all $f \in L^p(\mathbb{R}^n; \mathbb{C})$. When $q = p$ this is (1.3.23) since

$$1 - \frac{p}{p'} = 2 - p, \tag{1.3.31}$$

as is readily verified. In the general case of $p \le q \le p'$ we define $\theta \in [0, 1]$ via $q = (1 - \theta)p + \theta p'$, or equivalently

$$\theta = \frac{q - p}{p' - p} \text{ and } 1 - \theta = \frac{p' - q}{p' - p} = \frac{1 - q/p'}{1 - p/p'} = \frac{1 - q/p'}{2 - p}. \tag{1.3.32}$$

Then from (1.3.30), the basic Hausdorff-Young inequality from Theorem 1.3.2, and Hölder's inequality we may bound

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^q \frac{d\xi}{|\xi|^{n(1-q/p')}} &= \int_{\mathbb{R}^n} \left(\frac{\left|\hat{f}(\xi)\right|^p}{|\xi|^{n(2-p)}} \right)^{1-\theta} \left(\left|\hat{f}(\xi)\right|^{p'} \right)^{\theta} d\xi \\
&\le \left(\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^p \frac{d\xi}{|\xi|^{n(2-p)}} \right)^{1-\theta} \left(\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^{p'} d\xi \right)^{\theta} \le C \left\|f\right\|_{L^p}^{(1-\theta)p} \left\|f\right\|_{L^p}^{\theta p'} = C \left\|f\right\|_{L^p}^q , \quad (1.3.33)
\end{aligned}$$

and we deduce that

$$\left(\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^q \frac{d\xi}{|\xi|^{n(1-q/p')}} \right)^{1/q} \le C \left\|f\right\|_{L^p} . \quad (1.3.34)$$

This is (1.3.23).

□

1.3.3 Integral operators

Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces and $U \subseteq L^0(Y; \mathbb{F})$ be a subspace. We now turn our attention to maps $T : U \to L^0(X; \mathbb{F})$ of the form

$$Tf(x) = \int_Y K(x, y) f(y) d\nu(y), \quad (1.3.35)$$

where $K : X \times Y \to \mathbb{F}$ is a given measurable "kernel" function. Of course, we need to worry about hypotheses on K and f that will guarantee that this integral makes sense, and we will identify sufficient conditions for this. When the operator is well-defined it is obviously linear, and such linear maps are referred to as integral operators.

Our first result on integral operators assumes that the kernel satisfies some uniform integrability conditions.

Theorem 1.3.5. *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be $\sigma-$finite measure spaces, and let $1 \le p \le \infty$. Let $K : X \times Y \to \mathbb{F}$ be measurable with respect to $\mathfrak{M} \otimes \mathfrak{N}$ and suppose there exists a constant $A \ge 0$ such that*

$$\begin{aligned}
&\int_X |K(x, y)| \, d\mu(x) \le A \text{ for } \nu - a.e. \ y \in Y \\
&\int_Y |K(x, y)| \, d\nu(y) \le A \text{ for } \mu - a.e. \ x \in X.
\end{aligned} \quad (1.3.36)$$

Then the following hold.

1. *If $f \in L^p(Y; \mathbb{F})$ then for $\mu-$a.e. $x \in X$ the integral $\int_Y K(x, y) f(y) d\nu(y)$ is well-defined, and the function $Tf : X \to \mathbb{F}$ defined a.e. via*

$$Tf(x) = \int_Y K(x, y) f(y) d\nu(y) \quad (1.3.37)$$

is such that $Tf \in L^p(X; \mathbb{F})$ and

$$\left\|Tf\right\|_{L^p(X;\mathbb{F})} \le A \left\|f\right\|_{L^p(Y;\mathbb{F})} . \quad (1.3.38)$$

2. *The induced map* $T : L^p(Y;\mathbb{F}) \to L^p(X;\mathbb{F})$ *is linear and bounded.*

Proof. First note that it suffices to prove the first item, as the second follows immediately from the first. Second, note that we may reduce to proving the result with $\mathbb{F} = \mathbb{C}$ since $\mathbb{R} \subseteq \mathbb{C}$ and Tf is clearly real-valued when K and f are. If $p = 1$ or ∞, then the first item follows from Fubini-Tonelli or Hölder, respectively. Then the Riesz-Thorin interpolation theorem implies that the first item continues to hold for general $1 < p < \infty$. □

Remark 1.3.6. *We also could have applied Marcinkiewicz to prove the theorem, but the constant would have been worse.*

We can parlay the result of Theorem 1.3.5 into a proof of Young's inequality.

Theorem 1.3.7 (Young's inequality)**.** *Let* $1 \le p, q, r \le \infty$ *satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{1.3.39}$$

Then for each $f \in L^p(\mathbb{R}^n;\mathbb{F})$ *the linear map* $\Gamma_f : L^1(\mathbb{R}^n;\mathbb{F}) \cap L^{p'}(\mathbb{R}^n;\mathbb{F}) \cap S(\mathbb{R}^n;\mathbb{F}) \to L^0(\mathbb{R}^n;\mathbb{F})$ *given by*

$$\Gamma_f g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy \tag{1.3.40}$$

is well-defined and extends uniquely to a bounded linear map $\Gamma_f : L^q(\mathbb{R}^n;\mathbb{F}) \to L^r(\mathbb{R}^n;\mathbb{F})$*, and*

$$\|\Gamma_f g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q} \text{ for all } g \in L^q(\mathbb{R}^n;\mathbb{F}). \tag{1.3.41}$$

Proof. First note that the conditions on p, q, r imply that

$$\frac{1}{p} = \frac{1}{q'} + \frac{1}{r} \ge \frac{1}{r} \Rightarrow p \le r \le \infty. \tag{1.3.42}$$

If we define $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{F}$ via $K(x,y) = g(x-y)$, then we may use Theorem 1.3.5 with $A = \|g\|_{L^1}$ to see that

$$\|\Gamma_f g\|_{L^p} \le \|g\|_{L^1} \|f\|_{L^p}, \tag{1.3.43}$$

so $\Gamma_f g$ is well-defined on $L^1(\mathbb{R}^n;\mathbb{F})$. If $r = p$, then $q = 1$ and the result follows from this and a standard density argument.

On the other hand, Hölder's inequality shows that

$$\|\Gamma_f g\|_{L^\infty} \le \|g\|_{L^{p'}} \|f\|_{L^p} \text{ for all } g \in L^{p'}(\mathbb{R}^n;\mathbb{F}). \tag{1.3.44}$$

If $r = \infty$, then $q = p'$ and again the result follows from this and a standard density argument.

Suppose, then, that $p < r < \infty$ in which case we can define $\theta \in (0,1)$ via

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{\infty} = \frac{1-\theta}{p}. \tag{1.3.45}$$

This implies that

$$\frac{1-\theta}{1} + \frac{\theta}{p'} = 1 - \theta + \theta - \frac{\theta}{p} = 1 - \frac{\theta}{p} = 1 + \frac{1}{r} - \frac{1}{p} = \frac{1}{q}. \tag{1.3.46}$$

We may then apply Riesz-Thorin to deduce that the map Γ_f extends to a bounded linear operator $\Gamma_f : L^q(\mathbb{R}^n;\mathbb{F}) \to L^r(\mathbb{R}^n;\mathbb{F})$ with

$$\|\Gamma_f g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}, \tag{1.3.47}$$

and the result is proved. □

This suggests a definition.

Definition 1.3.8. *Let $1 \le p, q, r \le \infty$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{1.3.48}$$

We define the bilinear map $* : L^p(\mathbb{R}^n;\mathbb{F}) \times L^q(\mathbb{R}^n;\mathbb{F}) \to L^r(\mathbb{R}^n;\mathbb{F})$ via

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \tag{1.3.49}$$

By Young's inequality this is well-defined and bounded:

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}. \tag{1.3.50}$$

Remark 1.3.9. *If we replace $\mathbb{R}^n$ with $\mathbb{Z}^n$ and Lebesgue measure by counting measure in Young's inequality, then the argument pushes through without modification and shows that if we define*

$$f * g(m) = \sum_{k \in \mathbb{Z}} f(k)g(m-k), \tag{1.3.51}$$

then $f * g \in \ell^r(\mathbb{Z}^n;\mathbb{F})$ when $f \in \ell^p(\mathbb{Z}^n;\mathbb{F})$ and $g \in \ell^q(\mathbb{Z}^n;\mathbb{F})$ with (1.3.39) satisfied. Moreover, $\|f * g\|_{\ell^r} \le \|f\|_{\ell^p} \|g\|_{\ell^r}$.

In the above we have exploited Riesz-Thorin in order to get bounds on integral operators. It turns out that we can also use the Marcinkiewicz theorem, and this allows us to put weaker assumptions on the kernel. To prove this we fist need a lemma.

Lemma 1.3.10. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $f : X \to \mathbb{F}$ be measurable, $\tau \in \mathbb{R}_+$, and set*

$$E(\tau) = \{x \in X \mid |f(x)| > \tau\}. \tag{1.3.52}$$

Define $h_\tau, g_\tau : X \to V$ via

$$h_\tau = f\chi_{E(\tau)^c} + \tau\frac{f}{|f|}\chi_{E(\tau)} \text{ and } g_\tau = f - h_\tau = \frac{f}{|f|}(|f| - \tau)\chi_{E(\tau)}. \tag{1.3.53}$$

Then

$$d_{g_\tau}(t) = d_f(t+\tau) \text{ and } d_{h_\tau}(t) = \begin{cases} d_f(t) & \text{if } t < \tau \\ 0 & \text{if } \tau \le t. \end{cases} \tag{1.3.54}$$

Proof. First note that $|h_\tau(x)| \le \tau$ for all x, so $d_{h_\tau}(t) = 0$ for $t \ge \tau$. For $t < \tau$ we then compute

$$d_{h_\tau}(t) = \mu(\{x \in E(\tau)^c \mid |f(x)| > t\}) + \mu(E(\tau)) = \mu(\{x \in X \mid |f(x)| > t\}) = d_f(t), \tag{1.3.55}$$

which proves the equality for h_τ. On the other hand,

$$d_{g_\tau}(t) = \mu(\{x \in E(\tau) \mid \left|\frac{f(x)}{|f(x)|}(|f(x)| - \tau)\right| > t\}) = \mu(\{x \in X \mid |f(x)| - \tau > t\}) = d_f(t+\tau). \tag{1.3.56}$$

□

With the lemma in hand, we can now prove a variant of Theorem 1.3.5.

Theorem 1.3.11. *Let* $(X,\mathfrak{M},\mu)$ *and* $(Y,\mathfrak{N},\nu)$ *be* $\sigma-$*finite measure spaces, and let* $1 < p,q,r < \infty$ *be such that*

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}. \tag{1.3.57}$$

Let $K : X\times Y \to \mathbb{F}$ *be measurable with respect to* $\mathfrak{M}\otimes\mathfrak{N}$*, and suppose there exists a constant* $A \geq 0$ *such that*

$$\begin{aligned} |||K(\cdot,y)|||_{L^{q,\infty}(X;\mathbb{F})} &\leq A \text{ for } \nu-a.e.\ y\in Y \\ |||K(x,\cdot)|||_{L^{q,\infty}(Y;\mathbb{F})} &\leq A \text{ for } \mu-a.e.\ x\in X. \end{aligned} \tag{1.3.58}$$

Then the following hold.

1. *If* $f\in L^p(Y;\mathbb{F})$ *then for* $\mu-$*a.e.* $x\in X$ *the integral* $\int_Y K(x,y)f(y)d\nu(y)$ *is well-defined, and the function* $Tf : X\to\mathbb{F}$ *defined a.e. via*

$$Tf(x)=\int_Y K(x,y)f(y)d\nu(y) \tag{1.3.59}$$

   is such that $Tf\in L^p(X;\mathbb{F})$*.*

2. *The induced map* $T : L^p(Y;\mathbb{F})\to L^r(X;\mathbb{F})$ *is linear and bounded, and there exists a constant* $C>0$ *depending only on* p,q,r*, such that*

$$\|Tf\|_{L^r(X;\mathbb{F})}\leq CA\,\|f\|_{L^p(Y;\mathbb{F})} \text{ for all } f\in L^p(Y;\mathbb{F}) \tag{1.3.60}$$

Proof. First note that

$$\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1=\frac{1}{p}-\frac{1}{q'}=\frac{1}{q}-\frac{1}{p'}, \tag{1.3.61}$$

and this tells us that $p<q'$ and $q<p'$. Next note that the result is trivial if $A=0$, so we can assume that $A>0$.

Consider $0\neq f\in L^p(Y;\mathbb{F})$. Assume for now that $\|f\|_{L^p}=1$ and $A\leq 1$. Let $\tau>0$ and define

$$E(\tau)=\{(x,y)\in X\times Y \mid |K(x,y)|>\tau\}, \tag{1.3.62}$$

and

$$K_1=\frac{K}{|K|}(|K|-\tau)\chi_{E(\tau)} \text{ and } K_2=K-K_1. \tag{1.3.63}$$

Define the operators T_1 and T_2 with K replaced by K_1 and K_2, respectively.

By the Lemma 1.3.10 we may compute

$$\int_Y |K_1(x,y)|\,d\nu(y)=\int_0^\infty d_{K(x,\cdot)}(t+\tau)dt\leq\int_\tau^\infty t^{-q}dt=\frac{\tau^{1-q}}{q-1}, \tag{1.3.64}$$

and similarly

$$\int_X |K_1(x,y)|\,d\mu(x)\leq\frac{\tau^{1-q}}{q-1}. \tag{1.3.65}$$

Thus Theorem 1.3.5 tells us that $T_1 f$ is defined a.e. and

$$\|T_1 f\|_{L^p}\leq\frac{\tau^{1-q}}{q-1}\|f\|_{L^p}=\frac{\tau^{1-q}}{q-1}. \tag{1.3.66}$$

On the other hand, since $q < p'$ we may compute

$$\int_Y |K_2(x,y)|^{p'} \, d\nu(y) = p' \int_0^\tau t^{p'-1} d_{K(x,\cdot)}(t) dt \le p' \int_0^\tau t^{p'-1-q} dt = \frac{p' \tau^{p'-q}}{p'-q}. \tag{1.3.67}$$

This bound and Hölder's inequality imply that $T_2 f$ is defined for a.e. $x \in X$ and that

$$\|T_2 f\|_{L^\infty} \le \left(\frac{p' \tau^{p'-q}}{p'-q} \right)^{1/p'} \|f\|_{L^p} = \left(\frac{r}{q} \right)^{1/p'} \tau^{q/r}. \tag{1.3.68}$$

These two arguments combine to show that $Tf = T_1 f + T_2 f$ is well-defined a.e. in X.

Due to the decomposition $Tf = T_1 f + T_2 f$ we can bound, for any $t \in \mathbb{R}_+$,

$$d_{Tf}(t) \le d_{T_1 f}(t/2) + d_{T_2 f}(t/2), \tag{1.3.69}$$

and so if we choose

$$\tau = \left(\frac{t}{2} \right)^{r/q} \left(\frac{q}{r} \right)^{r/(qp')} \tag{1.3.70}$$

then (1.3.68) shows that $\|T_2 f\|_{L^\infty} \le t/2$ and hence $d_{T_2 f}(t/2) = 0$. With this choice made we then use Chebyshev to bound

$$\begin{aligned} d_{Tf}(t) \le d_{T_1 f}(t/2) \le \left(\frac{2 \|T_1 f\|_{L^p}}{t} \right)^p &\le \left(\frac{2\tau^{1-q}}{(q-1)t} \right)^p \\ &= \frac{2^{p-[(1-q)pr]/q}}{(q-1)^p} \left(\frac{q}{r} \right)^{[(1-q)(pr)]/(qp')} t^{-p+[(1-q)pr]/q} = \frac{C_p}{t^r} \end{aligned} \tag{1.3.71}$$

for some constant $C = C(p,q,r) > 0$, due to the algebraic calculation

$$\frac{(1-q)pr}{q} - p = p\left(-\frac{r}{q'} - 1 \right) = -p\frac{r}{p} = -r. \tag{1.3.72}$$

The bound

$$d_{Tf}(t) \le C \frac{1}{t^r} \tag{1.3.73}$$

was proved under the assumption that $\|f\|_{L^p} = 1$, but for general $f \neq 0$ we can apply this estimate to $f / \|f\|_{L^p}$ to deduce that

$$d_{Tf}(t) \le C \left(\frac{\|f\|_{L^p}}{t} \right)^r. \tag{1.3.74}$$

This yields the bound

$$|||Tf|||_{L^{r,\infty}} \le C \|f\|_{L^p} \tag{1.3.75}$$

for all p, r satisfying the relation to q in the hypotheses.

Since $1 < p < \infty$ we may choose $1 < p_0 < p < p_1 < \infty$ and define $1 < r_0 < r < r_1 < \infty$ via

$$\frac{1}{q} + \frac{1}{p_0} = 1 + \frac{1}{r_0} \text{ and } \frac{1}{q} + \frac{1}{p_1} = 1 + \frac{1}{r_1}. \tag{1.3.76}$$

Define $\theta \in (0,1)$ via

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \tag{1.3.77}$$

and note that

$$\frac{1-\theta}{r_0}+\frac{\theta}{r_1}=(1-\theta+\theta)\left(\frac{1}{q}-1\right)+\frac{1-\theta}{p_0}+\frac{\theta}{p_1}=\frac{1}{q}+\frac{1}{p}-1=\frac{1}{r}. \tag{1.3.78}$$

Further note that $p_i \le r_i$ for $i=1,2$ by construction, so $p \le r$.

The estimates above tell us that

$$|||Tf|||_{L^{r_i,\infty}(X;\mathbb{F})} \le C_i \|f\|_{L^{p_i}(Y;\mathbb{F})} \text{ for } i=1,2 \text{ and } f \in L^{p_i}(Y;\mathbb{F}) \tag{1.3.79}$$

for constants $C_i > 0$. Then according to this and the Marcinkiewicz interpolation theorem there exists a constant $C > 0$ such that

$$\|Tf\|_{L^r(X;\mathbb{F})} \le C \|f\|_{L^p(X;\mathbb{F})} \tag{1.3.80}$$

for all $f \in L^p(Y;\mathbb{F})$. Finally, this estimate was derived under the auxiliary assumption that $A \le 1$, but the general case follows from this special case applied to K/A. □

As an immediate consequence we get a very useful variant of Young's inequality.

Theorem 1.3.12 (Young's inequality, weak-strong form)**.** *Let $1 < p,q,r < \infty$ satisfy*

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}. \tag{1.3.81}$$

If $f \in L^p(\mathbb{R}^n;\mathbb{F})$ and $g \in L^{q,\infty}(\mathbb{R}^n;\mathbb{F})$, then the function $f * g : \mathbb{R}^n \to \mathbb{F}$ given by

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy \tag{1.3.82}$$

is well-defined a.e., measurable, and belongs to $L^r(\mathbb{R}^n;\mathbb{F})$. Moreover, there exists a constant $C = C(p,q,r) > 0$ such that

$$\|f * g\|_{L^q} \le C \|f\|_{L^p} |||g|||_{L^{q,\infty}} \tag{1.3.83}$$

for every $f \in L^p(\mathbb{R}^n;\mathbb{F})$ and $g \in L^{q,\infty}(\mathbb{R}^n;\mathbb{F})$.

Proof. This follows from Theorem 1.3.11 by setting $K(x,y) = g(x-y)$ and $A = |||g|||_{L^{q,\infty}}$. □

Remark 1.3.13. *This theorem can again be extended to functions defined on the integer lattice $\mathbb{Z}^n$. We will not need this, though, so we don't record it in a precise form.*

As a brief glimpse of the power of this generalized version of Young's inequality we present the following result on the Riesz potentials.

Theorem 1.3.14 (Hardy-Littlewood-Sobolev inequality)**.** *Let $\alpha \in \mathbb{R}$ satisfy $0 < \alpha < n$. For $f \in S_{fin}(\mathbb{R}^n;\mathbb{F})$ define the Riesz potential of f to be the measurable function $\mathcal{I}_\alpha f : \mathbb{R}^n \to \mathbb{F}$ given by*

$$\mathcal{I}_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\alpha} dy. \tag{1.3.84}$$

Then $\mathcal{I}_\alpha$ extends to a bounded linear operator $\mathcal{I}_\alpha : L^p(\mathbb{R}^n;\mathbb{F}) \to L^r(\mathbb{R}^n;\mathbb{F})$ for $1 < p < n/(n-\alpha)$ and

$$\frac{1}{r}=\frac{1}{p}-\frac{n-\alpha}{n}. \tag{1.3.85}$$

Proof. Note that

$$t < \frac{1}{|x|^\alpha} \Leftrightarrow |x| < \frac{1}{t^{1/\alpha}} \tag{1.3.86}$$

and so if we write $g(x) = 1/|x|^\alpha$ then

$$t^{n/\alpha} d_g(t) = \omega_n \tag{1.3.87}$$

and we deduce that $g \in L^{n/\alpha,\infty}$. We can then apply Theorem 1.3.12 to deduce that $\mathcal{I}_\alpha \in \mathcal{L}(L^p(\mathbb{R}^n;\mathbb{F}); L^r(\mathbb{R}^n;\mathbb{F}))$ whenever

$$\frac{1}{r} = \frac{1}{p} - \frac{n-\alpha}{n} \Leftrightarrow r = \frac{np}{n-p(n-\alpha)}. \tag{1.3.88}$$

□

These bounds for the Riesz potential cannot be improved. Indeed, we have the following example.

Example 1.3.15. Suppose that there exists $C > 0$ such that

$$\|\mathcal{I}_\alpha f\|_{L^r} \le C \|f\|_{L^p} \text{ for all } f \in L^p. \tag{1.3.89}$$

Pick $f \in L^p(\mathbb{R}^n)\backslash\{0\}$ such that $f \ge 0$ a.e., which implies that $\mathcal{I}_\alpha f \neq 0$. For $\lambda > 0$ consider $f_\lambda \in L^p$ given by $f_\lambda(x) = f(\lambda x)$. We then compute

$$\|f_\lambda\|_{L^p} = \lambda^{-n/p} \|f\|_{L^p}. \tag{1.3.90}$$

Also

$$\mathcal{I}_\alpha f_\lambda(x) = \int_{\mathbb{R}^n} \frac{f(\lambda y)}{|x-y|^\alpha} dy = \lambda^\alpha \int_{\mathbb{R}^n} \frac{f(\lambda y)}{|\lambda x - \lambda y|^\alpha} dy = \lambda^{\alpha-n} \mathcal{I}_\alpha f(\lambda x), \tag{1.3.91}$$

and so

$$\|\mathcal{I}_\alpha f_\lambda\|_{L^r} = \lambda^{\alpha-n-n/r} \|\mathcal{I}_\alpha f\|_{L^r}. \tag{1.3.92}$$

Thus

$$\lambda^{\alpha-n-n/r} \|\mathcal{I}_\alpha f\|_{L^r} \le \lambda^{-n/p} C \|f\|_{L^p}. \tag{1.3.93}$$

Since both of the norms are nonzero and we are free to send $\lambda \to 0$ and $\lambda \to \infty$ in this inequality, we see that we can avoid a contradiction if and only if

$$-\frac{n}{p} = -\frac{n}{r} + \alpha - n \Leftrightarrow \frac{1}{r} = \frac{1}{p} - \frac{n-\alpha}{n} \Leftrightarrow r = \frac{np}{n-p(n-\alpha)}. \tag{1.3.94}$$

This scaling argument shows that the relationship between p and r from the theorem is a necessary condition. Note that we need $1 \le p \le n/(n-\alpha)$ for this to make sense at all. The scaling argument does not address what happens at the endpoint cases, $p = 1$ and $p = n/(n-\alpha)$, though. We will show that the Riesz potentials are unbounded in these cases.

Let $p = 1$, which means that $r = n/\alpha$. Consider $f \in L^1(\mathbb{R}^n;\mathbb{F})\backslash\{0\}$ real valued such that $f \ge 0$ a.e. Pick $R > 0$ and $\delta > 0$ such that

$$\int_{B(0,R)} |f| \ge \delta. \tag{1.3.95}$$

Then for $|x| \ge R$ and $|y| \le R$ we have that $|x-y| \le |x| + |y| \le |x| + R \le 2|x|$. Thus, for $|x| \ge R$ we can bound

$$\mathcal{I}_\alpha f(x) \ge \int_{B(0,R)} \frac{f(y)}{|x-y|^\alpha} dy \ge \int_{B(0,R)} \frac{f(y)}{2^\alpha |x|^\alpha} dy \ge \frac{\delta}{2^\alpha |x|^\alpha}, \tag{1.3.96}$$

and so

$$\int_{B(0,R)^c} |\mathcal{I}_\alpha f(x)|^{n/\alpha}\, dx \geq \left(\frac{\delta}{2^\alpha}\right)^{n/\alpha} \int_{B(0,R)^c} \frac{dx}{|x|^n} = \infty, \tag{1.3.97}$$

which in particular means that $\mathcal{I}_\alpha f \notin L^r(\mathbb{R}^n;\mathbb{F})$.

