Γ-CONVERGENCE OF THE GINZBURG-LANDAU ENERGY

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1. Introduction – Difficulties with harmonic maps

Let us begin by recalling *Dirichlet's principle*. Let $n, m \geq 1$ be integers, $\Omega \subset \mathbb{R}^n$ be an open, bounded set with $\partial\Omega$ smooth, and $g \in C^\infty(\partial\Omega; \mathbb{R}^m)$. Define the space

$$H^1_g(\Omega;\mathbb{R}^m) := \{u \in H^1(\Omega;\mathbb{R}^m) \mid \ u|_{\partial\Omega} = g\}$$

and consider the *Dirichlet energy* $E : H^1_g(\Omega;\mathbb{R}^m) \to \mathbb{R}$ given by

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2.$$

Then Dirichlet's principle asserts that

$$E(u) = \min_{H^1_g(\Omega;\mathbb{R}^m)} E \iff u \text{ is a weak solution to} \begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

From elliptic regularity one immediately has that, should such a minimizer u exist, $u \in C^\infty(\bar{\Omega})$. Then u is in fact a smooth (analytic, even) harmonic function obtaining the boundary value g, i.e. a solution to the classical *Dirichlet problem* with boundary data g. A standard exercise in the *Direct Method of the Calculus of Variations* provides for the existence of a minimizer, i.e. $\exists\, u \in H^1_g(\Omega;\mathbb{R}^m)$ such that $E(u) = \min_{H^1_g(\Omega;\mathbb{R}^m)} E$ (via choosing a minimizing sequence and using compactness and lower semicontinuity of the norm). From this we deduce that minimizing the Dirichlet energy over $H^1_g(\Omega;\mathbb{R}^m)$ produces a very well-behaved minimizer.

Now we ask ourselves the following question: what if g takes value in $\mathcal{M} \subset \mathbb{R}^m$, where $\mathcal{M}$ is a smooth compact submanifold of $\mathbb{R}^m$ without boundary? For example, consider $\mathcal{M} = \mathbb{S}^{m-1} = \{x \in \mathbb{R}^m \text{ s.t. } |x| = 1\}$. Can we find a u minimizing the Dirichlet energy such that $u(x) \in \mathcal{M}$ for almost every $x \in \Omega$?

In order to study these questions we first define

$$H^1_g(\Omega;\mathcal{M}) := \{u \in H^1_g(\Omega;\mathbb{R}^m) \,|\, u(x) \in \mathcal{M} \text{ for a.e. } x \in \Omega\}.$$

Notice that $H^1_g(\Omega;\mathcal{M})$ has neither the structure of a vector space nor a convex set. Then we consider the minimization problem of finding $u \in H^1_g(\Omega;\mathcal{M})$ so that $E(u) = \min_{H^1_g(\Omega;\mathcal{M})} E$. This is called the *Harmonic Mapping Problem*.

Let's assume that $H^1_g(\Omega;\mathcal{M}) \neq \emptyset$. Then:

(1) A version of the direct method produces a minimizer. The only tricky part is showing that the limit extracted from the minimizing sequence, u, satisfies $u(x) \in \mathcal{M}$ a.e. This can be done by extracting a further subsequence that converges a.e. to u (first using Rellich to get L^2 convergence).

(2) The minimizer u satisfies a semilinear PDE that couples to the geometry of $\mathcal{M}$. By making variations of u that remain $\mathcal{M}-$valued one can show that

$$\begin{cases} -\Delta u(x) \perp T_{u(x)}\mathcal{M}, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

In particular, if $\mathcal{M} = \mathbb{S}^{m-1} \subset \mathbb{R}^m$, then this reduces to the PDE

$$\begin{cases} -\Delta u = u|\nabla u|^2, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

More generally, the right hand side of the equation involves the 2^{nd} fundamental form of $\mathcal{M}$.

(3) The regularity of u is not so clear. For instance, when $\mathcal{M} = \mathbb{S}^{m-1}$ we only have that $\Delta u \in L^1$, which is where the usual elliptic regularity theory breaks down.

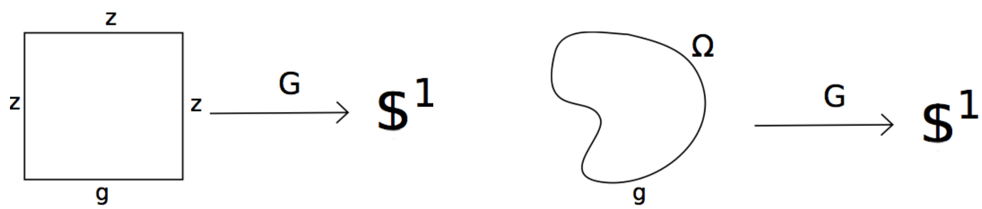
There is a huge literature devoted to this problem. A mathscinet search turns up over 3000 papers. Several books have been written on the subject. We refer to the books of Fanghua Lin and Changyou Wang [8] and Frédéric Hélein [5] for a survey of what is known. The general picture is that things are nowhere near as nice as in the case of $\mathbb{R}^m-$valued functions. For example,

(1) A result of Tristan Rivière [10] shows that there are sphere-valued (weak) harmonic maps that are everywhere discontinuous.

(2) There can be topological obstructions and non-existence of minimizers.

Let us now explore this second item in the simple case when $\Omega \subset \mathbb{R}^2$ and $\mathcal{M} = \mathbb{S}^1$.

Example 1.1. *Let $\Omega \subset \mathbb{R}^2$ be bounded, open, and simply connected with $\partial\Omega$ smooth. Set $\mathcal{M} = \mathbb{S}^1$ and note that $\partial\Omega \simeq \mathbb{S}^1$. This means that topology may play a role. Set deg$(g) :=$ winding$\# \in \mathbb{Z}$. Let us recall the notion of homotopy: let $z \in \partial\Omega$; then we say $g \sim 0 \iff \exists\, G : \partial\Omega \times [0,1] \to \mathbb{S}^1$ continuous with $G(x,0) = g(x)$ and $G(x,1) = z$. We can then easily prove that $g \sim 0 \iff G$ in Figure 1(a) is continuous (here the bottom of the square is identified with $\partial\Omega$ torn apart at $x \in \partial\Omega$, where $g(x) = z$) $\iff G$ in Figure 1(b) is continuous (here we exploit the fact that Ω is simply connected to collapse the boundary of the square to $\partial\Omega$).*



(a) G maps the square into $\mathbb{S}^1$ (b) G maps Ω into $\mathbb{S}^1$

FIGURE 1. The map G

Notice that if $H^1_g(\Omega;\mathbb{S}^1) \neq \emptyset$, then by the above argument there exist a minimizer u. A result due to C. Morrey (see chapter 3 of Hélein's book [5]) implies that $u \in C^0(\bar{\Omega})$. Thus $u = G$ is a continuous extension of g from Figure 1(b), and hence $g \sim 0$. From this we easily see that deg(g) $= 0$ (i.e. it cannot wrap around the circle at all!).

The previous example shows that $\deg(g) \neq 0 \Rightarrow H^1_g(\Omega;\mathbb{S}^1) = \emptyset$, and hence there can be no minimizer. The topology (degree) of the map g obstructs the existence of a minimizer. In fact, one can show the stronger result that

$$\deg(g) \neq 0 \iff H^1_g(\Omega;\mathbb{S}^1) = \emptyset.$$

2. Relaxation

Let us henceforth assume that $\Omega \subset \mathbb{R}^2$ is bounded, open, and simply connected with $\partial\Omega$ smooth, and that $g \in C^\infty(\partial\Omega;\mathbb{S}^1)$ satisfies $\deg(g) > 0$ (the case $\deg(g) < 0$ can be recovered via complex conjugation). In order to attack the topological obstruction problem we will introduce a modification of the problem by considering a "relaxation of the energy functional E."

Let's consider the *Ginzburg-Landau energy*

$$E_\epsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(1 - |u|^2)^2,$$

where $E_\epsilon : H^1_g(\Omega;\mathbb{R}^2) \to \mathbb{R}$ and $\epsilon \in (0,1)$. The first term is the usual Dirichlet energy, but the second is a "penalization term" that should force minimizers to be nearly $\mathbb{S}^1-$valued as $\epsilon \to 0$. Our goal then is to study this functional in the regime $\epsilon \to 0$ in order to extract some information about the (vacuous) $\mathbb{S}^1-$valued harmonic mapping problem.