Let $p = n/(n-\alpha)$, which means that $r = \infty$. Fix $1 < \beta < p$ and define $f : \mathbb{R}^n \to \mathbb{R}$ via

$$f(x) = \begin{cases} |x|^{-n/p} \left|\log |x|\right|^{-\beta/p} & \text{for } 0 < |x| < 1/e \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.98}$$

Then since $1 < \beta$,

$$\|f\|_{L^p}^p = \alpha_n \int_0^{1/e} \frac{dr}{r(\log(1/r))^\beta} = \int_e^\infty \frac{ds}{s(\log s)^\beta} < \infty, \tag{1.3.99}$$

so $f \in L^p(\mathbb{R}^n;\mathbb{F})$. However, if $0 < \delta < 1/e$, $\delta < |y| < 1/e$, and $|x| < \delta$, then $|x-y| \leq |x| + |y| < \delta + |y| < 2|y|$, and so

$$\begin{aligned} \mathcal{I}_\alpha f(x) &\geq \int_{\{\delta<|y|<1/e\}} \frac{f(y)}{|x-y|^\alpha} dy \geq \frac{1}{2^\alpha} \int_{\{\delta<|y|<1/e\}} \frac{f(y)}{|y|^\alpha} dy \\ &= \frac{1}{2^\alpha} \int_{\{\delta<|y|<1/e\}} \frac{dy}{|y|^{\alpha+n/p} [\log(1/|y|)]^{\beta/p}} = \frac{\alpha_n}{2^\alpha} \int_\delta^{1/e} \frac{r^{n-1} dr}{r^n [\log(1/r)]^{\beta/p}} = \frac{\alpha_n}{2^\alpha} \int_e^{1/\delta} \frac{ds}{s[\log s]^{\beta/p}}. \end{aligned} \tag{1.3.100}$$

Since $\beta/p < 1$,

$$\int_e^\infty \frac{ds}{s[\log s]^{\beta/p}} = \infty, \tag{1.3.101}$$

and we conclude that $\mathcal{I}_\alpha f \notin L^\infty(\mathbb{R}^n;\mathbb{F})$.

$\triangle$

2 Abstract interpolation theory

We have now seen that the Lebesgue and Lorentz spaces admit a rich interpolation theory that is extremely useful in the sense that it allows us to gain information about the mapping properties of certain types of operators by first understanding how the operator acts on two fixed pairs of such spaces. Our goal now is to generalize these techniques to the context of more abstract spaces, and to thereby distill the essential ideas at the core of what we've done so far. In turn, we can hope that this will provide a clearer perspective on the Lebesgue and Lorentz interpolation as well as equip us with tools analogous to the Marcinkiewicz and Riesz-Thorin theorems that we can use in a more general context.

A serious problem with this program immediately presents itself: in the context of Lebesgue and Lorentz spaces we naturally have a continuum of parameters with which can interpolate, namely the values $1 \leq p, q \leq \infty$, but in a more general context it's not obvious where such parameters will come from. This means that our starting point will not be the development of the interpolation theory for operators but rather the construction of the interpolation spaces themselves. Once we have these in hand, we can then turn to the question of building the corresponding operator interpolation theory.

Fortunately, these spaces *can* be constructed, and they *do* admit a beautiful and useful theory of operator interpolation. Once we have developed this in this section, we will return to the context of the previous section and demonstrate how what we saw there fits into this framework. Unfortunately, due to time constraints, we will not be able to fully develop this theory in these notes. Because of this, we have chosen to focus solely on the so-called real method of interpolation, which is, roughly-speaking, the abstract generalization of the Marcinkiewicz interpolation theorem. There is a corresponding abstract generalization of the Riesz-Thorin theorem known as the complex method that we will completely ignore.

2.1 Compatible pairs of Banach spaces

We begin by laying the groundwork for the construction of the interpolation spaces. The crucial observation was described in the overview: if $(X, \mathfrak{M}, \mu)$ is a measure space and $1 < p < \infty$, then

$$L^1(X;\mathbb{F}) \cap L^\infty(X;\mathbb{F}) \hookrightarrow L^p(X;\mathbb{F}) \hookrightarrow L^1(X;\mathbb{F}) + L^\infty(X;\mathbb{F}). \tag{2.1.1}$$

In fact, it's possible to prove the same result with $L^p(X;\mathbb{F})$ replaced by $L^{p,q}(X;\mathbb{F})$ for any $1 \le q \le \infty$. What this tells us is that the spaces we use for interpolation always nest between the extreme spaces $L^1(X;\mathbb{F}) \cap L^\infty(X;\mathbb{F})$ and $L^1(X;\mathbb{F}) + L^\infty(X;\mathbb{F})$. This suggests that if we want to extend our theory we should start by looking at pairs of Banach spaces X_0 and X_1 for which we can make sense of $X_0 \cap X_1$ and $X_0 + X_1$. We turn our attention to this now.

2.1.1 Reminders about the basics of Banach spaces

We first record a couple quick reminders about the structure of general Banach spaces. The first gives a characterization of completeness in terms of infinite sums.

Theorem 2.1.1. *Let X be a normed vector space. The following are equivalent.*

1. *X is Banach.*
2. *If $\{x_n\}_{n=\ell}^\infty \subseteq X$ and $\sum_{n=\ell}^\infty \|x_n\|$ is convergent in $\mathbb{R}$, then $\sum_{n=\ell}^\infty x_n$ is convergent in X.*

Proof. Suppose first that X is Banach and that $\sum_{n=\ell}^\infty \|x_n\|$ is convergent in $\mathbb{R}$. Let $\varepsilon > 0$. Since $\mathbb{R}$ is complete we may choose $N \ge \ell$ such that $m \ge n \ge N$ implies that

$$\left\| \sum_{k=n}^m x_k \right\| \le \sum_{k=n}^m \|x_k\| < \varepsilon \tag{2.1.2}$$

and hence the partial sums $\{\sum_{n=\ell}^N x_n\}_{N=\ell}^\infty$ are Cauchy and hence convergent due to the completeness of X. Hence $\sum_{n=\ell}^\infty x_n$ converges in X.

Now suppose that (2) holds and let $\{x_n\}_{n=\ell}^\infty \subseteq X$ be Cauchy. We may extract a subsequence $\{x_{n_k}\}_{k=\ell}^\infty$ such that $\left\|x_{n_{k+1}} - x_{n_k}\right\| < 2^{-k}$ for all $k \ge \ell$. Set $y_k = x_{n_{k+1}} - x_{n_k}$ for $k \ge \ell$. Then

$$\sum_{k=\ell}^\infty \|y_k\| \le \sum_{k=\ell}^\infty 2^{-k} < \infty, \tag{2.1.3}$$

and so by (2) we have that $\sum_{k=\ell}^\infty y_k$ converges in X. However,

$$\sum_{k=\ell}^K y_k = x_{n_{K+1}} - x_{n_\ell} \Rightarrow \lim_{k\to\infty} x_{n_k} = x_{n_\ell} + \sum_{k=\ell}^\infty y_k \in X, \tag{2.1.4}$$

i.e. the subsequence $\{x_{n_k}\}_{k=\ell}^{\infty}$ converges in X. Basic real analysis tells us that Cauchy sequences with convergent subsequences must be convergent, so we find that $\{x_n\}_{n=\ell}^{\infty}$ is convergent. Hence X is complete.

□

Our second result records two useful ways to check that we have an embedding $X \hookrightarrow Y$ when X and Y are Banach spaces.

Theorem 2.1.2. *Let X and Y be Banach spaces and suppose that $X \subseteq Y$ as a vector subspace. Then the following are equivalent.*

1. *$X \hookrightarrow Y$, i.e. the inclusion map $I : X \to Y$ is continuous.*
2. *There exists a constant $C > 0$ such that $\|x\|_Y \le C \|x\|_X$ for all $x \in X$.*
3. *If $\{x_n\}_{n=\ell}^{\infty} \subseteq X$, $x_n \to x$ in X, and $x_n \to y$ in Y, then $x = y$.*

Proof. The equivalence of the first and second follow from the equivalence of boundedness and continuity for linear maps, and the equivalence and the second and third follow from the closed graph theorem. □

2.1.2 Compatible Banach spaces and their sums and intersections

If we want to form the intersection $X_0 \cap X_1$ and the sum $X_0 + X_1$ for general pairs of spaces X_0 and X_1, then we clearly need that the vector space structure of these spaces are compatible in some way. In fact, it will be crucial to have a sort of topological compatibility condition as well. We define this notion now.

Definition 2.1.3. *Suppose that X_0, X_1 are Banach spaces over the field $\mathbb{F}$.*

1. *We say X_0 and X_1 are compatible if there exists a Hausdorff topological vector space Z over $\mathbb{F}$ such that $X_i \hookrightarrow Z$ is a subspace for $i = 0, 1$ and the inclusions are continuous.*
2. *Suppose that X_0, X_1 are compatible. We define*

$$X_0 \cap X_1 = \{z \in Z \mid z \in X_0 \text{ and } z \in X_1\} \tag{2.1.5}$$

and

$$X_0 + X_1 = \{z \in Z \mid z = x_0 + x_1 \text{ for } x_0 \in X_0 \text{ and } x_1 \in X_1\}. \tag{2.1.6}$$

Define $\|\cdot\|_{X_0 \cap X_1} : X_0 \cap X_1 \to \mathbb{R}$ via

$$\|x\|_{X_0 \cap X_1} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\} \tag{2.1.7}$$

and define $\|\cdot\|_{X_0 + X_1} : X_0 + X_1 \to \mathbb{R}$ via

$$\|x\|_{X_0 + X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} \mid x = x_0 + x_1 \text{ for } x_0 \in X_0 \text{ and } x_1 \in X_1\}. \tag{2.1.8}$$

Note: we will sometimes write $\|\cdot\|_{0+1} = \|\cdot\|_{X_0+X_1}$ and $\|\cdot\|_{0\cap 1} = \|\cdot\|_{X_0 \cap X_1}$ as shorthand.

We can now mimic the usual proof that $L^{p_0} \cap L^{p_1}$ and $L^{p_0} + L^{p_1}$ are complete in this more general context.

Theorem 2.1.4. *Suppose that X_0 and X_1 are compatible Banach spaces. Then the following hold.*

1. $\|\cdot\|_{X_0 \cap X_1}$ *is a norm, and* $X_0 \cap X_1$ *is a Banach space when equipped with this norm.*

2. $\|\cdot\|_{X_0 + X_1}$ *is a norm, and* $X_0 + X_1$ *is a Banach space when equipped with this norm.*

3. *We have the continuous embeddings*

$$X_0 \cap X_1 \hookrightarrow X_i \hookrightarrow X_0 + X_1 \text{ for } i = 0, 1. \tag{2.1.9}$$

Proof. We begin with the proof of the first item. The fact that $\|\cdot\|_{X_0 \cap X_1}$ is a norm follows directly from fact that $\|\cdot\|_{X_i}$ is a norm for $i = 0, 1$; we leave it as an exercise to check the details. Consider a Cauchy sequence $\{x_n\}_{n=\ell}^{\infty} \subset X_0 \cap X_1$. Then the definition of the norm shows that the sequence is Cauchy in both X_0 and in X_1, and since these are both assumed to be complete, there exist $x \in X_0$ and $y \in X_1$ such that $x_n \to x$ in X_0 and $x_n \to y$ in X_1 as $n \to \infty$. Due to the continuous embeddings $X_i \hookrightarrow Z$ for $i = 0, 1$ we have that $x_n \to x$ and $x_n \to y$ in Z, but limits in Z are unique since Z is Hausdorff, and hence $x = y \in X_0 \cap X_1$. From this we readily deduce that

$$\lim_{n \to \infty} \|x - x_n\|_{X_0 \cap X_1} = 0 \tag{2.1.10}$$

and so $X_0 \cap X_1$ is a Banach space.

We now turn to the proof of the second item, which is a bit more involved than the proof of the first. First we show that $\|\cdot\|_{X_0 + X_1}$ is a norm. Clearly $\|x\|_{X_0 + X_1} \geq 0$ and $\|0\|_{X_0 + X_1} = 0$. On the other hand, if $\|x\|_{X_0 + X_1} = 0$ then for every $n \in \mathbb{N}$ we can find $y_n \in X_0$ and $z_n \in X_1$ such that $x = y_n + z_n$ and

$$\|y_n\|_{X_0} + \|z_n\|_{X_1} < 2^{-n}. \tag{2.1.11}$$

Then $y_n \to 0$ in X_0 and $z_n \to 0$ in X_1 and since the inclusions $X_i \hookrightarrow Z$ are continuous we have that $x = y_n + z_n \to 0$ as $n \to \infty$ in Z, and so $x = 0$. Homogeneity and the triangle inequality are easy and left as an exercise to verify. Thus, $\|\cdot\|_{X_0 + X_1}$ is a norm.

We now prove completeness for $X_0 + X_1$. Consider a sequence $\{x_n\}_{n=\ell}^{\infty} \subseteq X_0 + X_1$ such that $\sum_{n=\ell}^{\infty} \|x_n\|_{X_0 + X_1} < \infty$. For each $n \geq \ell$ we can pick $y_n \in X_0$ and $z_n \in X_1$ such that

$$\|y_n\|_{X_0} + \|z_n\|_{X_1} < \|x_n\|_{X_0 + X_1} + 2^{-n}, \tag{2.1.12}$$

which in turn means that

$$\sum_{n=\ell}^{\infty} \|y_n\|_{X_0} + \sum_{n=\ell}^{\infty} \|z_n\|_{X_1} < \infty. \tag{2.1.13}$$

Since X_0 and X_1 are Banach, we can use Theorem 2.1.1 to see that we have the convergence

$$\sum_{n=\ell}^{\infty} y_n = y \text{ in } X_0 \text{ and } \sum_{n=\ell}^{\infty} z_n = z \text{ in } X_1. \tag{2.1.14}$$

Set $x = y + z \in X_0 + X_1$ and note that for $N \geq \ell$ we have

$$x - \sum_{n=\ell}^{N} x_n = \left(y - \sum_{n=\ell}^{N} y_n \right) + \left(z - \sum_{n=\ell}^{N} z_n \right), \tag{2.1.15}$$

which implies that

$$\left\| x - \sum_{n=\ell}^{N} x_n \right\|_{X_0 + X_1} \leq \left\| y - \sum_{n=\ell}^{N} y_n \right\|_{X_0} + \left\| z - \sum_{n=\ell}^{N} z_n \right\|_{X_1} \to 0 \text{ and } N \to \infty. \tag{2.1.16}$$

Thus, $\sum_{n=\ell}^{\infty} x_n = x \in X_0 + X_1$, where the sum convergences in the $X_0 + X_1$ norm, and so by Theorem 2.1.1 we conclude that $X_0 + X_1$ is Banach. This proves the second item.

Finally, we prove the third item. The subspace inclusions $X_0 \cap X_1 \subseteq X_i \subseteq X_0 + X_1$ are obvious, so we only need to prove continuity. We have the bound

$$\|x\|_{X_i} \leq \|x\|_{X_0 \cap X_1} \text{ for every } x \in X_0 \cap X_1, \tag{2.1.17}$$

which proves that $X_0 \cap X_1 \hookrightarrow X_i$. On the other hand, if $x \in X_i$ then $x = x + 0$ and so

$$\|x\|_{X_0+X_1} \leq \|x\|_{X_i}, \tag{2.1.18}$$

which shows that $X_i \hookrightarrow X_0 + X_1$.

□

Remark 2.1.5. *Notice that in the proof we don't really exploit the assumption that Z is a topological vector space in the sense that we never use the continuity of scalar multiplication. This shows that we could in principle weaken the assumptions on Z in the notion of compatibility and require that Z is merely an Abelian topological group in which X_0 and X_1 are subgroups with respect to the additive group structure of these vector spaces.*

2.1.3 Intermediate spaces

We now have the ability to form the Banach spaces $X_0 \cap X_1$ and $X_0 + X_1$ when X_0 and X_1 are compatible. As the next step we introduce the notion of a space intermediate to these, which generalizes the relations we saw in (2.1.1).

Definition 2.1.6. *Suppose that X_0 and X_1 are compatible Banach spaces. A Banach space X is said to be intermediate to X_0 and X_1 if*

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1. \tag{2.1.19}$$

Let's consider some examples.

Example 2.1.7. $X_0 \cap X_1$ and $X_0 + X_1$ are both trivially intermediate to X_0 and X_1. △

Example 2.1.8. Theorem 2.1.4 shows that both X_0 and X_1 are intermediate to X_0 and X_1. △

Example 2.1.9. From (2.1.1) we know that $L^p(X;\mathbb{F})$ is intermediate to $L^1(X;\mathbb{F})$ and $L^\infty(X;\mathbb{F})$ when $1 < p < \infty$. The same is true of $L^{p,q}(X;\mathbb{F})$ for $1 < p < \infty$ and $1 \leq q \leq \infty$. △

Given a pair of intermediate spaces Y_0 and Y_1 between compatible spaces X_0 and X_1 we clearly have that Y_0 and Y_1 are compatible, and so we can repeat our previous constructions. The next elementary result shows how the resulting spaces sit inside the existing ones.

Proposition 2.1.10. *Let X_0 and X_1 be compatible Banach spaces, and suppose that Y_0 and Y_1 are intermediate to X_0 and X_1. Then $Y_0 \cap Y_1$ and $Y_0 + Y_1$ are intermediate to X_0 and X_1 and we have the following embeddings:*

$$\begin{array}{ccccccc} & & & Y_0 & & & \\ & & \nearrow & & \searrow & & \\ X_0 \cap X_1 & \hookrightarrow & Y_0 \cap Y_1 & & & Y_0 + Y_1 & \hookrightarrow X_0 + X_1. \\ & & \searrow & & \nearrow & & \\ & & & Y_1 & & & \end{array} \tag{2.1.20}$$

Proof. Exercise. □

Given three topological vector spaces that nest via $X \hookrightarrow Y \hookrightarrow Z$ we have two natural operations we perform to construct new spaces: we can take the closure of X in Y, and we can take the closure of Y in Z. We now give this idea a special name in the context of intermediate spaces.

Definition 2.1.11. *Suppose that X_0 and X_1 are compatible Banach spaces and that X is intermediate to X_0 and X_1. We define the space $\underline{X}$ as the closure of $X_0 \cap X_1$ in X, and the space $\bar{X}$ as the closure of X is $X_0 + X_1$. The space $\underline{X}$ is called the lower closure and $\bar{X}$ is called the upper closure. Clearly, $\underline{X}$ and $\bar{X}$ are Banach, and we have the continuous inclusions*

$$X_0 \cap X_1 \hookrightarrow \underline{X} \hookrightarrow X \hookrightarrow \bar{X} \hookrightarrow X_0 + X_1, \tag{2.1.21}$$

which in particular means that $\underline{X}$ and $\bar{X}$ are intermediate to X_0 and X_1.

At this point the only spaces that we know are intermediate to a compatible pair X_0 and X_1 are X_0, X_1, $X_0 \cap X_1$, and $X_0 + X_1$. It turns out that we can characterize the upper and lower closures of these in terms of sum and intersection operations in a very nice way. Indeed, we have the following fundamental result.

Theorem 2.1.12. *Let X_0 and X_1 be compatible Banach spaces. Then the following hold.*

1. $\underline{X}_0 = X_0 \cap \bar{X}_1$ *and* $\underline{X}_1 = \bar{X}_0 \cap X_1$.
2. $\bar{X}_0 = X_0 + \underline{X}_1$ *and* $\bar{X}_1 = \underline{X}_0 + X_1$.
3. $\underline{X}_0 \cap \underline{X}_1 = \underline{X}_0 \cap X_1 = X_0 \cap \underline{X}_1 = X_0 \cap X_1$, $\underline{X}_0 \cap \bar{X}_1 = \underline{X}_0$, *and* $\bar{X}_0 \cap \underline{X}_1 = \underline{X}_1$.
4. $\bar{X}_0 + \bar{X}_1 = \bar{X}_0 + X_1 = X_0 + \bar{X}_1 = X_0 + X_1$, $\underline{X}_0 + \bar{X}_1 = \bar{X}_1$, *and* $\bar{X}_0 + \underline{X}_1 = \bar{X}_0$.
5. $\underline{X_0 + X_1} = \overline{X_0 \cap X_1} = \bar{X}_0 \cap \bar{X}_1 = \underline{X}_0 + \underline{X}_1$.

Proof. We begin with a bit of notation. If A and B are normed vector spaces and we have the subspace inclusion $A \subseteq B$, then we write $\mathrm{cl}(A, B) \subseteq B$ for the closure of A in the topology of B. We leave it as an exercise to verify that if $A \hookrightarrow B \hookrightarrow C$ for normed vector spaces A, B, and C, then

$$\mathrm{cl}(A, B) \hookrightarrow \mathrm{cl}(A, C) \text{ and } \mathrm{cl}(A, C) \hookrightarrow \mathrm{cl}(B, C). \tag{2.1.22}$$

From (2.1.22) and the fact that each X_i is intermediate to X_0 and X_1 we see that

$$\underline{X}_i = \mathrm{cl}(X_0 \cap X_1, X_i) \hookrightarrow \mathrm{cl}(X_i, X_i) = X_i \tag{2.1.23}$$

and

$$\underline{X}_i = \mathrm{cl}(X_0 \cap X_1, X_i) \hookrightarrow \mathrm{cl}(X_0 \cap X_1, X_0 + X_1) \hookrightarrow \mathrm{cl}(X_{1-i}, X_0 + X_1) = \bar{X}_{1-i} \tag{2.1.24}$$

so that

$$\underline{X}_i \hookrightarrow X_i \cap \bar{X}_{1-i}. \tag{2.1.25}$$

To complete the proof of the first item it then suffices to show that this embedding is surjective.

Fix $i \in \{0, 1\}$ and let $x \in X_i \cap \bar{X}_{1-i}$. Then for every $\varepsilon > 0$ there exists $x_\varepsilon \in X_{1-i}$ such that $\|x - x_\varepsilon\|_{0+1} < \varepsilon$, which in turn allows us to choose $y_\varepsilon \in X_i$ and $z_\varepsilon \in X_{1-i}$ such that $x - x_\varepsilon = y_\varepsilon + z_\varepsilon$ and

$$\|y_\varepsilon\|_i + \|z_\varepsilon\|_{1-i} < \varepsilon. \tag{2.1.26}$$

Upon rearranging, we find that

$$X_i \ni x - y_\varepsilon = x_\varepsilon + z_\varepsilon \in X_{1-i}, \text{ and hence } x_\varepsilon + z_\varepsilon \in X_0 \cap X_1. \tag{2.1.27}$$

Then

$$\|x - (x_\varepsilon + z_\varepsilon)\|_i = \|y_\varepsilon\|_i < \varepsilon \text{ for all } \varepsilon > 0 \tag{2.1.28}$$

and we deduce that $x \in \text{cl}(X_0 \cap X_1, X_i) = \underline{X}_i$. Thus the embedding (2.1.25) is surjective, and the first item is proved.

For the second item we first note that by (2.1.24) we have

$$X_{1-i} + \underline{X}_i \hookrightarrow X_{1-i} + \bar{X}_{1-i} = \bar{X}_{1-i}. \tag{2.1.29}$$

We claim that this embedding is a surjection. Indeed, for a fixed $i \in \{0, 1\}$ and $x \in \bar{X}_{1-i} \subseteq X_0 + X_1$ we can write $x = x_0 + x_1$ for $x_0 \in X_0$ and $x_1 \in X_1$. Then $x_i = x - x_{1-i} \in \bar{X}_{1-i} \cap X_i$, but by the first item $\bar{X}_{1-i} \cap X_i = \underline{X}_i$, so

$$x = x_{1-i} + x_i \in X_{1-i} + \underline{X}_i. \tag{2.1.30}$$

This proves the claim, which then completes the proof of the second item.