Although we have motivated the introduction of the Ginzburg-Landau functional via the study of harmonic maps, the functional is of interest in many other areas. For example it arises in the study of superconductors, where it appeared in the work of Ginzburg and Landau [4]. It also arises as one of the simplest examples of a Yang-Mills-Higgs gauge theory within the realm of particle physics. The "potential term" $(1 - |u|^2)^2$ is related to the "Higgs mechanism" in the standard model of particle physics. For more on this we refer to the book of Manton and Sutcliffe [9].

The first order of business in studying E_ϵ is to make sure that minimization is possible. We observe the following.

(1) Using the direct method we can deduce the existence of a minimizer v_ϵ. It's easy to compute the associated Euler-Lagrange equations, and we see that v_ϵ solves the problem

$$\begin{cases} -\Delta v_\epsilon = \dfrac{v_\epsilon}{\epsilon^2}(1-|v_\epsilon|^2), & \text{in } \Omega, \\ v_\epsilon = g, & \text{on } \partial\Omega. \end{cases}$$

(2) By elliptic regularity $v_\epsilon \in C^\infty(\bar{\Omega})$. The associated PDE for $|v_\epsilon|$ obeys a maximum principle, which yields the bound $|v_\epsilon| \leq 1$.

(3) For every $\delta \in (0,1)$ we have $|\{x \in \Omega \,|\, |v_\epsilon| < \delta\}| > 0$. Otherwise we can prove that $H^1_g(\Omega; \mathbb{S}^1) \neq \emptyset$, a contradiction.

(4) $E_\epsilon(v_\epsilon) \to +\infty$ as $\epsilon \to 0^+$. Otherwise bounded energy lets us construct, via compactness, a $u \in H^1_g(\Omega; \mathbb{S}^1)$, which is again a contradiction.

The latter two observations suggest that in order to extract information as $\epsilon \to 0$ we must:

(1) identify the (divergent) energy scale of E_ϵ consistent with that of minimizers,

(2) understand the relationship between the sets $\{x \in \Omega \,|\, |v_\epsilon| < \delta\}$ and the divergence of the energy.

3. Vortices

An analysis of the minimizers v_ϵ of E_ϵ as $\epsilon \to 0$ was first completed in 1994 in the seminal book by Bethuel, Brezis, and Hélein [3]. There they uncovered the connection between the divergence of E_ϵ and the appearance of *vortices*. Roughly speaking, a vortex is an isolated patch where $|u| \approx 0$, around which there is a nontrivial winding number.

The simplest example of a map with non-trivial winding number is given when $B_R = \{|x| < R\}$ and $g_1 : \partial B_R \to \mathbb{S}^1$ is given by $g_1(x) = \dfrac{x}{|x|}$. Then $\deg(g) = 1$. Consider the following (defined as in [3]):

$$I(\epsilon, R) := \min_{H^1_{g_1}(B_R;\mathbb{R}^2)} \left(\int_{B_R} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(1-|u|^2)^2 \right),$$

where g_1 is as above. By the observations of the previous section we know that $I(\epsilon, R)$ is well-defined and the minimum is achieved. Let us define

$$\varphi(t) := I(t, 1).$$

An easy scaling argument implies that

$$I(\epsilon, R) = \varphi(\frac{\epsilon}{R}) = I(1, \frac{R}{\epsilon}).$$

Lemma 3.1 (Bethuel-Brezis-Hélein)**.** *If $t_1 \leq t_2$ then $\varphi(t_1) \leq \pi \log\left(\dfrac{t_2}{t_1}\right) + \varphi(t_2)$.*

Proof. Let u_2 be the minimizer for $\varphi(t_2) = I(1, 1/t_2)$. Let

$$u_1(x) = \begin{cases} u_2(x), & |x| < \frac{1}{t_2} \\ \frac{x}{|x|}, & \frac{1}{t_2} \leq |x| < \frac{1}{t_1}. \end{cases}$$