The third and fourth items follows directly from the first and second, so it remains only to prove the fifth. From the first item we have

$$\underline{X}_0 + \underline{X}_1 = X_0 \cap \bar{X}_1 + \bar{X}_0 \cap X_1 \hookrightarrow \bar{X}_0 \cap \bar{X}_1 + \bar{X}_0 \cap \bar{X}_1 = \bar{X}_0 \cap \bar{X}_1. \tag{2.1.31}$$

Let $x \in \bar{X}_0 \cap \bar{X}_1$. By the second item we can then find $a \in X_0$, $b \in \underline{X}_1$, $c \in \underline{X}_0$, and $d \in X_1$ such that

$$x = a + b = c + d. \tag{2.1.32}$$

Upon rearranging, this implies that

$$a - c = d - b \in X_0 \cap X_1, \tag{2.1.33}$$

and hence

$$a = c + (a - c) \in \underline{X}_0 + X_0 \cap X_1 \subseteq \underline{X}_0, \tag{2.1.34}$$

so

$$x = a + b \in \underline{X}_0 + \underline{X}_1 \tag{2.1.35}$$

and thus the embedding (2.1.31) is surjective, so $\underline{X}_0 + \underline{X}_1 = \bar{X}_0 \cap \bar{X}_1$.

Next we note that (2.1.22) shows

$$\underline{X_0 + X_1} = \overline{X_0 \cap X_1} = \text{cl}(X_0 \cap X_1, X_0 + X_1) \hookrightarrow \text{cl}(X_i, X_0 + X_1) = \bar{X}_i \text{ for } i = 0, 1 \tag{2.1.36}$$

so

$$\overline{X_0 \cap X_1} \hookrightarrow \bar{X}_0 \cap \bar{X}_1. \tag{2.1.37}$$

To conclude we will show that this embedding is also surjective. Let $x \in \bar{X}_0 \cap \bar{X}_1$. Since $x \in \bar{X}_0$, for each $\varepsilon > 0$ we can find $x_\varepsilon \in X_0$, $a_\varepsilon \in X_0$, and $b_\varepsilon \in X_1$ such that $x - x_\varepsilon = a_\varepsilon + b_\varepsilon$ and $\|a_\varepsilon\|_0 + \|b_\varepsilon\|_1 < \varepsilon$. Similarly, since $x \in \bar{X}_1$, for each $\varepsilon > 0$ we can find $y_\varepsilon \in X_1$, $c_\varepsilon \in X_0$, and $d_\varepsilon \in X_1$ such that $x - y_\varepsilon = c_\varepsilon + d_\varepsilon$ and $\|c_\varepsilon\|_0 + \|d_\varepsilon\|_1 < \varepsilon$. Then

$$x_\varepsilon + a_\varepsilon + b_\varepsilon = x = y_\varepsilon + c_\varepsilon + d_\varepsilon, \tag{2.1.38}$$

and upon rearranging we see that

$$x_\varepsilon + a_\varepsilon - c_\varepsilon = -b_\varepsilon + y_\varepsilon + d_\varepsilon \in X_0 \cap X_1. \tag{2.1.39}$$

Then $x - (x_\varepsilon + a_\varepsilon - c_\varepsilon) = b_\varepsilon + c_\varepsilon$ and

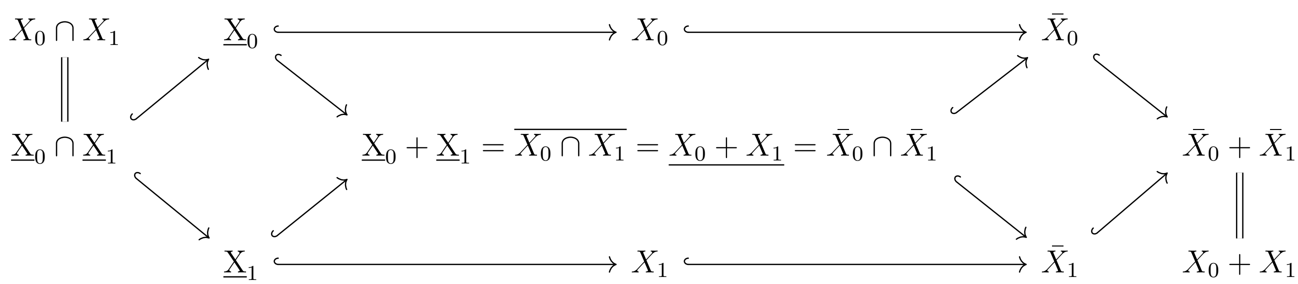
$$\|x - (x_\varepsilon + a_\varepsilon - c_\varepsilon)\|_{0+1} \le \|c_\varepsilon\|_0 + \|b_\varepsilon\|_1 < 2\varepsilon, \tag{2.1.40}$$

from which we deduce that $x \in \overline{X_0 \cap X_1}$, and (2.1.37) is surjective. Hence, $\overline{X_0 \cap X_1} = \bar{X}_0 \cap \bar{X}_1$, and the fifth item is proved. □

The final item of this theorem shows that there is a special role played by the single space

$$\underline{X_0 + X_1} = \overline{X_0 \cap X_1} = \bar{X}_0 \cap \bar{X}_1 = \underline{X}_0 + \underline{X}_1. \tag{2.1.41}$$

The following diagram summarizes the relation of this space to the others we have discussed.



(2.1.42)

We might hope at this point that we could iterate the constructions we've done so far to get even more spaces. For instance, we could take the upper closure of $\underline{X}_i$ or the lower closure of $\bar{X}_i$. However, the above diagram indicates another crucial fact: iterating these constructions doesn't produce anything else. Indeed, it immediately shows the following.

Corollary 2.1.13. *Let X_0 and X_1 be compatible Banach spaces. Then for $i \in \{0, 1\}$, the upper closure of $\underline{X}_i$, the lower closure of $\bar{X}_i$, and the space $\underline{X_0 + X_1} = \overline{X_0 \cap X_1} = \bar{X}_0 \cap \bar{X}_1 = \underline{X}_0 + \underline{X}_1$ coincide.*

The upshot of this analysis is that while the upper and lower closure operations do allow us to produce five new spaces intermediate to X_0 and X_1, we are stuck with these five and can proceed no further with the closure, sum, and intersection operations alone. If we want to construct more intermediate spaces, we need a new idea. We will turn to the development of this idea momentarily, but first we record another simple consequences of the above diagram.

Corollary 2.1.14. *Let X_0 and X_1 be compatible Banach spaces. The following hold.*

1. *If $X_i \hookrightarrow X_{1-i}$, then*

$$X_i \hookrightarrow \bar{X}_i = \underline{X}_{1-i} = \overline{X_0 \cap X_1} = \underline{X_0 + X_1} \hookrightarrow X_{1-i}. \tag{2.1.43}$$

2. *If $X_0 \cap X_1$ is a closed subspace of $X_0 + X_1$, then*

$$X_0 \cap X_1 = \underline{X}_0 = \underline{X}_1 = \underline{X_0 + X_1} = \overline{X_0 \cap X_1} = \bar{X}_0 \cap \bar{X}_1 = \underline{X}_0 + \underline{X}_1. \tag{2.1.44}$$

3. *If $X_i \subseteq X_{1-i}$ is a closed subspace, then*

$$X_i = \underline{X}_0 = \underline{X}_1 = \underline{X_0 + X_1} = \overline{X_0 \cap X_1} = \bar{X}_0 \cap \bar{X}_1 = \underline{X}_0 + \underline{X}_1 \subseteq X_{1-i}. \tag{2.1.45}$$

Proof. Exercise. □

2.1.4 The K and J functions

Our starting point for constructing more intermediate spaces to a given compatible pair of Banach spaces X_0 and X_1 is the simple realization that we can introduce a positive real parameter's worth of equivalent norms on the spaces $X_0 \cap X_1$ and $X_0 + X_1$. We give these a special name now.

Definition 2.1.15. *Let X_0 and X_1 be compatible Banach spaces. We define the maps $K : (X_0 + X_1) \times \mathbb{R}_+ \to \mathbb{R}$ and $J : (X_0 \cap X_1) \times \mathbb{R}_+ \to \mathbb{R}$ via*

$$K(x,t) = \inf\{\|x_0\|_0 + t\|x_1\|_1 \mid x = x_0 + x_1 \text{ for } x_i \in X_i, i = 0,1\} \tag{2.1.46}$$

and

$$J(x,t) = \max\{\|x\|_0, t\|x\|_1\}. \tag{2.1.47}$$

We begin our study of these new functions with the following proposition, which records the basic properties of the K function.

Proposition 2.1.16. *Let X_0 and X_1 be compatible Banach spaces. Then the following hold.*

1. *For each $t \in \mathbb{R}_+$ the map $X_0 + X_1 \ni x \mapsto K(x,t) \in \mathbb{R}$ is a norm that's equivalent to the usual norm.*

2. *For each $x \in X_0 + X_1$ the map $K(x,\cdot)$ is continuous, nondecreasing, and concave.*

3. *For each $x \in X_0 + X_1$, the map $\mathbb{R}_+ \ni t \mapsto K(x,t)/t \in \mathbb{R}$ is nonincreasing, and we have the inequalities*

$$\min\{1, t/s\}K(x,s) \le K(x,t) \le \max\{1, t/s\}K(x,s) \text{ for all } t,s \in \mathbb{R}_+. \tag{2.1.48}$$

4. *Let $\tilde{K} : (X_0 + X_1) \times \mathbb{R}_+ \to \mathbb{R}$ be the K map obtained by flipping the indices on X_0 and X_1, i.e.*

$$\tilde{K}(x,t) = \inf\{\|x_1\|_1 + t\|x_0\|_0 \mid x = x_0 + x_1 \text{ for } x_i \in X_i, i = 0,1\}. \tag{2.1.49}$$

Then

$$\frac{K(x,t)}{t} = \tilde{K}(x, 1/t) \text{ for all } x \in X_0 + X_1 \text{ and } t \in \mathbb{R}_+. \tag{2.1.50}$$

Proof. The fact that $K(\cdot,t)$ defines a norm is left as an exercise. Then the first item follows from the simple inequality

$$\min\{1,t\}\|x\|_{X_0+X_1} \le K(x,t) \le \max\{1,t\}\|x\|_{X_0+X_1}. \tag{2.1.51}$$

Now let $x = x_0 + x_1$ for $x_i \in X_i$ and suppose that $0 < t < s < \infty$. Then

$$K(x,t) \le \|x_0\|_0 + t\|x_1\|_1 \le \|x_0\|_0 + s\|x_1\|_1, \tag{2.1.52}$$

but since this holds for all such decompositions, we deduce that

$$K(x,t) \le K(x,s). \tag{2.1.53}$$

Similarly, for $\theta \in [0,1]$ we have that

$$\theta K(x,t) + (1-\theta)K(x,s) \le \theta[\|x_0\|_0 + t\|x_1\|_1] + (1-\theta)[\|x_0\|_0 + s\|x_1\|_1] = \|x_0\|_0 + [\theta t + (1-\theta)s]\|x_1\|_1, \tag{2.1.54}$$

and since this holds for all such decompositions, we deduce that

$$\theta K(x,t) + (1-\theta)K(x,s) \le K(x, \theta t + (1-\theta)s). \tag{2.1.55}$$

This proves all of the second item except continuity, which we delay momentarily.

For the third item again let $x = x_0 + x_1$ with $x_i \in X_i$ and let $s, t \in (0, \infty)$. If $t \le s$ then the first item shows that $K(x,t) \le K(x,s) = \max\{1, t/s\}K(x,s)$. On the other hand, if $s < t$, then

$$\frac{s}{t}K(x,t) \le \frac{s}{t}[\|x_0\|_0 + t\|x_1\|_1] = \frac{s}{t}\|x_0\|_0 + s\|x_1\|_1 \le \|x_0\|_0 + s\|x_1\|_1, \tag{2.1.56}$$

and since this holds for all such decompositions we find that $\frac{s}{t}K(x,t) \le K(x,s)$, which implies that $K(x,t) \le \max\{1, t/s\}K(x,s)$. This proves the second stated estimate, but the first follows by reversing the roles of s and t. This proves the third item.

Finally, we complete the proof of the second item by proving that $K(x,\cdot)$ is continuous. Fix $s \in (0,\infty)$. Then

$$K(x,s) = \lim_{t \to s} \min\{1, t/s\}K(x,s) = \lim_{t\to s} \max\{1, t/s\}K(x,s), \tag{2.1.57}$$

so the inequalities of the third item show that $K(x,s) = \lim_{t\to s} K(x,t)$, and we conclude that $K(x,\cdot)$ is continuous. We leave the proof of the fourth item as an exercise.

□

Next we record a corresponding result on the basic properties of the J function.

Proposition 2.1.17. *Let X_0 and X_1 be compatible Banach spaces. Then the following hold.*

1. *For each $t \in \mathbb{R}_+$ the map $X_0 \cap X_1 \ni x \mapsto J(x,t) \in \mathbb{R}$ is a norm that's equivalent to the usual norm on $X_0 \cap X_1$.*
2. *For each $x \in X_0 \cap X_1$ the map $J(x,\cdot)$ is continuous, nondecreasing, and convex.*
3. *For each $x \in X_0 \cap X_1$, the map $\mathbb{R}_+ \ni t \mapsto J(x,t)/t \in \mathbb{R}$ is nonincreasing, and we have the inequalities*

$$\min\{1, t/s\}J(x,s) \le J(x,t) \le \max\{1, t/s\}J(x,s) \text{ for all } t, s \in \mathbb{R}_+. \tag{2.1.58}$$

4. *Let $\tilde{J} : (X_0 \cap X_1) \times \mathbb{R}_+ \to \mathbb{R}$ be the J map obtained by flipping the indices on X_0 and X_1, i.e.*

$$\tilde{J}(x,t) = \max\{\|x\|_1, t\|x\|_0\}. \tag{2.1.59}$$

Then

$$\frac{J(x,t)}{t} = \tilde{J}(x, 1/t) \text{ for all } x \in X_0 \cap X_1 \text{ and } t \in \mathbb{R}_+. \tag{2.1.60}$$

Proof. We will only prove the second and third items and leave the first and fourth as exercises. If $s \le t$ then $s\|x\|_1 \le t\|x\|_1 \le J(x,t)$ and $\|x\|_0 \le J(x,t)$, so $J(x,s) \le J(x,t)$, which shows that $J(x,\cdot)$ is nondecreasing.

If $s,t \in \mathbb{R}_+$ and $\theta \in [0,1]$, then $[\theta s + (1-\theta)t]\|x\|_1 \le \theta J(x,s) + (1-\theta)J(x,t)$, while $\|x\|_0 = \theta\|x\|_0 + (1-\theta)\|x\|_0 \le \theta J(x,s) + (1-\theta)J(x,t)$, so $J(x, \theta s + (1-\theta)t) \le \theta J(x,s) + (1-\theta)J(x,t)$. Thus $J(x,\cdot)$ is convex.

If $t \le s$ then $J(x,t) \le J(x,s)$ by the above. On the other hand if $s < t$, then

$$\frac{s}{t}J(x,t) = \max\{(s/t)\left\|x\right\|_0, s\left\|x\right\|_1\} \le \max\{\left\|x\right\|_0, s\left\|x\right\|_1\} = J(x,s). \tag{2.1.61}$$

Thus $J(x,t) \le \max\{1, t/s\}J(x,s)$. The estimates of the third item follow from this and the inequality obtained by reversing t and s. This also shows that $\mathbb{R}_+ \ni t \mapsto J(x,t)/t \in \mathbb{R}$ is nondecreasing. Continuity follows directly from the bounds of the third item, and the proof of the second and third items is complete. □

The K and J functions are related in a useful way, as the following result shows.

Proposition 2.1.18. *Let X_0 and X_1 be compatible Banach spaces. The for each $x \in X_0 \cap X_1$ and $s, t \in \mathbb{R}_+$ we have that $K(x,s) \le \min\{1, s/t\}J(x,t)$.*

Proof. Since $x \in X_0 \cap X_1$ we have that $K(x,s) \le \left\|x\right\|_0 \le J(x,t)$ and $K(x,s) \le s\left\|x\right\|_1 = (s/t)t\left\|x\right\|_1 \le (s/t)J(x,t)$, so

$$K(x,s) \le \min\{J(x,t), (s/t)J(x,t)\} = \min\{1, s/t\}J(x,t). \tag{2.1.62}$$

□

It turns out that the K function's asymptotic behavior at 0 and ∞ encodes some useful information related to inclusion in spaces that we have seen before. In turn, this information is equivalent to an extremely useful decomposition that will be essential in our subsequent work with the K and J functions. We record this result now.

Theorem 2.1.19. *Let X_0 and X_1 be compatible Banach spaces. Then the following hold for $x \in X_0 + X_1$.*

1. $\lim_{t \to 0} K(x,t) = 0$ *if and only if* $x \in \bar{X}_1$.
2. $\lim_{t \to \infty} K(x,t)/t = 0$ *if and only if* $x \in \bar{X}_0$.
3. *The following are equivalent.*
   (a) $\lim_{t \to 0} K(x,t) = \lim_{t \to \infty} K(x,t)/t = 0$.
   (b) $x \in \overline{X_0 \cap X_1} = \underline{X_0 + X_1}$.
   (c) *There exists a sequence $\{x_n\}_{n \in \mathbb{Z}} \subseteq X_0 \cap X_1$ such that $x = \sum_{n \in \mathbb{Z}} x_n$, where the series converges in $X_0 + X_1$.*

   Moreover, if any (and hence all) of these holds, then for any $1 < r < \infty$ the sequence $\{x_n\}_{n \in \mathbb{Z}}$ in (c) can be chosen such that $J(x_n, r^n) \le 2(1+r)K(x, r^n)$ for each $n \in \mathbb{Z}$.

Proof. We begin by proving the first item. Suppose that $\lim_{t \to 0} K(x,t) = 0$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < t < \delta$ implies $K(x,t) < \varepsilon$. In particular, for every $\varepsilon > 0$ we can pick $y_\varepsilon \in X_0$ and $z_\varepsilon \in X_1$ such that $\left\|y_\varepsilon\right\|_0 + (\delta/2)\left\|z_\varepsilon\right\|_1 < \varepsilon$. In turn, this implies that

$$\left\|x - z_\varepsilon\right\|_{0+1} \le \left\|y_\varepsilon\right\|_0 < \varepsilon \text{ for every } \varepsilon > 0, \tag{2.1.63}$$

and since $z_\varepsilon \in X_1$ we deduce from this that $x \in \bar{X}_1$.

Conversely, assume that $x \in \bar{X}_1$. Then for every $\varepsilon > 0$ we can pick $x_\varepsilon \in X_1$ such that $\|x - x_\varepsilon\|_{0+1} < \varepsilon$. In turn, we may choose $y_\varepsilon \in X_0$ and $z_\varepsilon \in X_1$ such that $x - x_\varepsilon = y_\varepsilon + z_\varepsilon$ and $\|y_\varepsilon\|_0 + \|z_\varepsilon\|_1 < \varepsilon$. Then $x = y_\varepsilon + (x_\varepsilon + z_\varepsilon)$ and for any $t \in (0, \infty)$ we may estimate

$$K(x,t) \leq \|y_\varepsilon\|_0 + t\,\|x_\varepsilon + z_\varepsilon\|_1 < \varepsilon + t\,\|x_\varepsilon + z_\varepsilon\|_1 . \tag{2.1.64}$$

Hence,

$$\lim_{t\to 0} K(x,t) \leq \varepsilon \text{ for every } \varepsilon > 0, \text{ so } \lim_{t\to 0} K(x,t) = 0. \tag{2.1.65}$$

This completes the proof of the first item.

The second item follows from the first and the fourth item of Proposition 2.1.16. The equivalence of (a) and (b) in the third item follows from the first two items and Theorem 2.1.12. We will show that $(a) \Leftrightarrow (c)$ to conclude. These are clearly equivalent if $x = 0$, so we may assume without loss of generality that $x \neq 0$.

Suppose that (a) holds and let $1 < r < \infty$. For each $n \in \mathbb{Z}$ we can choose $y_n \in X_0$ and $z_n \in X_1$ such that $x = y_n + z_n$ and

$$\|y_n\|_0 + r^n\,\|z_n\|_1 < 2K(x, r^n). \tag{2.1.66}$$

Then (a) implies that

$$\lim_{n\to -\infty} \|y_n\|_0 = \lim_{n\to\infty} \|z_n\|_1 = 0. \tag{2.1.67}$$

For $n \in \mathbb{Z}$ define

$$x_n = y_n - y_{n+1} = z_{n+1} - z_n \in X_0 \cap X_1. \tag{2.1.68}$$

By construction, for each $n \in \mathbb{Z}$ we have the estimate

$$\begin{aligned} J(x_n, r^n) &\leq \max\{\|y_n\|_0 + \|y_{n+1}\|_0\,, r^n\,\|z_n\|_1 + r^n\,\|z_{n+1}\|_1\} \\ &\leq \max\{2K(x,r^n) + 2K(x,r^{n+1}), 2K(x,r^n) + \frac{2}{r}K(x,r^{n+1})\} \\ &\leq \max\{2K(x,r^n) + 2\max\{1,r\}K(x,r^n), 2K(x,r^n) + \frac{2}{r}\max\{1,r\}K(x,r^n)\} \\ &= 2(1+r)K(x,r^n). \end{aligned} \tag{2.1.69}$$

Moreover, for $N \in \mathbb{N}$ we may compute

$$x - \sum_{|n|\leq N} x_n = x - y_N + y_{-N+1} = z_N + y_{-N+1}, \tag{2.1.70}$$

which then implies that

$$\left\| x - \sum_{|n|\leq N} x_n \right\|_{0+1} \leq \|y_{-N+1}\|_0 + \|z_N\|_1 \to 0 \text{ as } N \to \infty. \tag{2.1.71}$$

Thus, (c) holds, and we have proved that $(a) \Rightarrow (c)$.

Now suppose that (c) holds. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $\left\| x - \sum_{|n|\leq N} x_n \right\|_{0+1} < \varepsilon$. Then we may estimate

$$K(x,t) \leq K(x - \sum_{|n|\leq N} x_n, t) + K(\sum_{|n|\leq N} x_n, t) \leq \max\{1,t\} \left\| x - \sum_{|n|\leq N} x_n \right\|_{0+1} + t \left\| \sum_{|n|\leq N} x_n \right\|_1 \tag{2.1.72}$$

in order to see that $\lim_{t\to 0} K(x,t) \le \varepsilon$ for every $\varepsilon > 0$ and hence $\lim_{t\to 0} K(x,t) = 0$. Similarly, we may bound

$$\frac{K(x,t)}{t} \le \frac{K(x - \sum_{|n|\le N} x_n, t)}{t} + \frac{K(\sum_{|n|\le N} x_n, t)}{t} \\ \le \frac{\max\{1,t\}}{t} \left\| x - \sum_{|n|\le N} x_n \right\|_{0+1} + \frac{1}{t} \left\| \sum_{|n|\le N} x_n \right\|_0 \tag{2.1.73}$$

in order to see that $\lim_{t\to\infty} K(x,t)/t \le \varepsilon$ for every $\varepsilon > 0$ and hence $\lim_{t\to\infty} K(x,t)/t = 0$. Thus, (a) holds, and we have proved that $(c) \Rightarrow (a)$, which completes the proof of the third item.

□

2.2 Interpolating between compatible Banach spaces

We now have all of the tools needed to begin constructing new intermediate spaces. In fact, with what we now know about the K and J functions, there are many options available to us. We begin our discussion with some heuristic considerations in order to justify the definition we will ultimately work with.

2.2.1 Heuristics

We have now seen that for each $t \in \mathbb{R}_+$ the function $K(\cdot,t)$ defines an equivalent norm on $X_0 + X_1$ when X_0 and X_1 are compatible Banach spaces. Moreover, since $K(x,\cdot)$ is continuous for each $x \in X_0 + X_1$, it's measurable for any choice of a Radon measure on $\mathbb{R}_+$. These facts suggest that we might use some sort of Lebesgue norms relative to a Radon measure μ on $\mathbb{R}_+$ to build new spaces intermediate to X_0 and X_1. This leads us to the following definition.

Definition 2.2.1. *Let X_0 and X_1 be compatible Banach spaces. Suppose that μ is a Radon measure on $\mathbb{R}_+$ and consider a continuous weight function $w : \mathbb{R}_+ \to \mathbb{R}_+$. Let $1 \le p \le \infty$. We define $\|\cdot\|_{X(\mu,w,p)} : X_0 + X_1 \to [0,\infty]$ via*

$$\|x\|_{X(\mu,w,p)} = \|wK(x,\cdot)\|_{L^p_\mu} < \infty, \tag{2.2.1}$$

where we recall that $L^p_\mu(\mathbb{R}_+)$ denotes L^p on $\mathbb{R}_+$ with respect to the measure μ. We define the space

$$X(\mu,w,p) = \{x \in X_0 + X_1 \mid \|x\|_{X(\mu,w,p)} < \infty\}. \tag{2.2.2}$$

Note that the weight w can be absorbed into the measure when $1 \le p < \infty$, so the main utility of the weight is seen when $p = \infty$.

Now, based on our prior discussion of intermediate spaces, the natural question that arises is when we can guarantee that $X(\mu,w,p)$ is intermediate. To address this question, first note that if $x \in X_0 \cap X_1$ then

$$K(x,t) \le \min\{\|x\|_0, t\|x\|_1\} \le \min\{1,t\} \|x\|_{X_0\cap X_1}, \tag{2.2.3}$$

and consequently

$$\|x\|_{X(\mu,w,p)} \le \|x\|_{X_0\cap X_1} \|w \min\{1,\cdot\}\|_{L^p_\mu}. \tag{2.2.4}$$

On the other hand, for $x \in X_0 + X_1$ we have $K(x,t) \geq \min\{1,t\} \|x\|_{X_0+X_1}$, so

$$\|x\|_{X(\mu,w,p)} \geq \|x\|_{X_0+X_1} \|w \min\{1,\cdot\}\|_{L^p_\mu}. \tag{2.2.5}$$

From these calculations we learn that if the space $X(\mu,w,p)$ is nontrivial, then we have the inclusion $w\min\{1,\cdot\} \in L^p_\mu(\mathbb{R}_+)$, and conversely this inclusion implies the embeddings

$$X_0 \cap X_1 \hookrightarrow X(\mu,w,p) \hookrightarrow X_0 + X_1. \tag{2.2.6}$$

To prove that $X(\mu,w,p)$ is intermediate, we also need to verify that it is complete. For this note that the assumption that μ is Radon guarantees that $w\max\{1,\cdot\}$ is locally in $L^p_\mu(\mathbb{R}_+)$.

Theorem 2.2.2. *Let X_0 and X_1 be compatible Banach spaces and $1 \leq p \leq \infty$. Suppose that μ is a Radon measure on $\mathbb{R}_+$, $w : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and $w\min\{1,\cdot\} \in L^p_\mu(\mathbb{R}_+)$. Then the following hold.*

1. *The space $X(\mu,w,p)$ is a Banach space intermediate to X_0 and X_1.*
2. *If $w\max\{1,\cdot\} \in L^p_\mu(\mathbb{R}_+)$, then $X(\mu,w,p) = X_0 + X_1$ with equivalent norms.*
3. *Suppose that $p < \infty$. If $\int_{\mathbb{R}_+} (w(t))^p d\mu(t) = \infty$, then $X(\mu,w,p) \hookrightarrow \bar{X}_1$. If $\int_{\mathbb{R}_+} (tw(t))^p d\mu(t) = \infty$, then $X(\mu,w,p) \hookrightarrow \bar{X}_0$. If $\int_{\mathbb{R}_+} (w(t))^p d\mu(t) = \int_{\mathbb{R}_+} (tw(t))^p d\mu(t) = \infty$, then $X(\mu,w,p) \hookrightarrow \overline{X_0 \cap X_1} = \underline{X_0 + X_1}$.*

Proof. From the above analysis we know that

$$\|x\|_{X_0+X_1} \|w\min\{1,\cdot\}\|_{L^p_\mu} \leq \|x\|_{X(\mu,w,p)} \leq \|x\|_{X_0\cap X_1} \|w\min\{1,\cdot\}\|_{L^p_\mu} \tag{2.2.7}$$

and so $X_0 \cap X_1 \hookrightarrow X(\mu,w,p) \hookrightarrow X_0 + X_1$. To conclude we only need to prove that $X(\mu,w,p)$ is complete.

Let $\{x_n\}_{n=\ell}^\infty \subseteq X(\mu,w,p)$ be Cauchy. According to (2.2.7), the sequence is also Cauchy in the Banach space $X_0 + X_1$, and so converges in $X_0 + X_1$ to some x. For $t \in \mathbb{R}_+$ we may estimate

$$K(x-x_n,t) \leq K(x_n - x_m, t) + K(x-x_m,t) \leq K(x_n-x_m,t) + \max\{1,t\}\|x-x_m\|_{X_0+X_1} \tag{2.2.8}$$

and

$$K(x,t) \leq K(x_n,t) + K(x-x_n,t). \tag{2.2.9}$$

Then for $1 < \lambda < \infty$ we may bound

$$\|wK(x-x_n,\cdot)\|_{L^p_\mu([\lambda^{-1},\lambda])} \leq \|x_n - x_m\|_{X(\mu,w,p)} + \|x-x_m\|_{X_0+X_1} \|w\max\{1,\cdot\}\|_{L^p_\mu([\lambda^{-1},\lambda])} \tag{2.2.10}$$

and

$$\|wK(x,\cdot)\|_{L^p_\mu([\lambda^{-1},\lambda])} \leq \|x_n\|_{X(\mu,w,p)} + \|x-x_n\|_{X_0+X_1} \|w\max\{1,\cdot\}\|_{L^p_\mu([\lambda^{-1},\lambda])}. \tag{2.2.11}$$

We take the limsup in (2.2.11) to see that

$$\|wK(x,\cdot)\|_{L^p_\mu([\lambda^{-1},\lambda])} \leq \limsup_{n\to\infty} \|x_n\|_{X(\mu,w,p)} < \infty. \tag{2.2.12}$$

Sending $\lambda \to \infty$ and employing the monotone convergence theorem if $p < \infty$, we deduce from this that $x \in X(\mu,w,p)$.

Now let $\varepsilon > 0$ and choose $N \geq \ell$ such that $n, m \geq N$ implies $\|x_n - x_m\|_{X(\mu,w,p)} < \varepsilon$. Then for $n \geq N$ we deduce from (2.2.10) that

$$\|wK(x - x_n, \cdot)\|_{L^p_\mu([\lambda^{-1},\lambda])} \leq \limsup_{m\to\infty} \left(\|x_n - x_m\|_{X(\mu,w,p)} + \|x - x_m\|_{X_0+X_1} \|w \max\{1, \cdot\}\|_{L^p_\mu([\lambda^{-1},\lambda])} \right) \leq \varepsilon \quad (2.2.13)$$

for all $1 < \lambda < \infty$. Again sending $\lambda \to \infty$ and using the monotone convergence theorem if $p < \infty$, we deduce that

$$n \geq N \Rightarrow \|x - x_n\|_{X(\mu,w,p)} = \|wK(x - x_n, \cdot)\|_{L^p_\mu(\mathbb{R}_+)} \leq \varepsilon, \quad (2.2.14)$$

which means that $x_n \to x$ in $X(\mu, w, p)$ as $n \to \infty$. Hence, $X(\mu, w, p)$ is complete, and the first item is proved.

Suppose now that $w \max\{1, \cdot\} \in L^p_\mu(\mathbb{R}_+)$. Then from the estimate $K(x, t) \leq \max\{1, t\} \|x\|_{X_0+X_1}$ we find that

$$\|x\|_{X(\mu,w,p)} \leq \|w \max\{1, \cdot\}\|_{L^p_\mu} \|x\|_{X_0+X_1} \quad (2.2.15)$$

for all $x \in X_0 + X_1$. This and (2.2.7) imply that $\|\cdot\|_{X(\mu,w,p)}$ and $\|\cdot\|_{X_0+X_1}$ are equivalent norms and $X(\mu, w, p) = X_0 + X_1$. This proves the second item.

We now prove the third item, assuming that $p < \infty$. Let $0 \neq x \in X(\mu, w, p)$. If $x \notin \bar{X}_1$, then Theorem 2.1.19 implies that there exists $\varepsilon > 0$ such that $K(x, t) > \varepsilon$ for all $t > 0$, and hence

$$\infty > \|x\|_{X(\mu,w,p)} \geq \varepsilon \left(\int_{\mathbb{R}_+} (w(t))^p d\mu(t) \right)^{1/p}. \quad (2.2.16)$$

Hence, if $\int_{\mathbb{R}_+} (w(t))^p d\mu(t) = \infty$, then $x \in \bar{X}_1$. Similarly, if $x \notin \bar{X}_0$, then Theorem 2.1.19 implies that there exists $\varepsilon > 0$ such that $K(x, t)/t > \varepsilon$ for all $t > 0$, and hence

$$\infty > \|x\|_{X(\mu,w,p)} = \left(\int_{\mathbb{R}_+} (tw(t))^p \left(\frac{K(x,t)}{t} \right)^p d\mu(t) \right)^{1/p} \geq \varepsilon \left(\int_{\mathbb{R}_+} (tw(t))^p d\mu(t) \right)^{1/p}. \quad (2.2.17)$$

In turn, this means that if $\int_{\mathbb{R}_+} (tw(t))^p d\mu(t) = \infty$, then $x \in \bar{X}_0$. This proves the first two assertions of the third item, and the third assertion follows from these and Theorem 2.1.12, which completes the proof of the third item.

□

Remark 2.2.3. *One consequence of this result can be stated nicely if we know a priori that $X_0 \cap X_1$ is nontrivial. Indeed, in this case we know that $X(\mu, w, p)$ is a nontrivial Banach space intermediate to X_0 and X_1 if and only if $w \min\{1, \cdot\} \in L^p_\mu(\mathbb{R}_+)$.*

The simplest candidate to consider is $w(t) = t^{-\alpha}$ for some $\alpha \in \mathbb{R}$ and μ standard Lebesgue measure on $\mathbb{R}_+$. Then it's easy to check that $w \min\{1, \cdot\} \in L^\infty(\mathbb{R}_+)$ if and only if $0 \leq \alpha \leq 1$, while for $1 \leq p < \infty$ we have that $w \min\{1, \cdot\} \in L^p(\mathbb{R}_+)$ if and only if

$$\int_0^1 \frac{dt}{t^{(\alpha-1)p}} < \infty \text{ and } \int_1^\infty \frac{dt}{t^{\alpha p}} < \infty, \quad (2.2.18)$$

which in turn is equivalent to

$$\frac{1}{p} < \alpha < 1 + \frac{1}{p}. \quad (2.2.19)$$

To ensure that we pick a useful weight for $p < \infty$, we then consider $\theta \in (0,1)$ and set

$$\alpha = (1-\theta)\frac{1}{p} + \theta\left(1 + \frac{1}{p}\right) = \frac{1}{p} + \theta, \tag{2.2.20}$$

which means that

$$\|x\|_{X(\mu,w,p)} = \left(\int_{\mathbb{R}_+} \left(\frac{K(x,t)}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/p}. \tag{2.2.21}$$

The upshot of this analysis is that the most natural choice is not Lebesgue measure, but rather the measure $\mu = dt/t$. This is actually a very nice choice on $\mathbb{R}_+$, as it is a Haar measure for the locally compact Abelian group $\mathbb{R}_+$. Thus, it seems like a good candidate to study is the space $X(\mu, ()^{-\theta}, p)$.

Note that if $p < \infty$, then

$$\int_{\mathbb{R}_+} \frac{t^p dt}{t^{1+p\theta}} = \int_{\mathbb{R}_+} \frac{dt}{t^{1+p\theta}} = \infty, \tag{2.2.22}$$

and so the third item of Theorem 2.2.2 shows that $X(\mu, ()^{-\theta}, p) \hookrightarrow \overline{X_0 \cap X_1}$. On the other hand,

$$\sup_{t>0} \frac{K(x,t)}{t^\theta} < \infty \Rightarrow \lim_{t\to 0} K(x,t) = \lim_{t\to\infty} \frac{K(x,t)}{t} = 0 \Rightarrow x \in \overline{X_0 \cap X_1}, \tag{2.2.23}$$

and again we find that $X(\mu, ()^{-\theta}, p) \hookrightarrow \overline{X_0 \cap X_1}$. Finally, note that $w \max\{1, \cdot\} \notin L^p_\mu(\mathbb{R}_+)$ for any $1 \le p \le \infty$, so the second item of the theorem does not apply and there is hope that $X(\mu, ()^{-\theta}, p)$ is strictly smaller than $X_0 + X_1$.

2.2.2 The interpolation spaces

The above heuristics lead us to change our notation slightly and make the following definitions.

Definition 2.2.4. *Let X_0 and X_1 be compatible Banach spaces.*

1. *For $1 \le p \le \infty$ and $\theta \in (0,1)$ we define the space*

$$(X_0, X_1)_{\theta,p} = \{x \in X_0 + X_1 \mid \|x\|_{\theta,p} < \infty\} \tag{2.2.24}$$

where $\|\cdot\|_{\theta,p} : X_0 + X_1 \to [0,\infty]$ is defined for $p < \infty$ by

$$\|x\|_{\theta,p} = \left(\int_{\mathbb{R}_+} \left(\frac{K(x,t)}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/p} \tag{2.2.25}$$

and for $p = \infty$ by

$$\|x\|_{\theta,\infty} = \sup_{t>0} \frac{K(x,t)}{t^\theta}. \tag{2.2.26}$$

Clearly, we have that $(X_0, X_1)_{\theta,p} = X(\mu, (\cdot)^{-\theta}, p)$ for $\mu = dt/t$, and so Theorem 2.2.2 implies that $(X_0, X_1)_{\theta,p}$ is a Banach space intermediate to X_0 and X_1 and that $(X_0, X_1)_{\theta,p} \hookrightarrow \overline{X_0 \cap X_1} = \underline{X_0 + X_1}$.

2. *For $p = \infty$ and $\theta \in \{0,1\}$ we define the space*

$$(X_0, X_1)_{\theta,\infty} = \{x \in X_0 + X_1 \mid \|x\|_{\theta,\infty} < \infty\} \tag{2.2.27}$$

where $\|\cdot\|_{\theta,\infty} : X_0 + X_1 \to [0,\infty]$ is defined by

$$\|x\|_{\theta,\infty} = \sup_{t>0} \frac{K(x,t)}{t^\theta}. \tag{2.2.28}$$

Clearly, $(X_0,X_1)_{\theta,\infty} = X(\mu,(\cdot)^{-\theta},\infty)$ for $\mu = dt/t$, and so Theorem 2.2.2 implies that $(X_0,X_1)_{\theta,p}$ is a Banach space intermediate to X_0 and X_1. However, in this case the available embeddings are $X_\theta \hookrightarrow (X_0,X_1)_{\theta,\infty} \hookrightarrow \overline{X_\theta}$ for $\theta \in \{0,1\}$ (the former is trivial and the latter follows from Theorem 2.1.19).

Three remarks are in order.

Remark 2.2.5. *When $1 \le p < \infty$, we don't include the endpoint cases $\theta \in \{0,1\}$ since for $0 \neq x \in X_0 + X_1$,*

$$\int_0^\infty (K(x,t))^p \frac{dt}{t} \ge (K(x,1))^p \int_1^\infty \frac{dt}{t} = \infty \tag{2.2.29}$$

and

$$\int_0^\infty \left(\frac{K(x,t)}{t}\right)^p \frac{dt}{t} \ge \left(\frac{K(x,1)}{1}\right)^p \int_0^1 \frac{dt}{t} = \infty. \tag{2.2.30}$$

However, when $p = \infty$, there is useful information encoded in the quantities

$$\|x\|_{0,\infty} = \sup_{t>0} K(x,t) = \lim_{t\to\infty} K(x,t) \text{ and } \|x\|_{1,\infty} = \sup_{t>0} \frac{K(x,t)}{t} = \lim_{t\to 0} \frac{K(x,t)}{t}. \tag{2.2.31}$$

Remark 2.2.6. *If we define $\tilde{K}(x,t) = K(x,t)/t$, then $\mathbb{R}_+ \ni t \mapsto \tilde{K}(x,t) \in [0,\infty]$ is nonincreasing, and*

$$\|x\|_{\theta,p} = \left\| (\cdot)^{1-\theta} \tilde{K}(x,\cdot) \right\|_{L^p_\mu}. \tag{2.2.32}$$

Since $\tilde{K}(x,\cdot)$ and its rearrangement coincide (exercise: verify this claim), we deduce that

$$\|x\|_{\theta,p} = \left|\left|\left| \tilde{K}(x,\cdot) \right|\right|\right|_{L^{1/(1-\theta),p}(\mathbb{R}_+)}, \tag{2.2.33}$$

and we naturally arrive at a nice connection between Lorentz spaces and our new spaces.

Remark 2.2.7. *For every choice of $\theta \in (0,1)$ and $1 \le p \le \infty$ we know that $(X_0,X_1)_{\theta,p} \hookrightarrow \overline{X_0 \cap X_1} = \underline{X_0 + X_1}$. This highlights the special role played by the latter space in interpolation theory: it serves as a container space for everything we construct using this method. Note, though, that in general we do not know that $(X_0,X_1)_{\infty,\theta}$ is contained in this space when $\theta \in \{0,1\}$. It is easy, though, to use Theorem 2.1.19 to see that*

$$X_\theta \hookrightarrow (X_0,X_1)_{\theta,\infty} \hookrightarrow \bar{X}_\theta. \tag{2.2.34}$$

Our next result establishes the fundamental embedding properties of our new spaces as we vary the parameter $1 \le p \le \infty$. The result should be contrasted with Theorem 1.1.39.

Theorem 2.2.8. *Let X_0 and X_1 be compatible Banach spaces, $1 \le p \le \infty$, and $\theta \in (0,1)$. Then the following hold.*

1. *We have that $(X_0,X_1)_{\theta,p} = (X_1,X_0)_{1-\theta,p}$ with identical norms.*

2. *If* $x \in (X_0, X_1)_{\theta,p}$*, then*

$$K(x,t) \le (\theta p)^{1/p} t^\theta \|x\|_{\theta,p} \text{ for all } t \in \mathbb{R}_+ \tag{2.2.35}$$

with the usual understanding that $(\theta p)^{1/p} = 1$ *for* $p = \infty$*.*

3. *If* $p < q \le \infty$*, then* $(X_0, X_1)_{\theta,p} \hookrightarrow (X_0, X_1)_{\theta,q}$ *and*

$$\|x\|_{\theta,q} \le (\theta p)^{1/p-1/q} \|x\|_{\theta,p} \text{ for all } x \in (X_0, X_1)_{\theta,p}. \tag{2.2.36}$$

Proof. We'll prove the first two results for $p < \infty$ and leave the case $p = \infty$ as an exercise. We use the fourth item of Proposition 2.1.16 and a change of variables $s = 1/t$ to compute

$$\begin{aligned} \|x\|^p_{(X_0,X_1)_{\theta,p}} &= \int_{\mathbb{R}_+} \left(\frac{K(x,t)}{t^\theta}\right)^p \frac{dt}{t} = \int_{\mathbb{R}_+} \left(s^\theta K(x,1/s)\right)^p \frac{ds}{s} \\ &= \int_{\mathbb{R}_+} \left(\frac{\tilde{K}(x,s)}{s^{1-\theta}}\right)^p \frac{ds}{s} = \|x\|^p_{(X_1,X_0)_{1-\theta,p}}, \end{aligned} \tag{2.2.37}$$

where $\tilde{K}$ is determined by switching the roles of X_0 and X_1. This proves the first result.

Next we use the fact that $K(x,\cdot)$ is nondecreasing from Proposition 2.1.16 to bound

$$\|x\|^p_{\theta,p} \ge \int_t^\infty \left(\frac{K(x,s)}{s^\theta}\right)^p \frac{ds}{s} \ge (K(x,t))^p \int_t^\infty \frac{ds}{s^{1+\theta p}} = \frac{(K(x,t))^p}{\theta p t^{\theta p}} \tag{2.2.38}$$

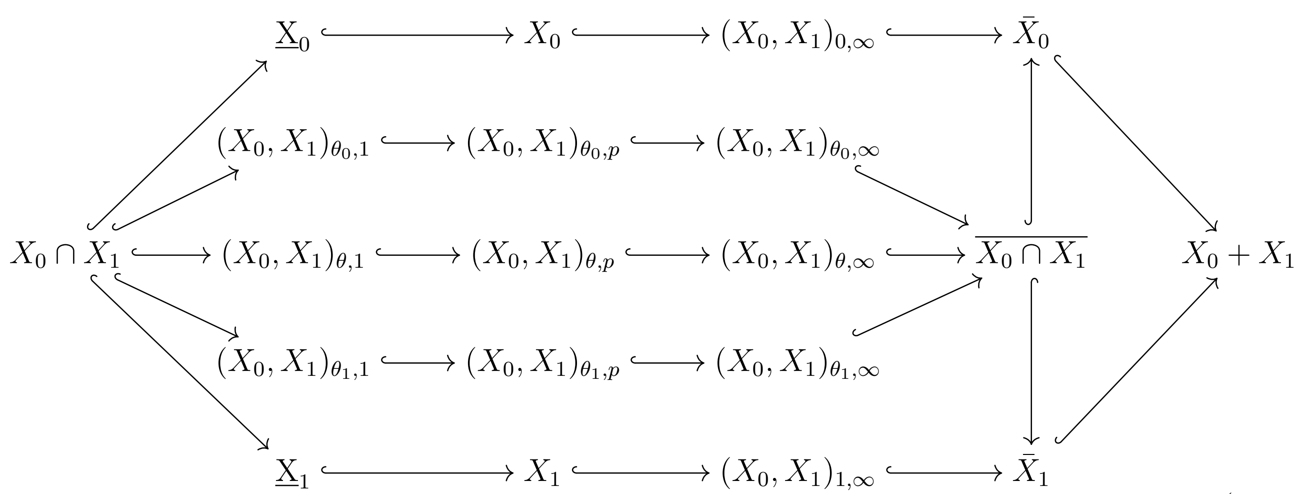
for every $t \in \mathbb{R}_+$. Upon rearranging, we then complete the proof of the second item.

Now suppose that $p < q \le \infty$. If $q = \infty$, then the second item implies that $\|x\|_{\theta,\infty} \le (\theta p)^{1/p} \|x\|_{\theta,p}$, so it remains to consider the case $q < \infty$. In this case we estimate

$$\|x\|_{\theta,q} = \left(\int_{\mathbb{R}_+} \left(\frac{K(x,t)}{t^\theta}\right)^{q-p} \left(\frac{K(x,t)}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/q} \le \|x\|^{1-p/q}_{\theta,\infty} \|x\|^{p/q}_{\theta,p} \le (\theta p)^{1/p-1/q} \|x\|_{\theta,p}, \tag{2.2.39}$$

which completes the proof of the third item. □

We can organize the results of Remark 2.2.7 and Theorem 2.2.8 as the following diagram, which highlights the fact that for each $\theta \in (0,1)$ we have a continuum of nested intermediate spaces indexed by $1 \le p \le \infty$, while for $\theta \in \{0,1\}$ we have outlier spaces. We typically arrange the spaces with θ increasing as we move down, so here $0 < \theta_0 < \theta < \theta_1 < 1$.



(2.2.40)

Next we study how our spaces depend on the parameter θ by examining the intersection of two of them.

Proposition 2.2.9. *Let X_0 and X_1 be compatible Banach spaces, $1 \le p \le \infty$, and $0 < \theta_0 < \theta < \theta_1 < 1$. Then $(X_0, X_1)_{\theta_0,p} \cap (X_0, X_1)_{\theta_1,p} \hookrightarrow (X_0, X_1)_{\theta,p}$, and*

$$\|x\|_{\theta,p} \le \|x\|_{\theta_0,p}^{\frac{\theta_1-\theta}{\theta_1-\theta_0}} \|x\|_{\theta_1,p}^{\frac{\theta-\theta_0}{\theta_1-\theta_0}} \tag{2.2.41}$$

for every $x \in (X_0, X_1)_{\theta_0,p} \cap (X_0, X_1)_{\theta_1,p}$.

Proof. Write $\theta = (1-\alpha)\theta_0 + \alpha\theta_1$ for $\alpha \in (0,1)$. If $p < \infty$ then we use Hölder's inequality to bound

$$\|x\|_{\theta,p}^p = \int_{\mathbb{R}_+} \left(\frac{K(x,t)}{t^\theta}\right)^p \frac{dt}{t} = \int_{\mathbb{R}_+} \left(\frac{K(x,t)}{t^{\theta_0}}\right)^{(1-\alpha)p} \left(\frac{K(x,t)}{t^{\theta_1}}\right)^{\alpha p} \frac{dt}{t} \le \|x\|_{\theta_0,p}^{(1-\alpha)p} \|x\|_{\theta_1,p}^{\alpha p}, \tag{2.2.42}$$

and the stated bound follows by solving for α and taking p^{th} roots. A similar argument works for $p = \infty$. □

One particularly interesting consequence of this result is the interpolation estimate recorded in the following.

Corollary 2.2.10. *Let X_0 and X_1 be compatible Banach spaces, $1 \le p \le \infty$, and $\theta \in [0,1]$. There exists a constant $C \in \mathbb{R}_+$ such that if $x \in X_0 \cap X_1$, then*

$$\|x\|_{\theta,p} \le C \|x\|_0^{1-\theta} \|x\|_1^\theta \text{ for all } x \in X_0 \cap X_1. \tag{2.2.43}$$

Proof. Exercise. □

Proposition 2.2.9 has the benefit of giving a nice constant on the right side but only works with the same value of p in all the spaces. Next we prove that the spaces can be improved at the expense of a worse constant on the right. This result should be contrasted with that of Theorem 1.1.41.