Then

$$
\begin{aligned}
\varphi(t_1) = I(1, 1/t_1) &\leq \frac{1}{2}\int_{B_{1/t_1}} |\nabla u_1|^2 + \frac{1}{4}\int_{B_{1/t_1}} \left(1 - |u_1|^2\right)^2 \\
&= \int_{B_{1/t_2}} |\nabla u_2|^2 + \frac{1}{4}\left(1 - |u_2|^2\right)^2 + \frac{1}{2}\int_{B_{1/t_1} \setminus B_{1/t_2}} \left|\nabla \frac{x}{|x|}\right|^2 \\
&= I(t_2) + \pi \log\left(\frac{t_2}{t_1}\right).
\end{aligned}
$$

□

By Lemma 3.1 we immediately get

$$
\epsilon < R \Rightarrow I(\epsilon, R) = \varphi\left(\frac{\epsilon}{R}\right) \leq \pi \log\left(\frac{R}{\epsilon}\right) + \varphi(1).
$$

This allows us to construct a generic upper bound for the energy of minimizers.

Theorem 3.2 (Bethuel-Brezis-Hélein). *Let $deg(g) = d > 0$. If $E_\epsilon(v_\epsilon) = \min_{H^1_g} E_\epsilon$ then $E_\epsilon(v_\epsilon) \leq \pi d \log \frac{1}{\epsilon} + C$ for a constant $C > 0$ depending on Ω and g.*

Sketch of the proof. Drill d holes out of the domain Ω as in Figure 2: find $R > 0$ and $\{a_1, \ldots, a_d\} \subset \Omega$ such that $B(a_i, R) \cap B(a_j, R) = \emptyset$ for $i \neq j$ and $B(a_i, R) \subset \Omega$. Let $\Omega_R = \Omega \setminus \bigcup_{i=1}^{d} B(a_i, R)$. Then because Ω_R is no longer simply connected, one may find $v \in C^\infty(\bar{\Omega}_R)$ such that

$$
v|_{\partial\Omega} = g \text{ and } v|_{\partial B(a_i, R)}(x) = \frac{(x - a_i)}{|x - a_i|}.
$$

Glue together translates of the minimizers of $I(\epsilon, R)$ to $B(a_i, R)$ with v to obtain that

$$
E_\epsilon(v_\epsilon) \leq \int_{\Omega_R} \frac{1}{2}|\nabla v|^2 + dI(\epsilon, R) \leq \pi d \log \frac{1}{\epsilon} + C.
$$

□

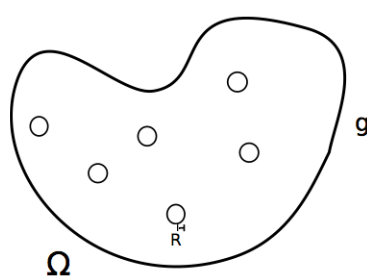


FIGURE 2. Drilling holes in Ω

Before moving on let's introduce some useful tools: we define the *current* $j(u) : \Omega \to \mathbb{R}^2$ as

$$j(u) := u_1 \nabla u_2 - u_2 \nabla u_1 = u^*(d\sigma_{\mathbb{S}^1}),$$

where $u = (u_1, u_2)$ and $u^*(d\sigma_{\mathbb{S}^1})$ is the pullback by u of the standard volume form on $\mathbb{S}^1$. We refer to the CNA Summer School mini-course of Bernard Dacorogna for more information on the pullback. The *Jacobian* $J(u) : \Omega \to \mathbb{R}$ is defined as

$$J(u) := \frac{1}{2}\mathrm{curl} j(u) = \det \nabla u.$$

Notice the following.

(1) If $|u|^2 = c$ in B open, then $\nabla u^T u = 0 \Rightarrow \det \nabla u = 0 \Rightarrow J(u) = 0$. Since we expect $|u| \approx 1$ in most of Ω, we expect $J(u)$ to concentrate in $\{u \approx 0\}$.

(2) If $u \in \mathbb{S}^1$ on ∂B_R, then

$$\int_{\partial B_R} j(u) \cdot \tau = 2\pi \deg(u, \partial B_R),$$

where τ is the unit tangent.