Theorem 2.2.11. *Let X_0 and X_1 be compatible Banach spaces, and $0 \le \theta_0 < \theta < \theta_1 \le 1$. Then $(X_0, X_1)_{\theta_0,\infty} \cap (X_0, X_1)_{\theta_1,\infty} \hookrightarrow (X_0, X_1)_{\theta,1}$, and*

$$\|x\|_{\theta,1} \le \left(\frac{1}{\theta_1-\theta} + \frac{1}{\theta-\theta_0}\right) \|x\|_{\theta_0,\infty}^{\frac{\theta_1-\theta}{\theta_1-\theta_0}} \|x\|_{\theta_1,\infty}^{\frac{\theta-\theta_0}{\theta_1-\theta_0}} \tag{2.2.44}$$

for every $x \in (X_0, X_1)_{\theta_0,\infty} \cap (X_0, X_1)_{\theta_1,\infty}$.

Proof. It suffices to prove the stated estimate, and for this we may also assume that $x \neq 0$. Due to the assumed inclusions, we have the bound

$$K(x,t) \le \min\{t^{\theta_0} \|x\|_{\theta_0,\infty}, t^{\theta_1} \|x\|_{\theta_1,\infty}\}. \tag{2.2.45}$$

Set

$$T = \left(\frac{\|x\|_{\theta_0,\infty}}{\|x\|_{\theta_1,\infty}}\right)^{1/(\theta_1-\theta_0)} \in \mathbb{R}_+. \tag{2.2.46}$$

Then

$$\begin{aligned}\int_{\mathbb{R}_+} \frac{K(x,t)}{t^\theta}\frac{dt}{t} &= \int_0^T \frac{t^{\theta_1}\|x\|_{\theta_1,\infty}}{t^\theta}\frac{dt}{t} + \int_T^\infty \frac{t^{\theta_0}\|x\|_{\theta_0,\infty}}{t^\theta}\frac{dt}{t} \\ &= \|x\|_{\theta_1,\infty}\frac{T^{\theta_1-\theta}}{\theta_1-\theta} + \|x\|_{\theta_1,\infty}\frac{1}{(\theta-\theta_0)T^{\theta-\theta_0}} = \left(\frac{1}{\theta_1-\theta} + \frac{1}{\theta-\theta_0}\right) \|x\|_{\theta_0,\infty}^{\frac{\theta_1-\theta}{\theta_1-\theta_0}} \|x\|_{\theta_1,\infty}^{\frac{\theta-\theta_0}{\theta_1-\theta_0}}\end{aligned} \tag{2.2.47}$$

□

Now that we have developed some of the essential properties of our interpolation spaces, we return to the question of whether these spaces admit a corresponding theory of operator interpolation. It turns out that they do, and the theory is relatively simple in comparison to the Marcinkiewicz and Riesz-Thorin theorems.

Theorem 2.2.12. *Let X_0 and X_1 be compatible Banach spaces and Y_0 and Y_1 be compatible Banach spaces over the same field. Suppose that $T : X_0+X_1 \to Y_0+Y_1$ is a linear map such that $T(X_i) \subseteq Y_i$ and $T \in \mathcal{L}(X_i;Y_i)$ for $i \in \{0,1\}$. Then for every $1 \le p \le \infty$ and $\theta \in (0,1)$ we have that $T \in \mathcal{L}((X_0,X_1)_{\theta,p},(Y_0,Y_1)_{\theta,p})$ and*

$$\|T\|_{\mathcal{L}((X_0,X_1)_{\theta,p},(Y_0,Y_1)_{\theta,p})} \le \|T\|_{\mathcal{L}(X_0;Y_0)}^{1-\theta} \|T\|_{\mathcal{L}(X_1;Y_1)}^{\theta}. \tag{2.2.48}$$

Proof. We write K_X for the K function associated to X_0 and X_1 and K_Y for the K function associated to Y_0 and Y_1. The result is trivial if $T = 0$, so we may assume that $T \neq 0$. Let $x = x_0 + x_1 \in (X_0,X_1)_{\theta,p}$. Then $Tx = Tx_0 + Tx_1$, $Tx_0 \in Y_0$, and $Tx_1 \in Y_1$, so for $t \in \mathbb{R}_+$

$$\begin{aligned} K_Y(Tx,t) &\le \|Tx_0\|_{Y_0} + t\|Tx_1\|_{Y_1} \le \|T\|_{\mathcal{L}(X_0;Y_0)} \|x_0\|_{X_0} + t\|T\|_{\mathcal{L}(X_1;Y_1)} \|x_1\|_{X_1} \\ &= \|T\|_{\mathcal{L}(X_0;Y_0)} \left(\|x_0\|_{X_0} + \frac{t\|T\|_{\mathcal{L}(X_1;Y_1)}}{\|T\|_{\mathcal{L}(X_0;Y_0)}} \|x_1\|_{X_1} \right). \end{aligned} \tag{2.2.49}$$

This holds for all such decompositions of x, and thus

$$K_Y(Tx,t) \le \|T\|_{\mathcal{L}(X_0;Y_0)} K_X(x, t\|T\|_{\mathcal{L}(X_1;Y_1)} / \|T\|_{\mathcal{L}(X_0;Y_0)}) \tag{2.2.50}$$

for $t \in \mathbb{R}_+$.

Let μ denote the measure dt/t on $\mathbb{R}_+$ and $f : \mathbb{R}_+ \to \mathbb{R}$ be measurable. For $\lambda \in \mathbb{R}_+$ write $f_\lambda(t) = f(\lambda t)$. Then we may compute

$$\left\|(\cdot)^{-\theta} f_\lambda(\cdot)\right\|_{L^p_\mu} = \lambda^\theta \left\|(\cdot)^{-\theta} f(\cdot)\right\|_{L^p_\mu}. \tag{2.2.51}$$

Combining (2.2.50) and (2.2.51), we then find that

$$\|Tx\|_{(Y_0,Y_1)_{\theta,p}} \le \|T\|_{\mathcal{L}(X_0;Y_0)} \left(\frac{\|T\|_{\mathcal{L}(X_1;Y_1)}}{\|T\|_{\mathcal{L}(X_0;Y_0)}} \right)^\theta \|x\|_{(X_0,X_1)_{\theta,p}} = \|T\|_{\mathcal{L}(X_0;Y_0)}^{1-\theta} \|T\|_{\mathcal{L}(X_1;Y_1)}^{\theta} \|x\|_{(X_0,X_1)_{\theta,p}}, \tag{2.2.52}$$

and this estimate completes the proof. □

In practice, we often verify the hypotheses of Theorem 2.2.12 with the help of the following lemma.

Lemma 2.2.13. *Let X_0 and X_1 be compatible Banach spaces and Y_0 and Y_1 be compatible Banach spaces over the same field. Suppose that for $i \in \{0,1\}$ we have $T_i \in \mathcal{L}(X_i;Y_i)$ such that $T_0 = T_1$ on $X_0 \cap X_1$. Then there exists a unique $T \in \mathcal{L}(X_0+X_1;Y_0+Y_1)$ such that $T|_{X_i} = T_i$ for $i \in \{0,1\}$. Moreover, we have the bound*

$$\|T\|_{\mathcal{L}(X_0+X_1;Y_0+Y_1)} \le \max\{\|T\|_{\mathcal{L}(X_0;Y_0)}, \|T\|_{\mathcal{L}(X_1;Y_1)}\}. \tag{2.2.53}$$

Proof. Suppose that $x = x_0 + x_1 = w_0 + w_1$ for $x_0, w_0 \in X_0$ and $x_1, w_1 \in X_1$. Then $x_0 - w_0 = w_1 - x_1 \in X_0 \cap X_1$ and so $T_0(x_0 - w_0) = T_1(w_1 - x_1)$ by hypothesis. Upon rearranging, this implies that $T_0(x_0)+T_1(x_1) = T_0(w_0)+T_1(w_1)$. From this we deduce that the mapping $T : X_0+X_1 \to Y_0+Y_1$ defined by $T(x_0 + x_1) = T_0(x_0) + T_1(x_1)$ is well-defined and linear. This map is bounded since if $x = x_0 + x_1$, then

$$\|Tx\|_{Y_0+Y_1} \le \|T_0x_0\|_{Y_0} + \|T_1x_1\|_{Y_1} \le \max\{\|T\|_{\mathcal{L}(X_0;Y_0)}, \|T\|_{\mathcal{L}(X_1;Y_1)}\} \left(\|x_0\|_{X_0} + \|x_1\|_{X_1}\right) \quad (2.2.54)$$

and since this holds for all such decompositions we have the bound

$$\|Tx\|_{Y_0+Y_1} \le \max\{\|T\|_{\mathcal{L}(X_0;Y_0)}, \|T\|_{\mathcal{L}(X_1;Y_1)}\} \|x\|_{X_0+X_1}. \quad (2.2.55)$$

This proves the existence of the desired T. Uniqueness follows since if S is any other such operator, the condition $S|_{X_i} = T_i$ implies that $Sx = Sx_0 + Sx_1 = T_0x_0 + T_1x_1 = Tx$ for all $x = x_0 + x_1 \in X_0 + X_1$. □

It's also possible to prove some interpolation results for nonlinear operators. We will demonstrate the basic principle in a particularly simple case in which the nonlinear map satisfies a Lipschitz-type condition. For more sophisticated versions see the paper of Tartar [6].

Theorem 2.2.14. *Let Y_0 and Y_1 be compatible Banach spaces, and let X_0 and X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$. Suppose that $f : X_0 \to Y_0$ is such that $f(X_1) \subseteq Y_1$ and there exist constants $A_0, A_1 \in \mathbb{R}_+$ such that*

$$\|f(x) - f(w)\|_{Y_0} \le A_0 \|x - w\|_{X_0} \text{ for all } x, w \in X_0 \quad (2.2.56)$$

and

$$\|f(x)\|_{Y_1} \le A_1 \|x\|_{X_1} \text{ for all } x \in X_1. \quad (2.2.57)$$

Then for $\theta \in (0, 1)$ and $1 \le p \le \infty$ we have that $f((X_0, X_1)_{\theta,p}) \subseteq (Y_0, Y_1)_{\theta,p}$, and

$$\|f(x)\|_{(Y_0,Y_1)_{\theta,p}} \le A_0^{1-\theta} A_1^{\theta} \|x\|_{(X_0,X_1)_{\theta,p}} \text{ for all } x \in (X_0, X_1)_{\theta,p}. \quad (2.2.58)$$

Proof. Write K_Y for the K function associated to Y_0 and Y_1 and K_X for the K function associated to X_0 and X_1. Let $x \in (X_0, X_1)_{\theta,p}$ and write $x = x_0 + x_1$ for $x_0 \in X_0$ and $x_1 \in X_1$. Then $f(x_1) \in Y_1$ and $f(x) - f(x_1) \in Y_0$ by hypothesis, so $f(x) = (f(x) - f(x_1)) + f(x_1) \in Y_0 + Y_1$. Consequently, for $t \in \mathbb{R}_+$ we may estimate

$$\begin{aligned} K_Y(f(x), t) \le \|f(x) - f(x_1)\|_{Y_0} + t \|f(x_1)\|_{Y_1} &\le A_0 \|x - x_1\|_{X_0} + tA_1 \|x_1\|_{X_1} \\ &= A_0 \left(\|x_0\|_{X_0} + \frac{tA_1}{A_0} \|x_1\|_{X_1} \right). \end{aligned} \quad (2.2.59)$$

This holds for all such decompositions of x, so we deduce that

$$K_Y(f(x), t) \le A_0 K_X(x, tA_1/A_0). \quad (2.2.60)$$

Arguing as in the proof of Theorem 2.2.12, we conclude from this that

$$\|f(x)\|_{(Y_0,Y_1)_{\theta,p}} \le A_0^{1-\theta} A_1^{\theta} \|x\|_{(X_0,X_1)_{\theta,p}}. \quad (2.2.61)$$

□

2.2.3 Some special cases

We now turn our attention to some special cases in which we know more about the relation between X_0 and X_1. We begin with a very simple result that shows that if $X_0 \cap X_1$ is trivial, then the interpolation spaces are also trivial.

Proposition 2.2.15. *Let X_0 and X_1 be compatible Banach spaces, $1 \le p \le \infty$, and $\theta \in (0,1)$. If $X_0 \cap X_1 = \{0\}$, then $(X_0, X_1)_{\theta,p} = \{0\}$.*

Proof. We know from Theorem 2.2.2 and Corollary 2.1.14 that $\{0\} = X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta,p} \hookrightarrow \{0\} = \overline{X_0 \cap X_1}$.

□

It often occurs in practice that $X_i \hookrightarrow X_{1-i}$, in which case $X_0 \cap X_1 = X_i$ and $X_0 + X_1 = X_{1-i}$. It is then possible to find equivalent norms on $(X_0, X_1)_{\theta,p}$ that are useful in the sense that they only involve strict subsets of $\mathbb{R}_+$. We record this result now.

Theorem 2.2.16. *Let X_0 and X_1 be compatible Banach spaces and $1 \le p \le \infty$. Suppose that $\theta \in (0,1)$ if $p < \infty$ and $\theta \in [0,1]$ if $p = \infty$. Then the following hold.*

1. *If $X_0 \hookrightarrow X_1$, then $X_0 \hookrightarrow (X_0, X_1)_{\theta,p} \hookrightarrow X_1$, and for any $T \in \mathbb{R}_+$ an equivalent norm on $(X_0, X_1)_{\theta,p}$ is given by the map*

$$X_1 \ni x \mapsto \begin{cases} \|x\|_1 + \left(\int_T^\infty (t^{-\theta} K(x,t))^p \frac{dt}{t}\right)^{1/p} \in [0,\infty] & \text{if } p < \infty \\ \|x\|_1 + \sup_{t>T} t^{-\theta} K(x,t) \in [0,\infty] & \text{if } p = \infty. \end{cases} \tag{2.2.62}$$

2. *If $X_1 \hookrightarrow X_0$, then $X_1 \hookrightarrow (X_0, X_1)_{\theta,p} \hookrightarrow X_0$, and for any $T \in \mathbb{R}_+$ an equivalent norm on $(X_0, X_1)_{\theta,p}$ is given by the map*

$$X_0 \ni x \mapsto \begin{cases} \|x\|_0 + \left(\int_0^T (t^{-\theta} K(x,t))^p \frac{dt}{t}\right)^{1/p} \in [0,\infty] & \text{if } p < \infty \\ \|x\|_0 + \sup_{0<t<T} t^{-\theta} K(x,t) \in [0,\infty] & \text{if } p = \infty. \end{cases} \tag{2.2.63}$$

Proof. If $X_0 \hookrightarrow X_1$, then $X_0 \cap X_1 = X_0$ and $X_0 + X_1 = X_1$. Similarly, if $X_1 \hookrightarrow X_0$, then $X_0 \cap X_1 = X_1$ and $X_0 + X_1 = X_0$. As such, we only need to prove the equivalence of the norms. For the first item we note that since $X_0 + X_1 = X_1$,

$$\min\{1,t\} \|x\|_1 \asymp \min\{1,t\} \|x\|_{X_0+X_1} \le K(x,t) \le t \|x\|_1, \tag{2.2.64}$$

and so for $0 < t < T$ we have that

$$K(x,t) \asymp t \|x\|_1. \tag{2.2.65}$$

Consequently,

$$\left(\int_0^T (t^{-\theta} K(x,t))^p \frac{dt}{t}\right)^{1/p} \asymp \|x\|_1 \asymp \sup_{0<t<T} t^{-\theta} K(x,t) \tag{2.2.66}$$

and the equivalence for the first item follows.

For the second item, we instead bound

$$\min\{1,t\} \|x\|_0 \asymp \min\{1,t\} \|x\|_{X_0+X_1} \le K(x,t) \le \|x\|_0 \tag{2.2.67}$$

to see that for $T \le t < \infty$,

$$K(x,t) \asymp \|x\|_0 \,. \tag{2.2.68}$$

Consequently,

$$\left(\int_T^\infty (t^{-\theta} K(x,t))^p \frac{dt}{t} \right)^{1/p} \asymp \|x\|_0 \asymp \sup_{T<t} t^{-\theta} K(x,t) \tag{2.2.69}$$

and the equivalence for the second item follows. □

As an immediate consequence of this new equivalence, we find a number of interesting embeddings.

Corollary 2.2.17. *Let X_0 and X_1 be Banach spaces and $0 < \theta_0 < \theta_1 < 1$. Then the following hold.*

1. *If $X_0 \hookrightarrow X_1$, then we have the embeddings $(X_0, X_1)_{\theta_0,\infty} \hookrightarrow (X_0, X_1)_{\theta_1,1}$, $(X_0, X_1)_{0,\infty} \hookrightarrow (X_0, X_1)_{\theta_0,1}$, and $(X_0, X_1)_{\theta_1,\infty} \hookrightarrow \underline{X}_1 = \bar{X}_0 \hookrightarrow (X_0, X_1)_{1,\infty} = X_1$.*
2. *If $X_1 \hookrightarrow X_0$, then we have the embeddings $(X_0, X_1)_{\theta_1,\infty} \hookrightarrow (X_0, X_1)_{\theta_0,1}$, $(X_0, X_1)_{1,\infty} \hookrightarrow (X_0, X_1)_{\theta_1,1}$, and $(X_0, X_1)_{\theta_0,\infty} \hookrightarrow \underline{X}_0 = \bar{X}_1 \hookrightarrow (X_0, X_1)_{0,\infty} = X_0$.*

Proof. In light of Corollary 2.1.14 and Theorem 2.2.16, applied with $T = 1$, it suffices to observe that

$$\int_1^\infty \frac{t^{\theta_0 - \theta_1} dt}{t} < \infty, \ \frac{1}{t^{\theta_1}} \le 1, \text{ and } \frac{1}{t} \le \frac{1}{t^{\theta_1}} \text{ for } t > 1 \tag{2.2.70}$$

for the first item, and

$$\int_0^1 \frac{t^{\theta_1 - \theta_0} dt}{t} < \infty, \ \frac{1}{t^{\theta_1}} \le \frac{1}{t}, \text{ and } 1 \le \frac{1}{t^{\theta_0}} \text{ for } 0 < t < 1 \tag{2.2.71}$$

for the second item. □

It's convenient to again organize what we know in a diagram. If $X_0 \hookrightarrow X_1$, then we have the following zig-zag embedding diagram with $0 < \theta_0 < \theta_1 < 1$.

$$\begin{array}{ccccc}
X_0 = \underline{X}_0 & \hookrightarrow & & \longrightarrow & (X_0,X_1)_{0,\infty} \\
 & \swarrow & & & \\
(X_0,X_1)_{\theta_0,1} & \hookrightarrow & (X_0,X_1)_{\theta_0,p} & \hookrightarrow & (X_0,X_1)_{\theta_0,\infty} \\
 & \swarrow & & & \\
(X_0,X_1)_{\theta_1,1} & \hookrightarrow & (X_0,X_1)_{\theta_1,p} & \hookrightarrow & (X_0,X_1)_{\theta_0,\infty} \\
 & \swarrow & & & \\
\underline{X}_1 = \bar{X}_0 & \hookrightarrow & & \longrightarrow & X_1 = (X_0,X_1)_{1,\infty}
\end{array} \tag{2.2.72}$$

On the other hand, if $X_1 \hookrightarrow X_0$, then we have the following zig-zag embedding diagram with

$0 < \theta_0 < \theta_1 < 1$.

$$
\begin{array}{ccccc}
\underline{X}_0 = \bar{X}_1 & \hookrightarrow & & & X_0 = (X_0, X_1)_{0,\infty} \\
(X_0, X_1)_{\theta_0,1} & \hookrightarrow & (X_0, X_1)_{\theta_0,p} & \hookrightarrow & (X_0, X_1)_{\theta_0,\infty} \\
(X_0, X_1)_{\theta_1,1} & \hookrightarrow & (X_0, X_1)_{\theta_1,p} & \hookrightarrow & (X_0, X_1)_{\theta_0,\infty} \\
X_1 = \underline{X}_1 & \hookrightarrow & & & (X_0, X_1)_{1,\infty}
\end{array}
\tag{2.2.73}
$$

These diagrams highlight the interesting fact that if $X_i \hookrightarrow X_{1-i}$, then $(X_0, X_1)_{i,\infty}$ is the smallest of the interpolation spaces we have constructed, and $(X_0, X_1)_{1-i,\infty}$ is the largest.

2.3 Further properties of interpolation spaces

We now aim to derive some more properties of our abstract interpolation spaces.

2.3.1 Equivalent norms

So far we have not really employed the J function in our analysis of the interpolation spaces. It turns out to play a role through the following theorem, which establishes the existence of an equivalent norm on $(X_0, X_1)_{\theta,p}$ that utilizes the J function. In the theorem we will also establish equivalence with a discretized version of our previous norm. Roughly speaking, the idea is that we can return to the context of Theorem 2.2.2 and exploit the monotonicity properties of $K(x, \cdot)$ to switch from the measure $\mu = dt/t$ and weight $w(t) = t^{-\theta}$ to the measure $\mu' = \sum_{n\in\mathbb{Z}} \delta_{r^n}$ for some $r > 1$, but with the same weight.

Theorem 2.3.1. *Let X_0 and X_1 be compatible Banach spaces, $1 \le p \le \infty$, and $\theta \in (0, 1)$. Fix $1 < r < \infty$. Then there exist constants $C_0, C_1, C_2, C_3 \in \mathbb{R}_+$ such that*

$$
\begin{aligned}
C_0 \|x\|_{\theta,p} &\le C_1 \left\|\{r^{-\theta n} K(x, r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \\
&\le \inf\{\left\|\{r^{-\theta n} J(x_n, r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \mid x = \sum_{n\in\mathbb{Z}} x_n \text{ for } \{x_n\}_{n\in\mathbb{Z}} \subseteq X_0 \cap X_1\} \\
&\le C_2 \left\|\{r^{-\theta n} K(x, r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \le C_3 \|x\|_{\theta,p}
\end{aligned}
\tag{2.3.1}
$$

for all $x \in X_0 + X_1$. Consequently, all three of these quantities define equivalent norms on $(X_0, X_1)_{\theta,p}$.

Proof. We will first prove the existence of constants $A_0, A_1 \in \mathbb{R}_+$ such that

$$
A_0 \|x\|_{\theta,p} \le \left\|\{r^{-\theta n} K(x, r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \le A_1 \|x\|_{\theta,p} \tag{2.3.2}
$$

for all $x \in X_0 + X_1$. Suppose initially that $p < \infty$. Since $r > 1$ we can write

$$
\int_{\mathbb{R}^+} (K(x,t))^p \frac{dt}{t^{1+\theta p}} = \sum_{n\in\mathbb{Z}} \int_{r^n}^{r^{n+1}} (K(x,t))^p \frac{dt}{t^{1+\theta p}}. \tag{2.3.3}
$$

For each $n \in \mathbb{Z}$ we can also compute

$$\int_{r^n}^{r^{n+1}} \frac{dt}{t^{1+\theta p}} = \frac{1}{\theta p}\left(\frac{1}{r^{n\theta p}} - \frac{1}{r^{(n+1)\theta p}}\right) = \frac{1}{r^{n\theta p}}\frac{r^{\theta p}-1}{\theta p r^{\theta p}} = \frac{1}{r^{(n+1)\theta p}}\frac{r^{\theta p}-1}{\theta p}. \tag{2.3.4}$$

From these and the fact that $K(x,\cdot)$ is nondecreasing, we deduce that

$$\begin{aligned}\frac{r^{\theta p}-1}{\theta p r^{\theta p}}\sum_{n\in\mathbb{Z}}\left(\frac{K(x,r^n)}{r^{n\theta}}\right)^p \le \int_{\mathbb{R}^+}(K(x,t))^p\frac{dt}{t^{1+\theta p}} &\le \frac{r^{\theta p}-1}{\theta p}\sum_{n\in\mathbb{Z}}\left(\frac{K(x,r^{n+1})}{r^{(n+1)\theta}}\right)^p \\ &= \frac{r^{\theta p}-1}{\theta p}\sum_{n\in\mathbb{Z}}\left(\frac{K(x,r^n)}{r^{n\theta}}\right)^p. \end{aligned} \tag{2.3.5}$$

This proves (2.3.2) when $p < \infty$. In the case $p = \infty$ we note that

$$\sup_{t>0}\frac{K(x,t)}{t^\theta} = \sup_{n\in\mathbb{Z}}\sup_{r^n<t\le r^{n+1}}\frac{K(x,t)}{t^\theta} \text{ and } \frac{K(x,r^n)}{r^{(n+1)\theta}} \le \frac{K(x,t)}{t^\theta} \le \frac{K(x,r^{n+1})}{r^{n\theta}} \text{ for } r^n < t \le r^{n+1} \tag{2.3.6}$$

and argue similarly. Thus, (2.3.2) is proved.

Next we will prove the existence of constants $B_0, B_1 \in \mathbb{R}_+$ such that

$$\begin{aligned} &B_0\left\|\{r^{-\theta n}K(x,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \\ &\quad\le \inf\{\left\|\{r^{-\theta n}J(x_n,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \mid x = \sum_{n\in\mathbb{Z}} x_n \text{ for } \{x_n\}_{n\in\mathbb{Z}} \subseteq X_0\cap X_1\} \\ &\quad\le B_1\left\|\{r^{-\theta n}K(x,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \end{aligned} \tag{2.3.7}$$

for all $x \in X_0 + X_1$, and once this is established the proof is complete.

Suppose initially that the infimum in the middle of (2.3.7) is finite, and pick any sequence $\{x_n\}_{n\in\mathbb{Z}} \subseteq X_0 \cap X_1$ such that $x = \sum_{n\in\mathbb{Z}} x_n$. For $n \in \mathbb{Z}$ we can use Propositions 2.1.16 and 2.1.18 to estimate

$$\begin{aligned} r^{-n\theta}K(x,r^n) \le r^{-n\theta}\sum_{m\in\mathbb{Z}}K(x_m,r^n) &\le r^{-n\theta}\sum_{m\in\mathbb{Z}}\min\{1,r^{n-m}\}J(x_m,r^m) \\ &= \sum_{m\in\mathbb{Z}}\min\{1,r^{n-m}\}r^{-(n-m)\theta}r^{-m\theta}J(x_m,r^m). \end{aligned} \tag{2.3.8}$$

Define the sequences $k = \{k_n\}_{n\in\mathbb{Z}}$, $d = \{d_n\}_{n\in\mathbb{Z}}$, $j = \{j_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}_+$ via

$$k_n = r^{-n\theta}K(x,r^n), d_n = \min\{1,r^n\}r^{-n\theta}, \text{ and } j_n = r^{-n\theta}J(x_n,r^n), \tag{2.3.9}$$

and note that

$$\|d\|_{\ell^1} = \sum_{n\in\mathbb{Z}} d_n = \sum_{0\le n} r^{-n\theta} + \sum_{n<0} r^{n(1-\theta)} < \infty. \tag{2.3.10}$$

Then the (2.3.8) implies that

$$k_n \le (d * j)_n \text{ for all } n \in \mathbb{Z}, \tag{2.3.11}$$

and so Young's inequality (see Remark 1.3.9) implies that

$$\left\|\{r^{-\theta n}K(x,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} = \|k\|_{\ell^p} \le \|d*j\|_{\ell^p} \le \|d\|_{\ell^1}\|j\|_{\ell^p} = \|d\|_{\ell^1}\left\|\{r^{-\theta n}J(x_n,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p}. \tag{2.3.12}$$

This holds for all such decompositions of x, and hence

$$\left\|\{r^{-\theta n}K(x,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \le \|d\|_{\ell^1} \inf\{\left\|\{r^{-\theta n}J(x_n,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \mid x = \sum_{n\in\mathbb{Z}} x_n \text{ for } \{x_n\}_{n\in\mathbb{Z}} \subseteq X_0\cap X_1\}. \tag{2.3.13}$$

Now suppose that $\left\|\{r^{-\theta n}K(x,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} < \infty$. According to (2.3.2), we then know that $x \in (X_0,X_1)_{\theta,p} \hookrightarrow \overline{X_0\cap X_1}$, and so the third item of Theorem 2.1.19 allows us to choose a sequence $\{w_n\}_{n\in\mathbb{Z}} \subseteq X_0\cap X_1$ such that $x=\sum_{n\in\mathbb{Z}} w_n$ and $J(w_n,r^n)\le 2(1+r)K(x,r^n)$ for $n\in\mathbb{Z}$. Then

$$\begin{aligned}\inf\{\left\|\{r^{-\theta n}J(x_n,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \mid x = \sum_{n\in\mathbb{Z}} x_n \text{ for } \{x_n\}_{n\in\mathbb{Z}} \subseteq X_0\cap X_1\} \\ \le \left\|\{r^{-\theta n}J(w_n,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p} \le 2(1+r)\left\|\{r^{-\theta n}K(x,r^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^p},\end{aligned} \tag{2.3.14}$$

and together with (2.3.13) this proves (2.3.7). □

Of course, the reason we care about the new equivalent norms is that they allow us to prove things that were out of reach with the original norm. To demonstrate what we can do with the J formulation we now record an amusing embedding result that complements that of Theorem 2.2.11.

Theorem 2.3.2. *Let X_0 and X_1 be compatible Banach spaces and $0<\theta_0<\theta<\theta_1<1$. Then*

$$(X_0,X_1)_{\theta,\infty} \hookrightarrow (X_0,X_1)_{\theta_0,1} + (X_0,X_1)_{\theta_1,1}. \tag{2.3.15}$$

Proof. Let $x\in(X_0,X_1)_{\theta,\infty}$ and write $x=\sum_{n\in\mathbb{Z}}x_n$ for $\{x_n\}_{n\in\mathbb{Z}}\subseteq X_0\cap X_1$. Define $y=\sum_{m<0}x_m$ and $z=\sum_{m\ge 0}x_m$. Then

$$\begin{aligned}2^{-n\theta_0}K(y,2^n) &\le \sum_{m<0}2^{-n\theta_0}K(x_m,2^n) \le \sum_{m<0}2^{-n\theta_0}\min\{1,2^{n-m}\}J(x_m,2^m)\\ &\le \sup_{m\in\mathbb{Z}}\frac{J(x_m,2^m)}{2^{m\theta}}\sum_{m<0}2^{-n\theta_0}\min\{1,2^{n-m}\}2^{m\theta},\end{aligned} \tag{2.3.16}$$

and

$$\sum_{m<0}2^{-n\theta_0}\min\{1,2^{n-m}\}2^{m\theta} \le \sum_{m<0}2^{-(n-m)\theta_0}\min\{1,2^{n-m}\}2^{m(\theta-\theta_0)}. \tag{2.3.17}$$

Hence,

$$2^{-n\theta_0}K(y,2^n)\le (d*e)_n \text{ for } n\in\mathbb{Z}, \tag{2.3.18}$$

where

$$d_n = 2^{-n\theta_0}\min\{1,2^n\} \text{ and } e_n = 2^{n(\theta-\theta_0)}\chi_{(-\infty,0)}(n). \tag{2.3.19}$$

Since $d,e\in\ell^1(\mathbb{Z})$ we may use Young's theorem to bound

$$\left\|\{2^{-n\theta_0}K(y,2^n)\}_{n\in\mathbb{Z}}\right\|_{\ell^1} \le \|d\|_{\ell^1}\|e\|_{\ell^1}\sup_{m\in\mathbb{Z}}\frac{J(x_m,2^m)}{2^{m\theta}}, \tag{2.3.20}$$

and so Theorem 2.3.1 allows us to pick a constant $C_0\in\mathbb{R}_+$ such that

$$\|y\|_{\theta_0,1}\le C_0\|x\|_{\theta,\infty}. \tag{2.3.21}$$

On the other hand,

$$2^{-n\theta_1}K(z,2^n) \le \sum_{m\ge 0} 2^{-n\theta_1}K(x_m,2^n) \le \sum_{m\ge 0} 2^{-n\theta_1}\min\{1,2^{n-m}\}J(x_m,2^m)$$
$$\le \sup_{m\in\mathbb{Z}} \frac{J(x_m,2^m)}{2^{m\theta}} \sum_{m\ge 0} 2^{-n\theta_0}\min\{1,2^{n-m}\}2^{m\theta}, \quad (2.3.22)$$

and

$$\sum_{m\ge 0} 2^{-n\theta_1}\min\{1,2^{n-m}\}2^{m\theta} = \sum_{m\ge 0} 2^{-(n-m)\theta_1}\min\{1,2^{n-m}\}2^{m(\theta-\theta_1)}. \quad (2.3.23)$$

Hence,

$$2^{-n\theta_1}K(y,2^n) \le (f*g)_n \text{ for } n\in\mathbb{Z}, \quad (2.3.24)$$

where

$$f_n = 2^{-n\theta_1}\min\{1,2^n\} \text{ and } g_n = 2^{n(\theta-\theta_1)}\chi_{[0,\infty)}(n). \quad (2.3.25)$$

Then $f,g\in\ell^1(\mathbb{Z})$, so again we can use Young's inequality and Theorem 2.3.1 to pick a constant $C_1\in\mathbb{R}_+$ such that

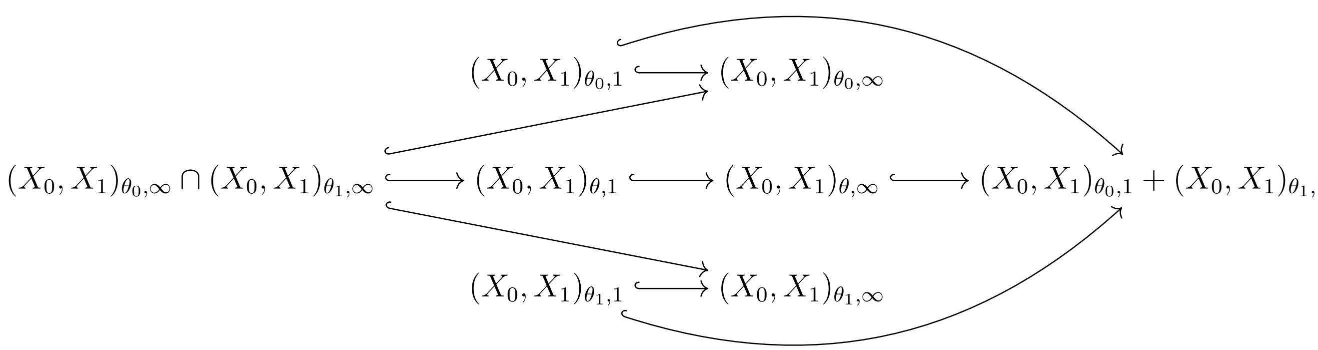
$$\|z\|_{\theta_1,1} \le C_1\|x\|_{\theta,\infty}. \quad (2.3.26)$$

We now know that $x=y+z$ with $y\in(X_0,X_1)_{\theta_0,1}$ and $z\in(X_0,X_1)_{\theta_1,1}$. Moreover, (2.3.21) and (2.3.26) show that

$$\|x\|_{(X_0,X_1)_{\theta_0,1}+(X_0,X_1)_{\theta_1,1}} \le \|y\|_{\theta_0,1}+\|z\|_{\theta_1,1} \le \max\{C_0,C_1\}\|x\|_{\theta,\infty}. \quad (2.3.27)$$

This proves the asserted embedding. □

We can summarize the results in Theorems 2.2.11 and 2.3.2 in the following diagram, in which $0<\theta_0<\theta<\theta_1<1$.



(2.3.28)

As another use of our new norm, we prove that $X_0\cap X_1$ is dense in $(X_0,X_1)_{\theta,p}$ for $1\le p<\infty$ and $\theta\in(0,1)$.

Theorem 2.3.3. *Let X_0 and X_1 be compatible Banach spaces, $1\le p<\infty$, and $\theta\in(0,1)$. Then $X_0\cap X_1$ is dense in $(X_0,X_1)_{\theta,p}$.*

Proof. Let $x\in(X_0,X_1)_{\theta,p}$ and pick $1<r<\infty$. According to Theorem 2.3.1, we can choose $\{x_n\}_{n\in\mathbb{Z}}\subseteq X_0\cap X_1$ such that $x=\sum_{n\in\mathbb{Z}}x_n$ (convergence in X_0+X_1) and

$$\left(\sum_{n\in\mathbb{Z}}(r^{-\theta n}J(x_n,r^n))^p\right)^{1/p} < \infty. \quad (2.3.29)$$

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$\left(\sum_{|n| \geq N} (r^{-\theta n} J(x_n, r^n))^p \right)^{1/p} < C_0 \varepsilon, \tag{2.3.30}$$

where $C_0 \in \mathbb{R}_+$ is the constant from Theorem 2.3.1. Then $\sum_{|n|<N} x_n \in X_0 \cap X_1$ and $x - \sum_{|n|<N} x_n = \sum_{|n| \geq N} x_n$, with this series again converging in $X_0 + X_1$, and so Theorem 2.3.1 implies that

$$C_0 \left\| x - \sum_{|n|<N} x_n \right\|_{\theta,p} \leq \left(\sum_{|n| \geq N} (r^{-\theta n} J(x_n, r^n))^p \right)^{1/p} < C_0 \varepsilon. \tag{2.3.31}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $X_0 \cap X_1$ is dense in $(X_0, X_1)_{\theta,p}$. □

2.3.2 Reiteration

For a pair of compatible Banach spaces X_0 and X_1, we now know that $(X_0, X_1)_{\theta_0,p_0}$ and $(X_0, X_1)_{\theta_1,p_1}$ are also compatible spaces. A natural question then arises: what happens if we interpolate between these new spaces? Do we get something new, or do we end up with another interpolation space? In order to answer this question we first need to introduce some machinery, starting with the following simple result.

Proposition 2.3.4. *Let X_0 and X_1 be compatible Banach spaces, $1 \leq p \leq \infty$, and $\theta \in (0,1)$. There exists a constant $C \in \mathbb{R}_+$ such that*

$$\|x\|_{\theta,p} \leq C \inf_{t>0} t^{-\theta} J(x,t) \text{ for all } x \in X_0 \cap X_1. \tag{2.3.32}$$

Proof. Suppose $x \in X_0 \cap X_1$. For any $m \in \mathbb{Z}$ we have that $x = \sum_{n \in \mathbb{Z}} x_n$, where $x_n = x$ if $n = m$ and $x_n = 0$ otherwise, so Theorem 2.3.1 with $r = 2$ implies that

$$C_0 \|x\|_{\theta,p} \leq \left\| \{2^{-\theta n} J(x_n, 2^n)\}_{n \in \mathbb{Z}} \right\|_{\ell^p} = \frac{J(x, 2^m)}{2^{\theta m}}. \tag{2.3.33}$$

Consequently,

$$C_0 \|x\|_{\theta,p} \leq \inf_{m \in \mathbb{Z}} \frac{J(x, 2^m)}{2^{\theta m}}. \tag{2.3.34}$$

However, for $2^n \leq t \leq 2^{n+1}$ we have that

$$\frac{1}{2^\theta} \inf_{m \in \mathbb{Z}} \frac{J(x, 2^m)}{2^{\theta m}} \leq \frac{J(x, 2^n)}{2^{(n+1)\theta}} \leq \frac{J(x,t)}{t^\theta}, \tag{2.3.35}$$

from which we deduce that

$$\|x\|_{\theta,p} \leq \frac{2^\theta}{C_0} \inf_{t>0} \frac{J(x,t)}{t^\theta}. \tag{2.3.36}$$

□

In light of the second item of Theorem 2.2.8 and the estimate of Proposition 2.3.4, we are led to introduce the following idea.

Definition 2.3.5. *Let X_0 and X_1 be compatible Banach spaces and X be intermediate to X_0 and X_1. Let $\theta \in [0,1]$.*

1. *We say that X is of $K-$type θ if there exists a constant $C > 0$ such that*

$$\sup_{t>0} t^{-\theta} K(x,t) \leq C \|x\|_X \text{ for all } x \in X. \tag{2.3.37}$$

2. *We say that X is of $J-$type θ if there exists a constant $C > 0$ such that*

$$\|x\|_X \leq C \inf_{t>0} t^{-\theta} J(x,t) \text{ for all } x \in X_0 \cap X_1. \tag{2.3.38}$$

3. *We say that X is of type θ if X is of $K-$type θ and of $J-$type θ.*

Let's consider a simple example.

Example 2.3.6. Let X_0 and X_1 be compatible Banach spaces. If $x \in X_0 + X_1$, then $K(x,t) \leq \|x\|_0$ for all $t \in \mathbb{R}_+$, and so X_0 and $\underline{X}_0$ are of $K-$type 0. Similarly, if $x \in X_0 \cap X_1$, then $\|x\|_0 \leq J(x,t)$ for all $t \in \mathbb{R}_+$, so X_0 and $\underline{X}_0$ are of $J-$type 0. Consequently, X_0 and $\underline{X}_0$ are of type 0.

On the other hand, $K(x,t) \leq t\|x\|_1$ for all $x \in X_0 + X_1$ and $t \in \mathbb{R}_+$, so X_1 and $\bar{X}_1$ are of $K-$type 1. If $x \in X_0 \cap X_1$ then $t\|x\|_1 \leq J(x,t)$ for all $t \in \mathbb{R}_+$, and so X_1 and $\underline{X}_1$ are of $J-$type 1 as well. Thus, X_1 is of type 1.

Let $\theta \in (0,1)$ and $1 \leq p \leq \infty$. We know from Theorem 2.2.8 and Proposition 2.3.4 that $(X_0, X_1)_{\theta,p}$ is of type θ. △

It turns out that we can exactly characterize which spaces are of $J-$type and $K-$type θ in terms of embeddings with familiar spaces. We record this now.

Proposition 2.3.7. *Let X_0 and X_1 be compatible Banach spaces and X be intermediate to X_0 and X_1. Then the following hold for every $\theta \in [0,1]$.*

1. *X is of $K-$type θ if and only if $X \hookrightarrow (X_0, X_1)_{\theta,\infty}$.*
2. *X is of $J-$type θ if and only if*

$$\begin{cases} (X_0, X_1)_{\theta,1} \hookrightarrow X & \text{if } \theta \in (0,1) \\ \underline{X}_\theta \hookrightarrow X & \text{if } \theta \in \{0,1\}. \end{cases} \tag{2.3.39}$$

3. *The following are equivalent for $\theta \in (0,1)$.*
   (a) *X is of type θ.*
   (b) *$(X_0, X_1)_{\theta,1} \hookrightarrow X \hookrightarrow (X_0, X_1)_{\theta,\infty}$.*
   (c) *X is intermediate to $(X_0, X_1)_{\theta,1}$ and $(X_0, X_1)_{\theta,\infty}$.*
4. *The following are equivalent for $\theta \in \{0,1\}$.*
   (a) *X is of type θ.*
   (b) *$\underline{X}_\theta \hookrightarrow X \hookrightarrow (X_0, X_1)_{\theta,\infty}$.*
   (c) *X is intermediate to $\underline{X}_\theta$ and $(X_0, X_1)_{\theta,\infty}$.*

Proof. The first item follows directly from the definition of $K-$type, and the third and fourth items follow from the first and second together with Theorem 2.2.8, so we only need to prove the second.

Suppose initially that $\theta \in (0,1)$. If $(X_0, X_1)_{\theta,1} \hookrightarrow X$, then there is a constant $C > 0$ such that $\|x\|_X \leq C \|x\|_{\theta,p}$ for all $x \in X_0 + X_1$, and so X is of $J-$type θ by Proposition 2.3.4. Conversely, suppose that X is of $J-$type θ. Let $x \in (X_0, X_1)_{\theta,1}$, and write $x = \sum_{n\in\mathbb{Z}} x_n$ for $\{x_n\}_{n\in\mathbb{Z}} \subseteq X_0 \cap X_1$. Then

$$\sum_{n\in\mathbb{Z}} \|x_n\|_X \leq C \sum_{n\in\mathbb{Z}} \frac{J(x_n, 2^n)}{2^{n\theta}} = C \|x\|_{\theta,1}, \tag{2.3.40}$$

and so Theorem 2.1.1 shows that $x = \sum_{n\in\mathbb{Z}} x_n$ with the series converging in X; moreover,

$$\|x\|_X \leq \sum_{n\in\mathbb{Z}} \|x_n\|_X \leq C \|x\|_{\theta,1}. \tag{2.3.41}$$

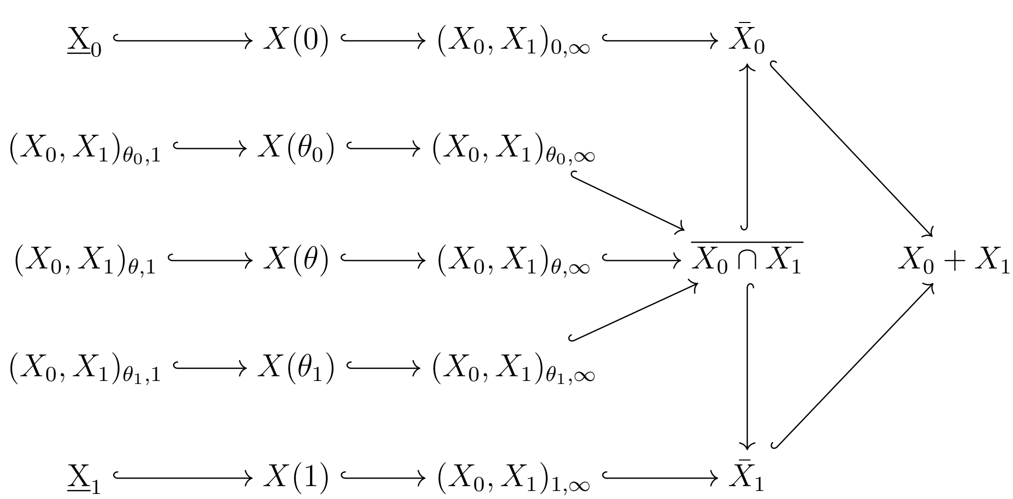
Hence, $(X_0, X_1)_{\theta,1} \hookrightarrow X$, and the second item is proved in the case $\theta \in (0,1)$.

Now assume that $\theta \in \{0,1\}$. Suppose $\underline{X}_\theta \hookrightarrow X$. Then there exists a constant $C > 0$ such that $\|x\|_X \leq C \|x\|_\theta$ since the norm on the space $\underline{X}_\theta$ is precisely $\|\cdot\|_0$. Thus, for $x \in X_0 \cap X_1$ we have that $\|x\|_X \leq C \|x\|_\theta \leq Ct^{-\theta} J(x,t)$ for all $t \in \mathbb{R}_+$, and we deduce that X is of $J-$type θ. Conversely, suppose that X is of $J-$type θ, so that there exists a constant $C > 0$ such that $\|x\|_X \leq Ct^{-\theta} J(x,t)$ for all $t \in \mathbb{R}_+$ and $x \in X_0 \cap X_1$. In particular, for a fixed $x \in X_0 \cap X_1$ we an send $t \to 0$ if $\theta = 0$ and $t \to \infty$ if $\theta = 1$ to see deduce from this that $\|x\|_X \leq C \|x\|_\theta$. Now let $x \in \underline{X}_\theta$ and pick a sequence $\{x_n\}_{n=\ell}^\infty \subseteq X_0 \cap X_1$ such that $x_n \to x$ in X_θ. For $m, n \geq \ell$ we then have that $\|x_n - x_m\|_X \leq C \|x_n - x_m\|_\theta$, which implies that $\{x_n\}_{n=\ell}^\infty$ is Cauchy in X, and hence convergent in X to x (thanks to the compatibility of X_0 and X_1). Hence,

$$\|x\|_X = \lim_{n\to\infty} \|x_n\|_X \leq C \lim_{n\to\infty} \|x_n\|_\theta = C \|x\|_\theta \text{ for all } x \in \underline{X}_\theta, \tag{2.3.42}$$

and we deduce that $\underline{X}_\theta \hookrightarrow X$. This completes the proof of the second item when $\theta \in \{0,1\}$. □

We can make a variant of the diagram (2.2.40) to indicate how to think about the location of the spaces of type $\theta \in [0,1]$ within the collection of interpolation spaces. In the following diagram we again write $0 < \theta_0 < \theta < \theta_1 < 1$ and indicate a generic space $X(\psi)$ as a space of type ψ, which must lie in the indicated position along the horizontal lines.



(2.3.43)

The notion of type is exactly what we need to answer the question raised above. We now state the answer as the important "reiteration theorem."

Theorem 2.3.8 (Reiteration theorem)**.** *Let X_0 and X_1 be compatible Banach spaces and Y_0 and Y_1 be intermediate spaces to X_0 and X_1. Suppose that $0 \le \theta_0 < \theta_1 \le 1$, $1 \le p \le \infty$, $\sigma \in (0,1)$, and $\theta = (1-\sigma)\theta_0 + \sigma\theta_1 \in (0,1)$. Then the following hold.*

1. *If Y_i is of $K-$type θ_i for $i \in \{0,1\}$, then*
$$(Y_0, Y_1)_{\sigma,p} \hookrightarrow (X_0, X_1)_{\theta,p}. \tag{2.3.44}$$

2. *If Y_i is of $J-$type θ_i for $i \in \{0,1\}$, then*
$$(X_0, X_1)_{\theta,p} \hookrightarrow (Y_0, Y_1)_{\sigma,p}. \tag{2.3.45}$$

3. *If Y_i is of type θ_i for $i \in \{0,1\}$, then*
$$(X_0, X_1)_{\theta,p} = (Y_0, Y_1)_{\sigma,p} \tag{2.3.46}$$
with equivalence of norms on these spaces.