(3) If $u \in \mathbb{S}^1$ on ∂B_R, then

$$\int_{B_R} J(u) = \int_{B_R} \frac{1}{2}\mathrm{curl} j(u) = \frac{1}{2}\int_{\partial B_R} j(u) \cdot \tau = \pi \deg(u, \partial B_R),$$

and hence formally $J \approx \pi \sum_{i=1}^{d} \delta_{a_i}$, where a_i are "vortex locations."

Theorem 3.2 establishes an upper bound for the order of the energy of minimizers of E_ϵ. In order to prove a $\Gamma-$convergence result we need a corresponding lower bound. To motivate this let's consider the idealized case in which $u : B_R \to \mathbb{R}^2$ satisfies $|u| = 1$ on $B_R \backslash B_\epsilon$ and $\deg(u, \partial B_r) = d$ for $\epsilon \le r \le R$. We may easily estimate $|j(u)| \le |u||\nabla u|$, and so the Cauchy-Schwarz inequality yields the estimate

$$\begin{aligned}
\int_{B_R} \frac{1}{2}|\nabla u|^2 &\ge \int_\epsilon^R \int_{\partial B_r} \frac{1}{2}|\nabla u|^2 dr = \int_\epsilon^R \int_{\partial B_r} \frac{1}{2}|u|^2|\nabla u|^2 dr \ge \int_\epsilon^R \int_{\partial B_r} \frac{1}{2}|j(u)|^2 dr \\
&\ge \int_\epsilon^R \frac{1}{4\pi r}\left(\int_{\partial B_r} |j(u)|\right)^2 dr \ge \int_\epsilon^R \frac{1}{4\pi r}\left(\int_{\partial B_r} j(u) \cdot \tau\right)^2 dr = \int_\epsilon^R \frac{1}{4\pi r}\left(2\pi \deg(u, \partial B_r)\right)^2 dr \\
&= \pi d^2 \log \frac{R}{\epsilon} \ge \pi |d| \log \frac{R}{\epsilon}.
\end{aligned}$$

From this computation we see that we should expect to be able to construct a lower bound for the energy at the scale $O(\log \frac{1}{\epsilon})$, which matches the scale identified in Theorem 3.2.

4. Γ-convergence

The $\Gamma-$convergence of E_ϵ was established in a number of papers using a variety of different techniques. Below we will attempt to give a crude summary of the main ideas and references to various papers. The references given are by no means exhaustive. For a concise approach to this problem we refer to the short paper by Alicandro and Ponsiglione [2].

As discussed in the CNA Summer School mini-course of Giovanni Leoni, we know that in order to establish a $\Gamma-$convergence result for E_ϵ we need three things:

(1) a compactness result,
(2) a general lower bound for the energy,
(3) a construction of sequence achieving a matching upper bound.

Motivated by the computations above we expect to be able to prove a lower bound in terms of degrees and location of the vortices. We also expect $J \approx \pi \sum_{i=1}^d \delta_{a_i}$, and so it's reasonable to look for a compactness result for $J(u)$. The following theorem guarantees these things. It comprises a version of results proved by various authors in various contexts. The lower bounds are originally due to Sandier [11] and Jerrard [6], the compactness is due to Jerrard and Soner [7], and variants with certain improvements were established, for example, by Sandier and Serfaty [12], Serfaty and Tice [13], and Tice [14].

Theorem 4.1 (Vortex balls construction). *Assume $E_\epsilon(u_\epsilon) \le C|\log\epsilon|$. Then there exists a finite disjoint collection of balls $\{B(a_i, r_i)\}_{i=1}^N$ such that*

(1) $B(a_i, r_i) \subset \Omega,\ \forall\, i = 1, \ldots, N,$
(2) $\dfrac{1}{|\log\epsilon|^2} \le \sum r_i := r = o(1),$ *as* $\epsilon \to 0^+,$
(3) $\{|u_\epsilon| \le 1/2\} \subset \bigcup_i B(a_i, r_i),$
(4) $\dfrac{1}{2}\displaystyle\int_{\cup B(a_i,r_i)} |\nabla u_\epsilon|^2 \ge \pi \sum |d_i| \log\frac{r}{\epsilon} - c,$ *where* $d_i = deg\left(\dfrac{u_\epsilon}{|u_\epsilon|}, \partial B(a_i, r_i)\right),$
(5) *for fixed* $\alpha \in (0,1)$, *up to extracting a subsequence we have*
$$||J(u_\epsilon) - \pi \sum_{i=1}^N d_i \delta_{a_i}||_{(C_c^{0,\alpha})^*} \to 0$$
as $\epsilon \to 0^+$.