Proof. First note that the first and second items imply the third, so we must only prove these two. In order to keep the association between the K and J functions and the pair of spaces clear, we will write

$$K_X(x,t) = \inf\{\|x\|_{X_0} + t\|x\|_{X_1} \mid x = x_0 + x_1\},\ J_X(x,t) = \max\{\|x\|_{X_0}, t\|x\|_{X_1}\} \tag{2.3.47}$$

and

$$K_Y(x,t) = \inf\{\|x\|_{Y_0} + t\|x\|_{Y_1} \mid x = x_0 + x_1\},\ J_Y(x,t) = \max\{\|x\|_{Y_0}, t\|x\|_{Y_1}\}. \tag{2.3.48}$$

Note also that σ satisfies
$$\sigma = \frac{\theta - \theta_0}{\theta_1 - \theta_0}. \tag{2.3.49}$$

We begin with the proof of the first item. Suppose that $x \in (Y_0, Y_1)_{\sigma,p} \hookrightarrow Y_0 + Y_1$ and let $x = y_0 + y_1$ for $y_i \in Y_i$. By hypothesis, there is a constant $C > 0$ such that

$$K_X(y,t) \le Ct^{\theta_i}\|y\|_{Y_i} \text{ for all } y \in X_0 + X_1 \text{ and } t \in \mathbb{R}_+. \tag{2.3.50}$$

Thus,

$$K_X(x,t) \le K_X(y_0,t) + K_X(y_1,t) \le Ct^{\theta_0}\|y_0\|_{Y_0} + Ct^{\theta_1}\|y_1\|_{Y_1} = Ct^{\theta_0}\left(\|y_0\|_{Y_0} + t^{\theta_1-\theta_0}\|y_1\|_{Y_1}\right), \tag{2.3.51}$$

and since this holds for all such decompositions, we find that

$$K_X(x,t) \le Ct^{\theta_0}K_Y(x, t^{\theta_1-\theta_0}). \tag{2.3.52}$$

If $p = \infty$, then this and (2.3.49) imply that

$$\sup_{t>0}\frac{K_X(x,t)}{t^\theta} \le C\sup_{t>0} t^{\theta_0-\theta}K_Y(x,t^{\theta_1-\theta_0}) = C\sup_{t>0}\frac{K_Y(x,t)}{t^\sigma}, \tag{2.3.53}$$

while if $p < \infty$ this, (2.3.49), and a change of variables show that

$$\left(\int_{\mathbb{R}_+} (t^{-\theta} K_X(x,t)) \frac{dt}{dt} \right)^{1/p} \le C \left(\int_{\mathbb{R}_+} (t^{\theta_0 - \theta} K_Y(x, t^{\theta_1 - \theta_0})) \frac{dt}{dt} \right)^{1/p}$$
$$= \frac{C}{(\theta_1 - \theta_0)^{1/p}} \left(\int_{\mathbb{R}_+} (t^{-\sigma} K_Y(x,t)) \frac{dt}{dt} \right)^{1/p}. \quad (2.3.54)$$

We deduce from these that $(Y_0, Y_1)_{\sigma,p} \hookrightarrow (X_0, X_1)_{\theta,p}$, which proves the first item.

We now turn to the proof of the second item. Suppose that $x \in (X_0, X_1)_{\theta,p}$ and write $x = \sum_{n\in\mathbb{Z}} x_n$ for $\{x_n\}_{n\in\mathbb{Z}} \subseteq X_0 \cap X_1$ and the series converging in $X_0 + X_1$. Then by Proposition 2.1.18,

$$K_Y(x, 2^{(\theta_1-\theta_0)n}) \le \sum_{m\in\mathbb{Z}} K_Y(x_m, 2^{(\theta_1-\theta_0)n}) \le \sum_{m\in\mathbb{Z}} \min\{1, 2^{(\theta_1-\theta_0)(n-m)}\} J_Y(x_m, 2^{(\theta_1-\theta_0)m}). \quad (2.3.55)$$

For each $m \in \mathbb{Z}$ we can use the fact that Y_i is of $J-$type θ_i to bound

$$J_Y(x_m, 2^{(\theta_1-\theta_0)m}) = \max\{\|x_m\|_{Y_0}, 2^{(\theta_1-\theta_0)m} \|x_m\|_{Y_1}\} \le C 2^{-m\theta_0} J_X(x_m, 2^m), \quad (2.3.56)$$

and we can use (2.3.49) to compute

$$2^{-(\theta_1-\theta_0)\sigma n} \min\{1, 2^{(\theta_1-\theta_0)(n-m)}\} 2^{-m\theta_0} = 2^{-(\theta-\theta_0)n} \min\{1, 2^{(\theta_1-\theta_0)(n-m)}\} 2^{-m\theta_0}$$
$$= 2^{-\theta(n-m)} \min\{2^{\theta_0(n-m)}, 2^{\theta_1(n-m)}\} 2^{-m\theta}. \quad (2.3.57)$$

Thus, we have the estimate

$$2^{-(\theta_1-\theta_0)\sigma n} K_Y(x, 2^{(\theta_1-\theta_0)n}) \le C \sum_{m\in\mathbb{Z}} d_{n-m} 2^{-m\theta} J_X(x_m, 2^m) \quad (2.3.58)$$

for

$$d_n = 2^{-\theta n} \min\{2^{\theta_0 n}, 2^{\theta_1 n}\}. \quad (2.3.59)$$

Since

$$\|d\|_{\ell^1} = \sum_{n\le 0} 2^{(\theta_1-\theta)n} + \sum_{0<n} 2^{-n(\theta-\theta_0)} < \infty \quad (2.3.60)$$

we can employ Young's inequality and Theorem 2.3.1 to bound

$$\|x\|_{(Y_0,Y_1)_{\sigma,p}} \le C \|d\|_{\ell^1} \left\| \{2^{-m\theta} J_X(x_m, 2^m)\}_{m\in\mathbb{Z}} \right\|_{\ell^p}. \quad (2.3.61)$$

This holds for all such decompositions of x, and so again Theorem 2.3.1 shows that

$$\|x\|_{(Y_0,Y_1)_{\sigma,p}} \le C \|d\|_{\ell^1} \|x\|_{(X_0,X_1)_{\theta,p}}. \quad (2.3.62)$$

We deduce from this that $(X_0, X_1)_{\theta,p} \hookrightarrow (Y_0, Y_1)_{\sigma,p}$, which completes the proof of the second item.
□

2.4 Examples and applications

We now turn our attention to characterizing the interpolation spaces that arise from various natural choices of X_0 and X_1.

2.4.1 Interpolation of Lebesgue spaces

We motivated our construction of the abstract interpolation spaces by examining $L^1(X;\mathbb{F})$ and $L^\infty(X;\mathbb{F})$ for a given measure space $(X,\mathfrak{M},\mu)$, and so it is natural to begin by studying what results when we use these in our abstract framework. The key idea is contained in the following beautiful theorem, which relates the decreasing rearrangement to the K function.

Theorem 2.4.1. *Let $(X,\mathfrak{M},\mu)$ be a measure space and $f \in L^0(X;\mathbb{F})$. Then the following are equivalent.*

1. *$f \in L^1(X;\mathbb{F}) + L^\infty(X;\mathbb{F})$.*
2. *There exists $t \in \mathbb{R}_+$ such that $\int_0^t f^\#(s)ds < \infty$.*
3. *For each $t \in \mathbb{R}_+$ we have that $\int_0^t f^\#(s)ds < \infty$.*

Moreover, if any of these (and hence all) holds, we have that

$$K(f,t) = \int_0^t f^\#(s)ds \text{ for all } t \in \mathbb{R}_+. \tag{2.4.1}$$

Proof. The third item trivially implies the second. If the second holds for some t, then by the monotonicity of $f^\#$ we must have that $f^\#(t) < \infty$ or else the integral would be infinite. If $0 < r \le t$, then $\int_0^r f^\#(s)ds \le \int_0^t f^\#(s)ds < \infty$. On the other hand, if $t < r < \infty$, then

$$\int_0^r f^\#(s)ds = \int_0^t f^\#(s)ds + \int_t^r f^\#(s)ds \le \int_0^t f^\#(s)ds + (r-t)f^\#(t) < \infty. \tag{2.4.2}$$

Thus, the second item implies the third.

Suppose now that the first item holds. Then $K(f,t) < \infty$ for every $t \in \mathbb{R}_+$ by Proposition 2.1.16. Let $f = g + h$ for $g \in L^1(X;\mathbb{F})$ and $h \in L^\infty(X;\mathbb{F})$. Then for $s \in \mathbb{R}_+$ and $0 < \varepsilon < 1$ we may may use Proposition 1.1.21 to bound $f^\#(s) \le g^\#((1-\varepsilon)s) + h^\#(\varepsilon s)$. Thus, for $t \in \mathbb{R}_+$ we have the estimate

$$\begin{aligned}\int_0^t f^\#(s)ds \le \int_0^t g^\#((1-\varepsilon)s)ds + \int_0^t h^\#(\varepsilon s)ds &\le \frac{1}{1-\varepsilon}\int_0^\infty g^\#(s)ds + th^\#(0) \\ &= \frac{1}{1-\varepsilon}\left\|g\right\|_{L^1} + t\left\|h\right\|_{L^\infty}.\end{aligned} \tag{2.4.3}$$

This holds for all $0 < \varepsilon < 1$ and all such decompositions, so we deduce that

$$\int_0^t f^\#(s)ds \le K(f,t). \tag{2.4.4}$$

Thus, the first item implies the third.

Now suppose the third item holds and let $t \in \mathbb{R}_+$. From the above analysis we know that $f^\#(t) < \infty$. Define $g, h \in L^0(X;\mathbb{F})$ via

$$g = \begin{cases} f - f^\#(t)\frac{f}{|f|} & \text{if } |f| > f^\#(t) \\ 0 & \text{if } |f| \le f^\#(t) \end{cases} \tag{2.4.5}$$

and

$$h = f - g = \begin{cases} f^{\#}(t)\frac{f}{|f|} & \text{if } |f| > f^{\#}(t) \\ f & \text{if } |f| \leq f^{\#}(t). \end{cases} \tag{2.4.6}$$

Define the set $E = \{x \in X \mid g(x) \neq 0\} = \{x \in X \mid |f(x)| > f^{\#}(t)\}$ and note that if $x \in E$, then $|f(x)| > 0$ and $|g(x)| = |f(x)| - f^{\#}(t)$. Proposition 1.1.21 then provides the estimate

$$\mu(E) = d_f(f^{\#}(t)) \leq t. \tag{2.4.7}$$

If there exists s such that $d_f(f^{\#}(t)) < s < t$, then we can use Proposition 1.1.21 again to see that

$$f^{\#}(t) \leq f^{\#}(s) \leq f^{\#}(d_f(f^{\#}(t))) \leq f^{\#}(t). \tag{2.4.8}$$

Hence, $f^{\#}(s) = f^{\#}(t)$ for all $s \in [\mu(E), t]$. We now have enough information to estimate $\|g\|_{L^1}$ and $\|h\|_{L^\infty}$. Indeed,

$$\|g\|_{L^1} = \int_E |g|\, d\mu = \int_E (|f| - f^{\#}(t)) d\mu = \int_X |f|\, \chi_E d\mu - \mu(E) f^{\#}(t) = \int_0^\infty (f\chi_E)^{\#}(s) ds - \mu(E) f^{\#}(t) \tag{2.4.9}$$

but Lemma 1.1.24 implies that $(f\chi_E)^{\#}(s) \leq f^{\#}(s)\chi_{(0,\mu(E))}(s)$ for $s \in \mathbb{R}_+$, and so

$$\|g\|_{L^1} \leq \int_0^{\mu(E)} f^{\#}(s) ds - \mu(E) f^{\#}(t). \tag{2.4.10}$$

On the other hand,

$$\begin{aligned} t\,\|h\|_{L^\infty} \leq t f^{\#}(t) = \mu(E) f^{\#}(t) + (t - \mu(E)) f^{\#}(t) = \mu(E) f^{\#}(t) + \int_{\mu(E)}^t f^{\#}(t) ds \\ = \mu(E) f^{\#}(t) + \int_{\mu(E)}^t f^{\#}(s) ds. \end{aligned} \tag{2.4.11}$$

Thus, $f \in L^1(X;\mathbb{F}) + L^\infty(X;\mathbb{F})$ and

$$K(f,t) \leq \|g\|_{L^1} + t\,\|h\|_{L^\infty} \leq \int_0^t f^{\#}(s) ds. \tag{2.4.12}$$

This completes the proof that the third item implies the first and that if any of the three items hold we have that $K(f,t) = \int_0^t f^{\#}(s) ds$ for all $t \in \mathbb{R}_+$. □

With the previous theorem in hand, we can now characterize the Lorentz spaces as the abstract interpolation spaces generated by interpolating between $L^1(X;\mathbb{F})$ and $L^\infty(X;\mathbb{F})$.

Theorem 2.4.2. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, $1 < p < \infty$, and $1 \leq q \leq \infty$. Then there exists a constant $C > 0$ such that*

$$|||f|||_{L^{p,q}} \leq \|f\|_{1/p',q} \leq C\, |||f|||_{L^{p,q}} \tag{2.4.13}$$

for every $f \in L^0(X;\mathbb{F})$. Consequently, we have the algebraic and topological identity

$$(L^1(X;\mathbb{F}), L^\infty(X;\mathbb{F}))_{1/p',q} = L^{p,q}(X;\mathbb{F}), \tag{2.4.14}$$

which in particular means that the Lorentz space $L^{p,q}(X;\mathbb{F})$ admits a norm that generates the same topology as the quasinorm and makes the space Banach.

Proof. Suppose that $f \in L^{p,q}(X;\mathbb{F})$. Then $f \in L^{p,\infty}(X;\mathbb{F})$ and so $f^{\#}(s) \le s^{-1/p} \left\lVert\!\left| f \right|\!\right\rVert_{L^{p,\infty}}$, which means that $\int_0^t f^{\#}(s)ds < \infty$ for every $t \in \mathbb{R}_+$. Then the previous theorem implies that

$$t^{1/p-1}K(f,t) = t^{1/p-1}\int_0^t f^{\#}(s)ds < \infty \text{ for all } t \in \mathbb{R}_+, \tag{2.4.15}$$

and we can use this and Hardy's inequality when $q < \infty$ to deduce that

$$\|f\|_{1/p',q} \le C \left\lVert\!\left| f \right|\!\right\rVert_{L^{p,q}} \tag{2.4.16}$$

for a constant $C = C(p,q) > 0$. Thus, $L^{p,q}(X;\mathbb{F}) \hookrightarrow (L^1(X;\mathbb{F}), L^\infty(X;\mathbb{F}))_{1/p',q}$.

Conversely, suppose $f \in (L^1(X;\mathbb{F}), L^\infty(X;\mathbb{F}))_{1/p',q}$. Then $f \in L^1(X;\mathbb{F}) + L^\infty(X;\mathbb{F})$, and so the previous theorem and the fact that $f^{\#}$ is nondecreasing imply that

$$t^{1/p}f^{\#}(t) \le t^{1/p-1}\int_0^t f^{\#}(s)ds = t^{1/p-1}K(f,t) < \infty \text{ for all } t \in \mathbb{R}_+. \tag{2.4.17}$$

Thus,

$$\left\lVert\!\left| f \right|\!\right\rVert_{L^{p,q}} \le \|f\|_{1/p',q}, \tag{2.4.18}$$

and we deduce that $(L^1(X;\mathbb{F}), L^\infty(X;\mathbb{F}))_{1/p',q} \hookrightarrow L^{p,q}(X;\mathbb{F})$. □

Remark 2.4.3. *Note that the theorem does apply to the spaces $L^{1,q}(X;\mathbb{F})$ for $1 \le q \le \infty$. In fact, it can be shows that when $q > 1$ these spaces are not normable, and we are stuck with the quasi-norm.*

As a consequence of this theorem and the reiteration theorem, we can interpolate between Lorentz spaces as well.

Corollary 2.4.4. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, $1 < p_0, p_1 < \infty$, and $1 \le q_0, q_1 \le \infty$. Let $1 \le q \le \infty$, $\theta \in (0,1)$, and define $1 < p_\theta < \infty$ via*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{2.4.19}$$

Then

$$(L^{p_0,q_0}(X;\mathbb{F}), L^{p_1,q_1}(X;\mathbb{F}))_{\theta,q} = L^{p_\theta,q}(X;\mathbb{F}) \tag{2.4.20}$$

and

$$(L^{p_0}(X;\mathbb{F}), L^{p_1}(X;\mathbb{F}))_{\theta,p_\theta} = L^{p_\theta}(X;\mathbb{F}). \tag{2.4.21}$$

Proof. First note that

$$\frac{1}{p_\theta'} = \frac{1-\theta}{p_0'} + \frac{\theta}{p_1'}. \tag{2.4.22}$$

We know from Theorem 2.4.2 that for $i \in \{0,1\}$

$$L^{p_i,q_i}(X;\mathbb{F}) = (L^1(X;\mathbb{F}), L^\infty(X;\mathbb{F}))_{1/p_i',q} \tag{2.4.23}$$

and is thus of type $1/p_i'$. Then Theorems 2.3.8 and 2.4.2 combine to show that

$$(L^{p_0,q_0}(X;\mathbb{F}), L^{p_1,q_1}(X;\mathbb{F}))_{\theta,q} = (L^1(X;\mathbb{F}), L^\infty(X;\mathbb{F}))_{1/p_\theta',q} = L^{p_\theta,q}(X;\mathbb{F}). \tag{2.4.24}$$

In particular, if we set $q_i = p_i$ and $q = p_\theta$, then we find the Lebesgue interpolation result

$$(L^{p_0}(X;\mathbb{F}), L^{p_1}(X;\mathbb{F}))_{\theta,p_\theta} = L^{p_\theta}(X;\mathbb{F}). \tag{2.4.25}$$

□

2.4.2 Interpolating between L^p and $W^{1,p}$

We now aim to interpolate between $L^p(\mathbb{R}^n;\mathbb{F})$ and and the Sobolev space $W^{1,p}(\mathbb{R}^n;\mathbb{F})$. To do this, we first need to recall the one definition of the Besov spaces (there are *many* available with different degrees of usefulness, depending on the area of intended use - Chapter 17 of Leoni's book [4] does a nice job of clarifying the relations among these definitions).

Definition 2.4.5. *We define the following.*

1. *Given a function $f : \mathbb{R}^n \to \mathbb{F}$ and $y \in \mathbb{R}^n$ define the difference quotient $\Delta_y f : \mathbb{R}^n \to \mathbb{F}$ via*

$$\Delta_y f(x) = f(x+y) - f(x). \tag{2.4.26}$$

2. *For $1 \le p \le \infty$, $1 \le q \le \infty$, and $s \in (0,1)$ the Besov space of regularity s and integrability parameters p,q is the space*

$$B^{s,p}_q(\mathbb{R}^n;\mathbb{F}) = \{[f] \mid f : \mathbb{R}^n \to \mathbb{F} \text{ is measurable and } \|f\|_{B^{s,p}_q} < \infty\}, \tag{2.4.27}$$

where

$$\|f\|_{B^{s,p}_q} = \|f\|_{L^p} + [f]_{B^{s,p}_q}, \tag{2.4.28}$$

and

$$[f]_{B^{s,p}_q} = \left(\int_{\mathbb{R}^n} \frac{\|\Delta_y f\|^q_{L^p}}{|y|^{n+sq}} dy \right)^{1/q} < \infty \tag{2.4.29}$$

when $1 \le q < \infty$ and

$$[f]_{B^{s,p}_\infty} = \sup_{0 \ne y \in \mathbb{R}^n} \frac{\|\Delta_y f\|_{L^p}}{|y|^s} \tag{2.4.30}$$

when $q = \infty$.

The space $B^{s,p}_q(\mathbb{R}^n;\mathbb{F})$ is a Banach space when endowed with this norm, a fact that we leave as an exercise to verify. Note that when $p = q < \infty$, we may use a change of variables to see that

$$[f]_{B^{s,p}_p} = \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} dx dy \right)^{1/p}, \tag{2.4.31}$$

which shows that $B^{s,p}_p(\mathbb{R}^n;\mathbb{F}) = W^{s,p}(\mathbb{R}^n;\mathbb{F})$, where the latter is the fractional Sobolev space of regularity s and integrability p. We now prove that the Besov spaces are the interpolation spaces between L^p and $W^{1,p}$.

Theorem 2.4.6. *Now let $1 \le p < \infty$, $1 \le q \le \infty$, and $\theta \in (0,1)$. Then*

$$(L^p(\mathbb{R}^n;\mathbb{F}), W^{1,p}(\mathbb{R}^n;\mathbb{F}))_{\theta,q} = B^{\theta,p}_q(\mathbb{R}^n;\mathbb{F}) \tag{2.4.32}$$

with equivalence of norms.

Proof. Suppose first that $f \in B^{\theta,p}_q(\mathbb{R}^n;\mathbb{F})$. Let $\eta \in C^\infty_c(\mathbb{R}^n)$ be such that $\operatorname{supp}(\eta) \subseteq B(0,1)$, $\eta \ge 0$ is radial, and $\int_{\mathbb{R}^n} \eta = 1$. For $t \in \mathbb{R}_+$ write $\eta_t(x) = t^{-n}\eta(x/t)$. Then we write

$$f = (f - f * \eta_t) + f * \eta_t =: g_t + h_t. \tag{2.4.33}$$

Since η is radial, we can compute

$$g_t(x) = \int_{\mathbb{R}^n} (f(x) - f(x-y))t^{-n}\eta(y/t)dy = \int_{\mathbb{R}^n} (f(x) - f(x+y))t^{-n}\eta(y/t)dy$$
$$= \int_{\mathbb{R}^n} -\Delta_y f(x) t^{-n}\eta(y/t)dy. \quad (2.4.34)$$

From this and Minkowski's integral inequality, we have that

$$\|g_t\|_{L^p} \leq \int_{\mathbb{R}^n} \|\Delta_y f\|_{L^p} t^{-n}\eta(y/t)dy. \quad (2.4.35)$$

If $q = \infty$, then

$$t^{-\theta} \|g_t\|_{L^p} \leq \int_{\mathbb{R}^n} \|\Delta_y f\|_{L^p} t^{-n-\theta}\eta(y/t)dy \leq \sup_{y\neq 0} \frac{\|\Delta_y f\|_{L^p}}{|y|^\theta} \int_{\mathbb{R}^n} |y/t|^\theta t^{-n}\eta(y/t)dy$$
$$\leq \|f\|_{B^{\theta,p}_\infty} \int_{\mathbb{R}^n} |y|^\theta \eta(y)dy = C(\eta) \|f\|_{B^{\theta,p}_\infty}. \quad (2.4.36)$$

On the other hand, if $q < \infty$, then we may use the normalization $\int \eta = 1$ together with Hölder's inequality (or Jensen's inequality) to bound

$$\|g_t\|^q_{L^p} \leq \int_{\mathbb{R}^n} \|\Delta_y f\|^q_{L^p} t^{-n}\eta(y/t)dy, \quad (2.4.37)$$

which implies, thanks to Fubini-Tonelli, that

$$\int_0^\infty \frac{\|g_t\|^q_{L^p}}{t^{1+\theta q}}dt \leq \int_{\mathbb{R}^n} \|\Delta_y f\|^q_{L^p} \int_0^\infty t^{-n-1-\theta q}\eta(y/t)dtdy. \quad (2.4.38)$$

Since $\operatorname{supp}(\eta) \subseteq B(0,1)$ we then see from this that

$$\int_0^\infty \frac{\|g_t\|^q_{L^p}}{t^{1+\theta q}}dt \leq \|\eta\|_{L^\infty} \int_{\mathbb{R}^n} \|\Delta_y f\|^q_{L^p} \int_{|y|}^\infty t^{-n-1-\theta q}dtdy = C(\eta, q, \theta, n) \int_{\mathbb{R}^n} \frac{\|\Delta_y f\|^q_{L^p}}{|y|^{n+\theta q}}dy$$
$$\leq C \|f\|^q_{B^{\theta,p}_q}. \quad (2.4.39)$$

For the term h_t we begin by bounding

$$\|h_t\|_{L^p} \leq \|f\|_{L^p} \quad (2.4.40)$$

with Young's inequality. We know that $\int_{\mathbb{R}^n} \nabla\eta(z)dz = 0$ and $\nabla\eta(z) = -\nabla\eta(-z)$, so we can compute

$$\nabla h_t(x) = \int_{\mathbb{R}^n} -(f(x+y) - f(x))t^{-n-1}\nabla\eta(y/t)dy \quad (2.4.41)$$

in order to bound, again with the help of Minkowski,

$$\|\nabla h_t\|_{L^p} \leq \int_{\mathbb{R}^n} \|\Delta_y f\|_{L^p} t^{-n-1} |\nabla\eta(y/t)| dy. \quad (2.4.42)$$

If $q = \infty$, then

$$t^{1-\theta} \|\nabla h_t\|_{L^p} \leq \|f\|_{B^{\theta,p}_\infty} \int_{\mathbb{R}^n} |y|^\theta |\nabla\eta(y)| dy = C(\eta) \|f\|_{B^{\theta,p}_\infty}. \quad (2.4.43)$$

On the other hand, $q < \infty$, then we us Hölder's inequality on (2.4.42) to bound

$$\|\nabla h_t\|_{L^p}^q \le Ct^{-q/q'} \int_{\mathbb{R}^n} \|\Delta_y f\|_{L^p}^q t^{-n-1} |\nabla \eta(y/t)| \, dy, \tag{2.4.44}$$

where we have used the computation

$$\left(\int_{\mathbb{R}^n} t^{-n-1} |\nabla \eta(y/t)| \, dy \right)^{1/q'} = t^{-1/q'} \|\nabla \eta\|_{L^1}^{1/q'} = Ct^{-1/q'} \text{ when } 1 < q. \tag{2.4.45}$$

This and Fubini-Tonelli then show us that

$$\begin{aligned} \int_0^\infty (t \|\nabla h_t\|_{L^p})^q \frac{dt}{t^{1+\theta q}} &\le C \int_0^\infty t^{-q/q'+q-1-\theta q} \int_{\mathbb{R}^n} \|\Delta_y f\|_{L^p}^q t^{-n-1} |\nabla \eta(y/t)| \, dy dt \\ &\le C \int_{\mathbb{R}^n} \|\Delta_y f\|_{L^p}^q \int_0^\infty t^{-n-1-\theta q} |\nabla \eta(y/t)| \, dt dy \le C \int_{\mathbb{R}^n} \|\Delta_y f\|_{L^p}^q \int_{|y|}^\infty t^{-n-1-\theta q} dt dy \\ &\le C \int_{\mathbb{R}^n} \frac{\|\Delta_y f\|_{L^p}^q}{|y|^{n+\theta q}} dy \le C \|f\|_{B_q^{\theta,p}}^q . \end{aligned} \tag{2.4.46}$$

Now, if $q = \infty$, then we may combine (2.4.36), (2.4.40), and (2.4.43) with Theorem 2.2.16 to bound

$$\begin{aligned} \|f\|_{\theta,\infty} \asymp \|f\|_{L^p} + \sup_{0<t<1} t^{-\theta} K(f,t) \asymp \|f\|_{L^p} + \sup_{0<t<1} \left(t^{-\theta} \|g_t\|_{L^p} + t^{1-\theta} \|h_t\|_{L^p} + t^{1-\theta} \|\nabla h_t\|_{L^p} \right) \\ \le C \|f\|_{B_\infty^{\theta,p}} . \end{aligned} \tag{2.4.47}$$

On the other hand, if $q < \infty$, then we instead use (2.4.39), (2.4.40), and (2.4.43) with Theorem 2.2.16 to bound

$$\begin{aligned} \|f\|_{\theta,q}^q \asymp \|f\|_{L^p}^q + \int_0^1 (K(f,t))^q \frac{dt}{t^{1+\theta q}} \le \|f\|_{L^p}^q + \int_0^1 (\|g_t\|_{L^p}^q + t^q \|h_t\|_{L^p}^q + t^q \|\nabla h_t\|_{L^p}^q) \frac{dt}{t^{1+\theta q}} \\ \le C \|f\|_{B_q^{\theta,p}}^q . \end{aligned} \tag{2.4.48}$$

Thus, in either case we deduce that

$$B_q^{\theta,p}(\mathbb{R}^n;\mathbb{F}) \hookrightarrow (L^p(\mathbb{R}^n;\mathbb{F}); W^{1,p}(\mathbb{R}^n;\mathbb{F}))_{\theta,q}. \tag{2.4.49}$$

To conclude, it remains to prove the reverse inclusion.

Suppose that $f \in (L^p(\mathbb{R}^n;\mathbb{F}); W^{1,p}(\mathbb{R}^n;\mathbb{F}))_{\theta,q}$ and write $f = g + h$ for $g \in L^p(\mathbb{R}^n;\mathbb{F})$ and $h \in W^{1,p}(\mathbb{R}^n;\mathbb{F})$. From the triangle inequality we have that

$$\|f\|_{L^p} \le \|g\|_{L^p} + \|h\|_{L^p} \le \|g\|_{L^p} + \|h\|_{W^{1,p}} \le K(f,1) \le C \|f\|_{\theta,q} , \tag{2.4.50}$$

where in the last inequality we have used Theorem 2.2.8. On the other hand, for $y \in \mathbb{R}^n$ we may write $\Delta_y f = \Delta_y g + \Delta_y h$ and use the translation invariance of Lebesgue measure to estimate

$$\|\Delta_y f\|_{L^p} \le \|\Delta_y g\|_{L^p} + \|\Delta_y h\|_{L^p} \le 2 \|g\|_{L^p} + \|\Delta_y h\|_{L^p} . \tag{2.4.51}$$

Next note that if $\psi \in C^1(\mathbb{R}^n;\mathbb{F}) \cap W^{1,p}(\mathbb{R}^n;\mathbb{F})$, then by the fundamental theorem of calculus,

$$\Delta_y \psi(x) = \int_0^1 y \cdot \nabla \psi(x + ty) dt, \tag{2.4.52}$$

and so Minkowski's inequality shows that

$$\|\Delta_y \psi\|_{L^p} \le |y| \int_0^1 \|\nabla \psi\|_{L^p} \, dt = |y| \, \|\nabla \psi\|_{L^p} \,. \tag{2.4.53}$$

However, basic Sobolev theory shows that smooth functions are dense in $W^{1,p}(\mathbb{R}^n;\mathbb{F})$, so this estimate continues to hold for general $\psi \in W^{1,p}(\mathbb{R}^n;\mathbb{F})$ by an approximation argument.

Using (2.4.53) on h, we find that

$$\|\Delta_y h\|_{L^p} \le |y| \, \|h\|_{W^{1,p}} \,, \tag{2.4.54}$$

and so (2.4.51) shows that

$$\|\Delta_y f\|_{L^p} \le 2 \left(\|g\|_{L^p} + |y| \, \|h\|_{W^{1,p}} \right) \tag{2.4.55}$$

for all decompositions $f = g + h$ with $g \in L^p(\mathbb{R}^n;\mathbb{F})$ and $h \in W^{1,p}(\mathbb{R}^n;\mathbb{F})$. Thus, for $y \neq 0$ we have that

$$\|\Delta_y f\|_{L^p} \le 2K(f, |y|). \tag{2.4.56}$$

If $q = \infty$, this implies that

$$[f]_{B^{\theta,p}_\infty} = \sup_{y \neq 0} \frac{\|\Delta_y f\|_{L^p}}{|y|^\theta} \le 2 \sup_{t>0} \frac{K(f,t)}{t^\theta} = 2 \, \|f\|_{\theta,\infty} \,. \tag{2.4.57}$$

On the other hand, if $q < \infty$, it instead implies that

$$[f]^q_{B^{\theta,p}_q} = \int_{\mathbb{R}^n} \frac{\|\Delta_y f\|^q_{L^p}}{|y|^{n+\theta q}} dy \le 2 \int_{\mathbb{R}^n} \frac{(K(f,|y|))^q}{|y|^{n+\theta q}} dy = 2\alpha_n \int_0^\infty \frac{(K(f,r))^q}{r^{1+\theta q}} dr \le C \, \|f\|^q_{\theta,q} \,. \tag{2.4.58}$$

Then (2.4.57) and (2.4.58) combine with (2.4.50) to show that

$$\|f\|_{B^{\theta,p}_q} \le C \, \|f\|_{\theta,q} \,, \tag{2.4.59}$$

and we deduce that

$$(L^p(\mathbb{R}^n;\mathbb{F}); W^{1,p}(\mathbb{R}^n;\mathbb{F}))_{\theta,q} \hookrightarrow B^{\theta,p}_q(\mathbb{R}^n;\mathbb{F}), \tag{2.4.60}$$

which completes the proof.

□

Theorem 2.4.6 allows us to deduce some simple properties of Besov spaces with minimal effort. We consider two examples of this now.

Example 2.4.7. Using Corollary 2.2.10 and Theorem 2.4.6 in conjunction, we derive the interpolation estimate

$$\|f\|_{B^{s,p}_q} \le C \, \|f\|^{1-s}_{L^p} \, \|f\|^s_{W^{1,p}} \text{ for all } f \in W^{1,p}(\mathbb{R}^n;\mathbb{F}), \tag{2.4.61}$$

where $C \in \mathbb{R}_+$ is a constant depending on the parameters. △

Example 2.4.8. Suppose that $1 < p < n$. Then by the Gagliardo-Nirenberg-Sobolev inequality, we know that $W^{1,p}(\mathbb{R}^n;\mathbb{F}) \hookrightarrow L^{p^*}(\mathbb{R}^n;\mathbb{F})$, where

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{2.4.62}$$

This tells us that the identity map $I : L^p(\mathbb{R}^n;\mathbb{F}) + W^{1,p}(\mathbb{R}^n;\mathbb{F}) \to L^p(\mathbb{R}^n;\mathbb{F}) + L^{p^*}(\mathbb{R}^n;\mathbb{F})$ satisfies $I \in \mathcal{L}(L^p; L^p)$ and $\mathcal{L}(W^{1,p}; L^p)$. Then Theorem 2.2.12 shows that I is a bounded linear map from $(L^p(\mathbb{R}^n;\mathbb{F}), W^{1,p}(\mathbb{R}^n;\mathbb{F}))_{s,q} = B^{s,p}_q(\mathbb{R}^n;\mathbb{F})$ to $(L^p(\mathbb{R}^n;\mathbb{F}), L^{p^*}(\mathbb{R}^n;\mathbb{F}))_{\theta,q} = L^{r,q}(\mathbb{R}^n;\mathbb{F})$ for

$$\frac{1}{r} = \frac{1-s}{p} + \frac{s}{p^*} = \frac{1}{p} - \frac{s}{n}. \tag{2.4.63}$$

This and Theorems 2.4.2 and 2.4.6 then provide a constant $C > 0$ such that

$$\|f\|_{L^{r,q}} \leq C \|f\|_{B^{s,p}_q} \text{ for all } f \in B^{s,p}_q(\mathbb{R}^n;\mathbb{F}). \tag{2.4.64}$$

In other words, we have the subcritical embedding of the Besov spaces into the Lorentz spaces: $B^{s,p}_q(\mathbb{R}^n;\mathbb{F}) \hookrightarrow L^{r,q}(\mathbb{R}^n;\mathbb{F})$. Since we trivially have the embedding $B^{s,p}_q(\mathbb{R}^n;\mathbb{F}) \hookrightarrow L^p(\mathbb{R}^n;\mathbb{F})$, we have that $B^{s,p}_q(\mathbb{R}^n;\mathbb{F}) \hookrightarrow L^p(\mathbb{R}^n;\mathbb{F}) \cap L^{r,q}(\mathbb{R}^n;\mathbb{F})$, and we can can then use Theorem 1.1.41 to further deduce that $B^{s,p}_q(\mathbb{R}^n;\mathbb{F}) \hookrightarrow L^{t,1}(\mathbb{R}^n;\mathbb{F})$ for all $p < t < r$.

In particular, if we take $q = p$ and note that $p \leq r$, then we get the fractional Sobolev embedding $W^{s,p}(\mathbb{R}^n;\mathbb{F}) \hookrightarrow L^{r,p}(\mathbb{R}^n;\mathbb{F}) \cap L^p(\mathbb{R}^n;\mathbb{F}) \hookrightarrow L^r(\mathbb{R}^n;\mathbb{F}) \cap L^p(\mathbb{R}^n;\mathbb{F})$.

△

Our final example computes the interpolation spaces between Besov and fractional Sobolev spaces.

Example 2.4.9. Let $1 \leq p < \infty$, $1 \leq q_0, q_1 \leq \infty$, and $0 < s_0, s_1 < 1$. We know from Theorem 2.4.6 that for $i \in \{0,1\}$,

$$B^{s_i,p}_{q_i}(\mathbb{R}^n;\mathbb{F}) = (L^p(\mathbb{R}^n;\mathbb{F}), W^{1,p}(\mathbb{R}^n;\mathbb{F}))_{s_i,q} \tag{2.4.65}$$

and is thus of type s_i. Let $1 \leq q \leq \infty$, $\theta \in (0,1)$, and define $1 < s_\theta < 1$ via

$$s_\theta = (1-\theta)s_0 + \theta s_1. \tag{2.4.66}$$

Then Theorems 2.3.8 and 2.4.2 combine to show that

$$(B^{s_0,p}_{q_0}(\mathbb{R}^n;\mathbb{F}), B^{s_1,p}_{q_1}(\mathbb{R}^n;\mathbb{F}))_{\theta,q} = (L^p(\mathbb{R}^n;\mathbb{F}), W^{1,p}(\mathbb{R}^n;\mathbb{F}))_{s_\theta,q} = B^{s_\theta,p}_q(\mathbb{R}^n;\mathbb{F}). \tag{2.4.67}$$

Thus, when we interpolate between Besov spaces with the same first integrability index we get another Besov space with the same first integrability index. Note that q_0 and q_1 play no role in determining the type, so if we set $q_0 = q_1 = p$, then we find the fractional Sobolev space interpolation result

$$(W^{s_0,p}(\mathbb{R}^n;\mathbb{F}), W^{s_1,p}(\mathbb{R}^n;\mathbb{F}))_{\theta,q} = B^{s_\theta,p}_q(\mathbb{R}^n;\mathbb{F}), \tag{2.4.68}$$

which in particular means that

$$(W^{s_0,p}(\mathbb{R}^n;\mathbb{F}), W^{s_1,p}(\mathbb{R}^n;\mathbb{F}))_{\theta,p} = W^{s_\theta,p}(\mathbb{R}^n;\mathbb{F}). \tag{2.4.69}$$

The above results extend also to the endpoints since $L^p(\mathbb{R}^n;\mathbb{F})$ is of type 0 and $W^{1,p}(\mathbb{R}^n;\mathbb{F})$ is of type 1. Indeed, we have that

$$(L^p(\mathbb{R}^n;\mathbb{F}), B^{s_1,p}_{q_1}(\mathbb{R}^n;\mathbb{F}))_{\theta,q} = B^{\theta s_1,p}_q(\mathbb{R}^n;\mathbb{F}) \text{ and } (L^p(\mathbb{R}^n;\mathbb{F}), W^{s_1,p}(\mathbb{R}^n;\mathbb{F}))_{\theta,p} = W^{\theta s_1,p}(\mathbb{R}^n;\mathbb{F}) \tag{2.4.70}$$

as well as

$$(B^{s_0,p}_{q_0}(\mathbb{R}^n;\mathbb{F}), W^{1,p}(\mathbb{R}^n;\mathbb{F}))_{\theta,q} = B^{(1-\theta)s_0+\theta,p}_q(\mathbb{R}^n;\mathbb{F}) \tag{2.4.71}$$

and

$$(W^{s_0,p}(\mathbb{R}^n;\mathbb{F}), W^{1,p}(\mathbb{R}^n;\mathbb{F}))_{\theta,p} = W^{(1-\theta)s_0+\theta,p}(\mathbb{R}^n;\mathbb{F}). \tag{2.4.72}$$

△

2.4.3 Interpolating between C^0 and C^1

It turns out that the technique we used above works equally well with $L^p(\mathbb{R}^n;\mathbb{F})$ replaced by $C_b^0(\mathbb{R}^n;\mathbb{F})$ and $W^{1,p}(\mathbb{R}^n;\mathbb{F})$ replaced by $C_b^1(\mathbb{R}^n;\mathbb{F})$. Here we recall that

$$C_b^0(\mathbb{R}^n;\mathbb{F}) = \{f : \mathbb{R}^n \to \mathbb{F} \mid f \text{ is continuous and bounded}\}, \text{ and}$$
$$C_b^k(\mathbb{R}^n;\mathbb{F}) = \{f : \mathbb{R}^n \to \mathbb{F} \mid f \text{ is k-times differentiable and } \partial^\alpha f \in C_b^0(\mathbb{R}^n;\mathbb{F}) \text{ for all } |\alpha| \le k\} \tag{2.4.73}$$

are Banach spaces with the norms

$$\|f\|_{C_b^0} = \sup_{x\in\mathbb{R}^n} |f(x)| \text{ and } \|f\|_{C_b^k} = \max_{|\alpha|\le k} \|\partial^\alpha f\|_{C_b^0}. \tag{2.4.74}$$

For $0 < \alpha \le 1$ and $k \in \mathbb{N}$ we also define

$$C_b^{k,\alpha}(\mathbb{R}^n;\mathbb{F}) = \{f \in C_b^k(\mathbb{R}^n;\mathbb{F}) \mid [f]_{C^{k,\alpha}} < \infty\}, \tag{2.4.75}$$

where

$$[f]_{C^{k,\alpha}} = \sup_{|\alpha|\le k} \sup_{x\neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\alpha}, \tag{2.4.76}$$

and we define $\|f\|_{C_b^{k,\alpha}} = \|f\|_{C_b^k} + [f]_{C^{k,\alpha}}$. The spaces $C_b^{k,\alpha}(\mathbb{R}^n;\mathbb{F})$ are Banach when endowed with these norms.

Theorem 2.4.10. *For $\theta \in (0,1)$ we have that*

$$(C_b^0(\mathbb{R}^n;\mathbb{F}), C_b^1(\mathbb{R}^n;\mathbb{F}))_{\theta,\infty} = (C_b^0(\mathbb{R}^n;\mathbb{F}), C_b^{0,1}(\mathbb{R}^n;\mathbb{F}))_{\theta,\infty} = C_b^{0,\theta}(\mathbb{R}^n;\mathbb{F}) \tag{2.4.77}$$

with equivalence of norms.

Proof. We will only prove that $(C_b^0(\mathbb{R}^n;\mathbb{F}), C_b^1(\mathbb{R}^n;\mathbb{F}))_{\theta,\infty} = C_b^{0,\theta}(\mathbb{R}^n;\mathbb{F})$. The second identity follows from essentially the same argument; we leave it as an exercise to check the details.

First note that $C_b^1(\mathbb{R}^n;\mathbb{F}) \hookrightarrow C_b^0(\mathbb{R}^n;\mathbb{F})$, so according to Theorem 2.2.16, we can use the equivalent norm

$$\|f\|_{C_b^0} + \sup_{0<t<1} t^{-\theta} K(f,t) \tag{2.4.78}$$

in place of the usual one. We will do so, and by abuse of notation, continue to refer to this quantity as $\|f\|_{\theta,\infty}$.

Let $f \in C_b^{0,\theta}(\mathbb{R}^n;\mathbb{C})$ and let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier. For $0 < t < 1$ write

$$f = (f - f * \eta_t) + f * \eta_t =: g + h. \tag{2.4.79}$$

Then

$$g(x) = \int_{\mathbb{R}^n} (f(x) - f(x-y)) t^{-n} \eta(y/t) dy \tag{2.4.80}$$

and we can estimate this via

$$\|g\|_{C_b^0} \le [f]_{C^{0,\theta}} \int_{\mathbb{R}^n} |y|^\theta t^{-n} \eta(y/t) \le C(\eta,\theta) t^\theta \|f\|_{C_b^{0,\theta}} \tag{2.4.81}$$

for

$$C(\eta,\theta) = \int_{\mathbb{R}^n} |x|^\theta \eta(x) dx < \infty. \tag{2.4.82}$$

On the other hand, since $0 < t < 1$,

$$\|h\|_{C_b^0} \le \|f\|_{C_b^0} \le t^{\theta-1} \|f\|_{C_b^{0,\theta}}. \tag{2.4.83}$$

Since

$$\int_{\mathbb{R}^n} \partial_i \eta(x) dx = 0 \text{ for all } 1 \le i \le n, \tag{2.4.84}$$

we have that

$$\partial_i h(x) = \int_{\mathbb{R}^n} f(y) t^{-n-1} \partial_i \eta((x-y)/t) dy = \int_{\mathbb{R}^n} (f(y) - f(x)) t^{-n-1} \partial_i \eta((x-y)/t) dy, \tag{2.4.85}$$

and so

$$\max_{1\le i\le n} \|\partial_i h\|_{C_b^0} \le [f]_{C^{0,\theta}} \max_{1\le i\le n} \int_{\mathbb{R}^n} |x-y|^\theta \, t^{-n-1} \left|\partial_i \eta((x-y)/t)\right| dy \le C'(\eta,\theta) t^{\theta-1} \|f\|_{C_b^{0,\theta}} \tag{2.4.86}$$

for

$$C'(\eta,\theta) = \sup_{1\le i\le n} \int_{\mathbb{R}^n} |x|^\theta \left|p_i \eta(x)\right| dx < \infty. \tag{2.4.87}$$

Hence, for $0 < t < 1$ we have that

$$K(f,t) \le \|g\|_{C_b^0} + t \|h\|_{C_b^1} \le C t^\theta \|f\|_{C_b^{0,\theta}} \tag{2.4.88}$$

for some constant $C = C(\eta,\theta) > 0$. From this we deduce that

$$\|f\|_{\theta,\infty} \le \|f\|_{C_b^0} + C \|f\|_{C_b^{0,\theta}} \le C \|f\|_{C_b^{0,\theta}}. \tag{2.4.89}$$

On the other hand, suppose now that $f \in (C_b^0(\mathbb{R}^n;\mathbb{F}), C_b^1(\mathbb{R}^n;\mathbb{F}))_{\theta,\infty}$ and write $f = g + h$ for $g \in C_b^0(\mathbb{R}^n;\mathbb{F})$ and $h \in C_b^1(\mathbb{R}^n;\mathbb{F})$. Then

$$|f(x) - f(y)| \le |g(x) - g(y)| + |h(x) - h(y)| \le 2\|g\|_{C_b^0} + \|h\|_{C_b^1} |x-y|, \tag{2.4.90}$$

and since this holds for all such decompositions we deduce that

$$|f(x) - f(y)| \le 2K(f, |x-y|). \tag{2.4.91}$$

Thus, for $0 < |x-y| < 1$ we have that

$$\frac{|f(x) - f(y)|}{|x-y|^\theta} \le 2 \sup_{0<t<1} t^{-\theta} K(f,t), \tag{2.4.92}$$

while for $1 \le |x-y| < \infty$ we have that

$$\frac{|f(x) - f(y)|}{|x-y|^\theta} \le |f(x) - f(y)| \le 2\|f\|_{C_b^0}. \tag{2.4.93}$$

Hence,

$$\|f\|_{C_b^0} + [f]_{C^{0,\theta}} \le 3\left(\|f\|_{C_b^0} + \sup_{0<t<1} t^{-\theta} K(f,t)\right) \le 3\|f\|_{\theta,\infty}. \tag{2.4.94}$$

□

Let's consider an example based on this result and reiteration.

Example 2.4.11. Let $0 < \theta_0, \theta_1, \sigma < 1$ and write $\theta_\sigma = (1-\sigma)\theta_0 + \sigma\theta_1$. From Theorems 2.4.10 and 2.3.8 we have the identities

$$(C^{0,\theta_0}(\mathbb{R}^n;\mathbb{F}), C^{0,\theta_1}(\mathbb{R}^n;\mathbb{F}))_{\sigma,\infty} = C^{0,\theta_\sigma}(\mathbb{R}^n;\mathbb{F}), \tag{2.4.95}$$

$$(C^{0}(\mathbb{R}^n;\mathbb{F}), C^{0,\theta_1}(\mathbb{R}^n;\mathbb{F}))_{\sigma,\infty} = C^{0,\sigma\theta_1}(\mathbb{R}^n;\mathbb{F}), \tag{2.4.96}$$

and

$$(C^{0,\theta_0}(\mathbb{R}^n;\mathbb{F}), C^{1}(\mathbb{R}^n;\mathbb{F}))_{\sigma,\infty} = C^{0,(1-\sigma)\theta_0+\sigma}(\mathbb{R}^n;\mathbb{F}). \tag{2.4.97}$$

△

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