The construction of a "recovery sequence" achieving the upper bound can be carried out by essentially following the sketch of Theorem 3.2. This leads to the zeroth-order $\Gamma-$convergence result for E_ϵ.

Theorem 4.2 (Zeroth-order $\Gamma-$limit). *Let $deg(g) = d > 0$. The following hold.*

(1) *Fix $\alpha \in (0,1)$. Let $u_\epsilon \in H^1_g(\Omega;\mathbb{R}^2)$ satisfy $E_\epsilon(u_\epsilon) \le M \log\frac{1}{\epsilon}$ for some $M > 0$. Then up to the extraction of a subsequence, $J(u_\epsilon) \to \pi \sum_{i=1}^N d_i\delta_{a_i}$ in $(C_c^{0,\alpha})^*$, where $d_i \in \mathbb{Z}\setminus\{0\}$, $a_i \in \Omega$,*

and $\sum d_i = d$. *Moreover,*

$$\liminf_{\epsilon\to 0} \frac{E_\epsilon(u_\epsilon)}{|\log\epsilon|} \geq \pi \sum_{i=1}^{N} |d_i|.$$

In particular, $N \leq \sum_{i=1}^N |d_i| \leq M/\pi$.

(2) *Given* $(a_i, d_i) \in (\Omega \times \mathbb{Z}\setminus\{0\})^N$, *there exist* $u_\epsilon \in H^1_g(\Omega;\mathbb{R}^2)$ *such that* $J(u_\epsilon) \to \pi \sum_{i=1}^{N} d_i\delta_{a_i}$ *and*

$$\limsup_{\epsilon\to 0} \frac{E_\epsilon(u_\epsilon)}{|\log\epsilon|} = \pi \sum_{i=1}^{N} |d_i|.$$

This theorem says that the zeroth-order $\Gamma-$limit of E_ϵ simply counts the "total vorticity," i.e. that if the limiting "vorticity measure" is $\mu = \sum_{i=1}^N d_i\delta_{a_i}$ then the $\Gamma-$limit is $||\mu||$. This result was significantly generalized to the case of $u : \mathbb{R}^{n+k} \supset \Omega \to \mathbb{R}^k$ for $n \geq 0$, $k \geq 2$ in the work of Alberti, Baldo, and Orlandi [1].

The zeroth-order convergence essentially only sees the energy within the vortex balls. What about the energy outside?

Let $\tilde\Omega = \Omega \setminus \cup B(a_i, r_i)$, and consider the minimization problem

$$\min\left\{\int_{\tilde\Omega} \frac{1}{2}|\nabla u|^2 \,|\, u : \tilde\Omega \to \mathbb{S}^1,\ u = g \text{ on } \partial\Omega,\ \deg(u, \partial B_i) = d_i\right\}.$$

Write $u = e^{i\psi}$ (identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the natural way); then the minimizer satisfies

$$\begin{cases} \Delta\psi = 0, & \text{in } \tilde\Omega, \\ \frac{\partial\psi}{\partial\tau} = \text{ given by } g, & \text{on } \partial\Omega, \\ \int_{B_i} \frac{\partial\psi}{\partial\tau} = 2\pi d_i, & \text{on } \partial B_i. \end{cases}$$

Let's introduce $\nabla\varphi = \nabla^\perp\psi$. Then

$$\begin{cases} \Delta\varphi = 0, & \text{in } \tilde\Omega, \\ \frac{\partial\varphi}{\partial\nu} = \text{ given}, & \text{on } \partial\Omega, \\ \int_{B_i} \frac{\partial\varphi}{\partial\nu} = 2\pi d_i, & \text{on } \partial B_i. \end{cases}$$

As $r \to 0$ we formally expect $\varphi \sim \Phi_0$, where

$$\begin{cases} \Delta\Phi_0 = 2\pi\sum d_i\delta_{a_i}, & \text{in } \Omega, \\ \frac{\partial\Phi_0}{\partial\nu} = \text{ given}, & \text{on } \partial\Omega. \end{cases}$$

We may write

$$\Phi_0(x) = 2\pi\sum d_i \log|x - a_i| + R(x),$$

where R is a smooth function. Then we expect

$$\min \int_{\tilde{\Omega}} \frac{1}{2}|\nabla u|^2 = \int_{\tilde{\Omega}} \frac{1}{2}|\nabla \psi|^2 \approx \int_{\tilde{\Omega}} \frac{1}{2}|\nabla \varphi|^2 \approx \int_{\tilde{\Omega}} \frac{1}{2}|\nabla \Phi_0|^2$$
$$= \pi \sum |d_i| \log \frac{1}{r} + W(a_1, \dots, a_N; d_1, \dots, d_N) + O(1),$$

where

$$W(a_1, \dots, a_N; d_1, \dots, d_N) = -\sum_{i \neq j} d_i d_j \log |a_i - a_j| + O(1)$$

is the *renormalized energy.* Notice that for vortices with degrees of the same sign, W increases as $|a_i - a_j|$ decreases, while for vortices with degrees of opposite sign, W decreases as $|a_i - a_j|$ decreases. This is interpreted as a "repulsion" between like-signed vortices and an "attraction" between opposite-signed vortices, which is akin to the behavior of electrons. This and the localized particle-like behavior of vortices are important in the particle physics versions of the Ginzburg-Landau model (again we refer to [9]).

By accounting for the energy outside of the vortex balls and employing the renormalized energy, we may prove a first-order $\Gamma-$convergence result for E_ϵ.

Theorem 4.3 (First-order $\Gamma-$limit)**.** *Let deg$(g) = d > 0$. Then the following hold.*

(1) *If* $E_\epsilon(u_\epsilon) \leq \pi M \log \frac{1}{\epsilon} + C$ *for some integer* $M > 0$*, then up to a subsequence* $J(u_\epsilon) \to \pi \sum_{i=1}^N d_i \delta_{a_i}$ *in* $(C_c^{0,\alpha})^*$*. Here* $d_i \in \mathbb{Z} \setminus \{0\}$*,* $\sum d_i = d$*,* $\sum |d_i| \leq M$*. If* $\sum |d_i| = M$*, then* $|d_i| = 1 \forall\, i$ *and* $N = M$*.*

(2) *If* $E_\epsilon(u_\epsilon) \leq \pi M \log \frac{1}{\epsilon} + C$ *for some integer* $M > 0$ *and* $J(u_\epsilon) \to \pi \sum_{i=1}^M d_i \delta_{a_i}$ *in* $(C_c^{0,\alpha})^*$ *for* $d_i = \pm 1$*, then*

$$\liminf_{\epsilon \to 0} \left(E_\epsilon(u_\epsilon) - \pi M \log \frac{1}{\epsilon} \right) \geq W(a_1, \dots, a_M; d_1, \dots, d_M) + M\gamma,$$

where W *is the renormalized energy and* $\gamma > 0$ *is an explicit constant, computed in* [3].

(3) *Given* $\mu = \pi \sum_{i=1}^M d_i \delta_{a_i}$*,* $d_i = \pm 1$*, there exist* u_ϵ *such that* $J(u_\epsilon) \to \mu$ *and*

$$\limsup_{\epsilon \to 0} \left(E_\epsilon(u_\epsilon) - \pi M \log \frac{1}{\epsilon} \right) = W(a_1, \dots, a_M; d_1, \dots, d_M) + M\gamma.$$

If $M = d = \deg(g)$ in this theorem then in fact $d_i = 1$ for $i = 1, \dots, M = d$. In this case we know, since minimizers of E_ϵ converge to minimizers of the $\Gamma-$limit, that the vortices associated to the minimizers v_ϵ all have degree 1 and minimize the renormalized energy W. This result was proved directly, i.e. without the $\Gamma-$convergence theory, by Bethuel, Brezis, and Hélein in their book [3].

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